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Efficient Implementation with Interdependent Valuations and Maxmin Agents

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We consider a single object allocation problem with multidimensional signals and interdependent valuations. When agents' signals are statistically independent, Jehiel and Moldovanu [Efficient design with interdependent valuations, *Econometrica*, 69(5):1237-1259, 2001] show that efficient and Bayesian incentive compatible mechanisms generally do not exist. In this paper, we extend the standard model to accommodate maxmin agents and obtain necessary as well as sufficient conditions under which efficient allocations can be implemented. In particular, we derive a condition that quantifies the amount of ambiguity necessary for efficient implementation. We further show that under some natural assumptions on the preferences, this necessary amount of ambiguity becomes sufficient. Finally, we provide a definition of informational size such that given any nontrivial amount of ambiguity, efficient allocations can be implemented if agents are sufficiently informationally small.

Keywords: Efficient implementation, ambiguity aversion, multidimensional signal, interdependent valuation.

JEL classification: D61, D82

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1 Introduction

One of the fundamental problems in mechanism design is the conflict between efficiency and incentive compatibility. That is, there are situations in which efficient allocations are not implementable. A prominent impossibility result is obtained by Jehiel and Moldovanu (2001): in a general mechanism design setting with multidimensional signals and interdependent valuations, if signals are statistically independent, then except in some special, nongeneric cases, ex post efficient and interim incentive compatible mechanisms do not exist.[1] This result is obtained under the standard assumption that agents are expected utility maximizers. However, there is both experimental and empirical evidence challenging the expected utility assumption: due to lack of knowledge about the environment, agents may perceive ambiguity, that is, they might not have a unique prior that fully describes the uncertainty that they face, and moreover, agents desire strategies that are robust to their ambiguity.[2] Thus, the primary question we ask is: does the conflict between efficiency and incentive compatibility extend to environments with ambiguity averse agents? In the case of maxmin expected utility (Gilboa and Schmeidler (1989)), our answer is: No. That is, we show that the presence of ambiguity aversion overturns the impossibility result of Jehiel and Moldovanu (2001). In particular, we extend the Myersonian approach to a single object allocation problem with maxmin agents and explicitly identify necessary and sufficient conditions for efficient implementation. In addition, we provide conditions under which the efficient allocation is implementable with a small amount of ambiguity.

Our first step is to derive a necessary condition for an allocation rule to be implementable which generalizes the envelope formula familiar from Bayesian mech-

[1] Jehiel and Moldovanu (2001) generalize earlier results by Maskin (1992) and Dasgupta and Maskin (2000).

[2] Experimental results on the Ellsberg paradox reveal that agents exhibit ambiguity averse behavior in many situations (e.g., Ellsberg (1961), Halevy (2007)). Aryal et al. (2018) find empirical evidence of ambiguity in U.S. timber auctions.

anism design. This condition quantifies a nontrivial amount of ambiguity, which we call Minimal Ambiguity, that is necessary for efficient and incentive compatible mechanisms to exist. That some ambiguity is necessary is consistent with the impossibility result obtained by Jehiel and Moldovanu (2001) in the sense that without ambiguity the requirements of efficiency and incentive compatibility become incompatible.

Our next step is to identify conditions under which this necessary amount of ambiguity is sufficient for efficient and incentive compatible mechanisms to exist. A key observation is that if Minimal Ambiguity is satisfied, we can construct efficient mechanisms that satisfy local incentive compatibility constraints. Thus, our question becomes: under what conditions does local incentive compatibility imply global incentive compatibility? In Bayesian settings, Myerson (1981) showed that under a monotonicity condition, global incentive compatibility constraints can be obtained from adding up a sequence of local incentive compatibility constraints. To extend the classic Myersonian approach to environments with maxmin agents, we need to address two issues. The first is to identify the monotonicity condition in our setting. The other is to deal with the nonadditivity of the maxmin representation: the belief used in each constraint is endogenously determined and, hence, the sum of these local constraints can differ from the global one. Regarding the first issue, the desired monotonicity condition turns out to be a multidimensional extension of the familiar single-crossing condition from one-dimensional settings.[3] Regarding the second issue, if each agent's valuation function is linear in his own signal, such nonadditivity does not arise. Otherwise, the linearity condition on valuation functions can be replaced by two other restrictions on preferences: agents' valuation functions satisfy a familiar increasing differences condition and agents' preferences satisfy the comonotonic independence axiom of Schmeidler (1989).

Another contribution of the paper is to identify conditions under which the

[3]See Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Bergemann and Välimäki (2002).

amount of ambiguity sufficient for efficient implementation can be arbitrarily small. Specifically, we link the required size of ambiguity perceived by an agent to his informational size, a notion studied by McLean and Postlewaite (2002, 2004, 2015a,b). Intuitively, an agent is informationally small if his private information has a small marginal effect on other agents' valuations. We show that given any nontrivial amount of ambiguity, efficient allocations can be implemented if agents are sufficiently informationally small. One instance in which informational smallness arises naturally is when the number of agents is large. As a result, efficient and incentive compatible mechanisms exist in a large economy even when each agent perceives only a small amount of ambiguity. An immediate consequence of this result is that complete ambiguity is generally not necessary for implementing efficient allocations when agents have quasilinear utilities.[4]

The paper is organized as follows. In Section 2, we describe the general model. In Sections 3 and 4, we derive necessary and sufficient conditions for an allocation rule to be implementable. We then apply these results to the implementation of the efficient allocation rule in Section 5. In Section 5.1, we present results on informational smallness. In Section 6, we present a simple example to illustrate the main insight for our results. We conclude with discussion and related literature in the final section.

2 The Model

Information structure. Suppose that there is a single object to be allocated among N agents, indexed by $i \in \mathcal{I} := \{1, ..., N\}$. We assume that agent i observes a signal (or type) $s^i = (s^i_1, ..., s^i_N)$ drawn from a space $S^i \subseteq \mathbb{R}^N$. The coordinate s^i_j represents agent i's *one-dimensional* piece of information affecting agent j's valua-

[4]Complete ambiguity means each agent's set of beliefs contains *all* probability measures over the other agents' signals.

tion for the object.[5] This information structure is used by many models that appear in the literature on mechanism design with multidimensional signals, including Jehiel and Moldovanu (2001) and Jehiel et al. (1996).

We assume that each agent i's signal space S^i is compact and convex, and that it has a nonempty interior and a piecewise smooth boundary. We further assume that each S^i is a sublattice in $\mathbb{R}^N$ according to the usual product order. Let $S := \times_{i=1}^N S^i$ with s as generic element and let $S^{-i} := \times_{j \neq i} S^j$ with s^{-i} as generic element. For every $i, j \in \mathcal{I}$, let $S_j^i := \{s_j^i | s^i \in S^i\}$ and $S_j^{-i} := \times_{l \neq i} S_j^l$.

Given the information structure defined above, agent i's valuation for the object is given by $v^i(s_i^1, ..., s_i^N) \in \mathbb{R}_+$. We assume that each v^i is continuously differentiable and $\frac{\partial v^i}{\partial s_i^j} > 0$ for every $j \in \mathcal{I}$. Moreover, we assume that the family of functions $\{\frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i}\}_{s_i^{-i} \in S_i^{-i}}$ is equicontinuous at every s_i^i.

Observe that given $s_i^{-i} \in S_i^{-i}$, v^i is solely a function of s_i^i. Thus, for every $s_i^i \in S_i^i$, agent i is indifferent among all signals whose i^{th} components are equal to s_i^i. This leads to the following notation:

$$e(s_i^i) := \{t^i \in S^i | t_i^i = s_i^i\} \quad \forall s_i^i \in S_i^i, \forall i \in \mathcal{I}.$$

Notice that the set $e(s_i^i)$ is a singleton in two special cases: the case of private valuations and the case of one-dimensional signals.

Mechanisms. An **allocation rule** is a function $p : S \to \mathbb{R}^N$ such that for every $s \in S$, $0 \leq p^i(s) \leq 1$ and for every $s \in S$, $\sum_{i=1}^N p^i(s) = 1$. For reported signals s, the term $p^i(s)$ is the probability that agent i is awarded the object. An allocation rule p is **efficient** if

$$p^i(s) > 0 \Rightarrow i \in \operatorname*{argmax}_j v^j(s_j^1, ..., s_j^N) \quad \forall s \in S.$$

A **transfer scheme** is a function $x : S \to \mathbb{R}^N$, where $x^i(s)$ represents the transfer to agent i given the reports s. A **direct mechanism** is a pair (p, x) where p is an

[5]The essential assumption is s_i^i, the part of agent i's information affecting his own valuation, is one-dimensional. All of our results extend to environments where s_j^i is multidimensional for $j \neq i$.

allocation rule and x is a transfer scheme. A direct mechanism is **efficient** if the associated allocation rule is efficient.

Interim utilities. Let Σ^{-i} be the Borel algebra on S^{-i} and $\mathcal{F}^i$ be a set of probability measures on (S^{-i}, Σ^{-i}). This set represents agent i's beliefs about the other agents' signals. A key assumption here is agent i's set of beliefs $\mathcal{F}^i$ is independent of the realization of his signal.[6] We assume that $\mathcal{F}^i$ is weak* compact and convex.

We assume that agents have quasilinear preferences. Given a direct mechanism (p, x), agent i's **interim utility** from reporting t^i when his signal is s^i and everyone else reports truthfully is

$$u^i(t^i, s^i) := \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} \left(p^i(t^i, s^{-i}) v^i(s^i_i, s^{-i}_i) + x^i(t^i, s^{-i})\right) dF^i.$$

The function $\mu^i : S^i \to \mathbb{R}$ defined by $\mu^i(s^i) := u^i(s^i, s^i)$, is called agent i's **indirect utility function** associated with (p, x). Notice that when an agent is not awarded the object and receives zero transfer, his utility is normalized to be zero.

Environment. An environment is a tuple $\langle \mathcal{I}, \{v^i\}_{i=1}^N, \{S^i\}_{i=1}^N, \{\mathcal{F}^i\}_{i=1}^N \rangle$. We assume that the environment is common knowledge, but the realizations of the signals are private information.[7]

Next we define completeness of sets of signals under an allocation rule.

Definition 1. The collection $\{S^i\}_{i=1}^N$ is **complete under the allocation rule** p if for every $i \in \mathcal{I}$ and every $s^i_i \in S^i_i$, there exist signals $\bar{s}^i(s^i_i) = (\bar{s}^i_1(s^i_i), ..., \bar{s}^i_N(s^i_i)), \underline{s}^i(s^i_i) = (\underline{s}^i_1(s^i_i), ..., \underline{s}^i_N(s^i_i)) \in e(s^i_i)$ such that

$$p^i\left(\bar{s}^i(s^i_i), s^{-i}\right) \geq p^i\left(t^i, s^{-i}\right) \geq p^i\left(\underline{s}^i(s^i_i), s^{-i}\right) \quad \forall t^i \in e(s^i_i), \forall s^{-i} \in S^{-i}.$$

In words, the collection $\{S^i\}_{i=1}^N$ is complete under an allocation rule p if within each $e(s^i_i)$, there exist a "best" signal and a "worst" signal that generate respectively the highest and the lowest probability of obtaining the object, regardless of

[6] A discussion about what would happen if an agent's set of beliefs can depend on the realization of his signal is given in Section 7.

[7] A discussion about relaxing the assumption that the sets $\mathcal{F}^i$ are common knowledge is given in Section 7.

the signals of the others. To simplify the analysis, throughout this paper we will focus on allocation rules under which the collection $\{S^i\}_{i=1}^N$ is complete. Clearly, the collection $\{S^i\}_{i=1}^N$ is complete under any allocation rule in the case of private valuations and the case of one-dimensional signals, as the set $e(s_i^i)$ is a singleton in those two cases. In general, the collection $\{S^i\}_{i=1}^N$ is complete under monotonic allocation rules, in the sense that keeping agent i's valuation fixed while increasing the valuations of other agents will not increase agent i's probability of obtaining the object. For example, we show in Section 5 that the collection $\{S^i\}_{i=1}^N$ is complete under the efficient allocation rule.[8]

3 Interim Incentive Compatible Mechanisms

We first derive a first-order condition which quantifies a necessary amount of ambiguity for an allocation rule to be implementable. In Section 3.2, we present conditions under which the first-order condition is sufficient for implementability.

By the revelation principle, it is without loss of generality to restrict attention to incentive compatible direct mechanisms. A direct mechanism (p, x) is **interim incentive compatible** if

$$\mu^i(s^i) = u^i(s^i, s^i) \geq u^i(t^i, s^i) \quad \forall s^i, t^i \in S^i, \forall i \in \mathcal{I}.^9$$

An allocation rule p is **implementable** if there exists a transfer scheme x such that the direct mechanism (p, x) is interim incentive compatible.

[8] More generally, the collection $\{S^i\}_{i=1}^N$ is complete under an allocation rule p if for every $i \in \mathcal{I}$ and every $s_i^i \in S_i^i$, there exists a continuous function $\alpha : \times_{j \neq i} S_j^i \to \mathbb{R}^K$ for some $K \in \mathbb{N}$ such that $\hat{p}^i(s_i^i, \alpha(s_1^i, ..., s_{i-1}^i, s_{i+1}^i, ..., s_N^i), s^{-i}) := p^i(s^i, s^{-i})$ is monotone in α for every $s^{-i} \in S^{-i}$. The efficient allocation rule corresponds to the case in which $K = N - 1$ and α is the identity function for all $i \in \mathcal{I}$ and $s_i^i \in S_i^i$. A trivial example is when the allocation rule p depends on s^i only through s_i^i. Another example is when agent i's probability of obtaining the object increases in his externality, which corresponds to the case when $\alpha(s_1^i, ..., s_{i-1}^i, s_{i+1}^i, ..., s_N^i) = \max_{j \neq i} s_j^i$ (e.g., the agent with the largest externality is awarded the object, that is, $p^i(s) > 0 \Rightarrow \max_{j \neq i} s_j^i \geq \max_{k \neq l} s_l^k$).

[9] Observe that the definition of interim incentive compatibility only invokes pure strategies. This is without loss of generality if either of the following assumptions holds: (i) agents cannot reduce ambiguity by randomizing ex ante; (ii) agents cannot commit to the results of their randomizations. For a more detailed discussion about these assumptions see Saito (2015) and Ke and Zhang (2017).

3.1 First-order Condition

We start by deriving a necessary condition for a mechanism to be interim incentive compatible, which generalizes the envelope formula familiar from Bayesian mechanism design. Recall that in a Bayesian environment, the envelope formula yields an expression for the derivative of an agent's equilibrium utility with respect to his signal in any interim incentive compatible mechanism (e.g., Theorem 3.1 in Jehiel and Moldovanu (2001)). However, when agents are ambiguity averse, the envelope theorem may fail due to the nondifferentiability of the interim utility functions.[10] Instead, we establish lower and upper bounds for the derivative of an agent's equilibrium utility.

Lemma 3.1. *If p is implementable, then for every $i \in \mathcal{I}$, the associated indirect utility function μ^i is a Lipschitz continuous function of s_i^i; its derivative is defined almost everywhere and satisfies the following inequalities*

$$\min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i\left(\bar{s}^i(s_i^i), s^{-i}\right) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i \leq \frac{\partial \mu^i(s^i)}{\partial s_i^i} \leq \max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i\left(\underline{s}^i(s_i^i), s^{-i}\right) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i.$$

One implication of Lemma 3.1 is that an agent's equilibrium utility associated with a given allocation rule may not be uniquely determined up to a constant. Thus, the payoff equivalence result may fail in the presence of ambiguity aversion.[11] However, it follows from Lemma 3.1 that the range of indirect utilities is determined by the allocation rule.

Another immediate implication of Lemma 3.1 is that a minimum requirement on the sets of beliefs must be imposed in order for the allocation rule p to be implementable. We now define the key concept of the paper—Minimal Ambiguity.

Definition 2. The collection $\{\mathcal{F}^i\}_{i=1}^N$ satisfies **Minimal Ambiguity under the allo-**

[10]It is well known that maxmin preferences have "kinks at certainty". See, for example, Dow and Werlang (1992).

[11]This point has already been noted by Bodoh-Creed (2012) and Wolitzky (2016). Carbajal and Ely (2013) state a similar result in a more general mechanism design setting.

cation rule p if for every $i \in \mathcal{I}$ and every $s_i^i \in S_i^i$,

$$\min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i\left(\bar{s}^i(s_i^i), s^{-i}\right) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i \leq \max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i\left(\underline{s}^i(s_i^i), s^{-i}\right) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i. \quad (1)$$

Since the inequality in (1) is necessary for implementation, it is easily seen that when agents are Bayesian, an allocation rule p is implementable only if p depends on s^i only through s_i^i. For example, this is the case when valuations are private or when signals are one-dimensional. However, agent i's information s_j^i for $j \neq i$ generally affects his probability of being awarded the object. Consequently, Minimal Ambiguity is violated when agents are Bayesian. For example, the signals s_j^i for $j \neq i$ are clearly relevant to determining the efficient allocation. Thus, given the efficient allocation rule p, there exist $s_i^i \in S_i^i$ and $s^{-i} \in S^{-i}$ such that $p^i\left(\bar{s}^i(s_i^i), s^{-i}\right) > p^i\left(\underline{s}^i(s_i^i), s^{-i}\right)$. As a result, a nontrivial amount of ambiguity is necessary for Minimal Ambiguity to hold and, hence, for the efficient allocation rule p to be implementable.

To see explicitly how the inequality in (1) quantifies a minimal amount of ambiguity for implementation, we consider two particular specifications of sets of priors that offer scalar parametrizations of ambiguity aversion and that are commonly used in the literature on robust Bayesian analysis.

Example 1. [ε-contamination[12]] We refer to ε-contamination if agent i's set of beliefs $\mathcal{F}^i$ is given by

$$C_\varepsilon(F_*^i) := \{(1-\varepsilon)F_*^i + \varepsilon G^i | G^i \in \Delta(S^{-i})\}$$

where $F_*^i \in \Delta(S^{-i})$ and $\varepsilon \in [0,1]$.[13] Intuitively, agent i puts a weight of $1-\varepsilon$ on the other agents' signals being drawn from the distribution F_*^i, but puts ε weight that the signals could be drawn from any other distribution. Thus, $1-\varepsilon$ can be interpreted as the agent's "degree of confidence" in the belief F_*^i. The larger ε is,

[12] The axiomatic foundation for the ε-contamination model is provided by Kopylov (2016). Bose et al. (2006) and Bose and Daripa (2009) adopt this formulation to study the problem of optimal auction design.

[13] For any measurable space (Ω, Σ), let $\Delta(\Omega)$ denote the set of all probability measures on (Ω, Σ).

the more ambiguity averse the agent is.[14] For simplicity, suppose that $\frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i}$ is independent of s_i^{-i} for every $i \in \mathcal{I}$. Then when agents' preferences are represented by ε-contamination, the inequality in (1) reduces to

$$\varepsilon \geq \frac{\int_{S^{-i}} \left(p^i\left(\bar{s}^i(s_i^i), s^{-i}\right) - p^i\left(\underline{s}^i(s_i^i), s^{-i}\right) \right) dF_*^i}{1 + \int_{S^{-i}} \left(p^i\left(\bar{s}^i(s_i^i), s^{-i}\right) - p^i\left(\underline{s}^i(s_i^i), s^{-i}\right) \right) dF_*^i} \quad \forall s_i^i \in S_i^i, \forall i \in \mathcal{I}.$$

Recall that by construction, $p^i\left(\bar{s}^i(s_i^i), s^{-i}\right) \geq p^i\left(\underline{s}^i(s_i^i), s^{-i}\right)$ for every $s_i^i \in S_i^i$, $s^{-i} \in S^{-i}$, and $i \in \mathcal{I}$. Clearly, if p depends on s^i only through s_i^i, that is, $p^i\left(\bar{s}^i(s_i^i), s^{-i}\right) - p^i\left(\underline{s}^i(s_i^i), s^{-i}\right) = 0$, then Minimal Ambiguity is satisfied for all $\varepsilon \geq 0$; otherwise, Minimal Ambiguity establishes a positive lower bound on ε for implementing the allocation rule p.

Example 2. In the second specification, agent i's set of beliefs $\mathcal{F}^i$ is an entropy-constrained ball. Fix a focal belief $F_*^i \in \Delta(S^{-i})$ for every $i \in \mathcal{I}$. For any belief $G^i \in \Delta(S^{-i})$, its relative entropy is $R(G^i \parallel F_*^i) \in [0, \infty]$, where

$$R(G^i \parallel F_*^i) := \int_{S^{-i}} \ln \frac{dG^i}{dF_*^i} dG^i$$

if G^i is absolutely continuous with respect to F_*^i and ∞ otherwise. Though $R(G^i \parallel F_*^i)$ is not a metric, it is a measure of the distance between G^i and F_*^i. In particular, $R(G^i \parallel F_*^i) = 0$ if and only if $G^i = F_*^i$. Let each agent i's set of beliefs be

$$\mathcal{F}^i = \{G^i \in \Delta(S^{-i}) | R(G^i \parallel F_*^i) \leq \lambda\}$$

for some $\lambda \geq 0$. Similarly, if p depends on s^i only through s_i^i, then Minimal Ambiguity is satisfied for all $\lambda \geq 0$. When $\lambda = \infty$, $\mathcal{F}^i = \Delta(S^{-i})$ for all i and Minimal Ambiguity is always satisfied. In general, there exists a threshold $\underline{\lambda} \geq 0$ such that Minimal Ambiguity is satisfied if and only if $\lambda \geq \underline{\lambda}$.

[14] Following Ghirardato and Marinacci (2002), we say that the agent with the set of priors $\mathcal{F}$ is more ambiguity averse than the agent with the set of priors $\mathcal{F}'$ if $\mathcal{F} \supseteq \mathcal{F}'$.

3.2 When is the First-order Condition Sufficient?

As in the standard Myersonian approach to Bayesian mechanism design problems, we first construct mechanisms that satisfy the first-order condition given by Lemma 3.1. Intuitively, this first-order condition can be interpreted as local incentive compatibility. In a Bayesian environment, a monotonicity condition is then used to ensure that local incentive compatibility implies global incentive compatibility. However, with maxmin agents, monotonicity alone may no longer suffice to establish the sufficiency of local incentive compatibility. To tackle this problem, we establish a technical condition under which the Myersonian first-order approach applies in our setting. We want to emphasize that the main goal of this paper is to identify natural conditions on primitives of the model so that this technical condition is satisfied, and this is addressed in Section 4.

3.2.1 Monotonicity

We start with the definition of monotonicity in our setting. Recall first that for every $i \in \mathcal{I}$ and every $s_i^i \in S_i^i$, the signals $\bar{s}^i(s_i^i) = (\bar{s}_1^i(s_i^i), ..., \bar{s}_N^i(s_i^i)), \underline{s}^i(s_i^i) = (\underline{s}_1^i(s_i^i), ..., \underline{s}_N^i(s_i^i)) \in e(s_i^i)$ are chosen such that

$$p^i\left(\bar{s}^i(s_i^i), s^{-i}\right) \geq p^i\left(t^i, s^{-i}\right) \geq p^i\left(\underline{s}^i(s_i^i), s^{-i}\right) \quad \forall t^i \in e(s_i^i), \forall s^{-i} \in S^{-i}.$$

Definition 3. The allocation rule p satisfies **Monotonicity** if for every $i \in \mathcal{I}$, every $s_i^i, t_i^i \in S_i^i$ such that $s_i^i < t_i^i$, and every $s^{-i} \in S^{-i}$,

$$p^i\left(\bar{s}^i(s_i^i), s^{-i}\right) \leq p^i\left(\bar{s}^i(t_i^i), s^{-i}\right) \quad \text{and} \quad p^i\left(\underline{s}^i(s_i^i), s^{-i}\right) \leq p^i\left(\underline{s}^i(t_i^i), s^{-i}\right).$$

Under the efficient allocation rule, Monotonicity becomes a multidimensional version of the single-crossing condition familiar from one-dimensional settings. To see this, recall that the efficient allocation rule requires that agent i should be awarded the object when $v^i(s_i^i, s_i^{-i}) > v^j(s_j^i, s_j^{-i})$ for every $j \neq i$. Thus, a sufficient condition for Monotonicity is

$$\frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} \geq \frac{\partial v^j(\bar{s}_j^i(s_i^i), s_j^{-i})}{\partial s_j^i} \frac{d\bar{s}_j^i(s_i^i)}{ds_i^i} \quad \text{and} \quad \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} \geq \frac{\partial v^j(\underline{s}_j^i(s_i^i), s_j^{-i})}{\partial s_j^i} \frac{d\underline{s}_j^i(s_i^i)}{ds_i^i}, \quad (2)$$

at any point where $v^i(s_i^i, s_i^{-i}) = v^j(s_j^i, s_j^{-i}) = \max_l v^l(s_l^i, s_l^{-i})$, and $\frac{d\bar{s}_j^i(s_i^i)}{ds_i^i}$ and $\frac{d\underline{s}_j^i(s_i^i)}{ds_i^i}$ exist. As is clear from condition (2) above, whether $p^i\left(\bar{s}^i(s_i^i), s^{-i}\right)$ and $p^i\left(\underline{s}^i(s_i^i), s^{-i}\right)$ increase in s_i^i depends both on how responsive agent j's valuation is to agent i's signal and on the shape of agent i's signal space S^i. In a setting in which signals are one-dimensional, only the former matters and condition (2) reduces to the standard single-crossing condition, which says that one agent's signal has a greater marginal effect on his own valuation than on that of any other agent; in a setting in which signals are multidimensional, the shape of the signal space also plays an important role. Therefore, Monotonicity imposes joint restrictions on valuation functions and signal spaces.

Two examples are provided below to help in understanding when condition (2) is satisfied and, consequently, Monotonicity is satisfied by the efficient allocation rule.

Example 3. Suppose that s_i^i and s_j^i are independently distributed for all $i \in \mathcal{I}$ and $j \neq i$.[15] Since $\frac{\partial v^j(s_j^i, s_j^{-i})}{\partial s_j^i} > 0$, we have $\bar{s}_j^i(s_i^i) = \min S_j^i$ and $\underline{s}_j^i(s_i^i) = \max S_j^i$ for all $j \neq i$. Thus, $\frac{d\bar{s}_j^i(s_i^i)}{ds_i^i} = \frac{d\underline{s}_j^i(s_i^i)}{ds_i^i} = 0$. By the assumption that $\frac{\partial v^i(s_i^i, s_i^j)}{\partial s_i^i} > 0$, condition (2) is always satisfied.

Example 4. We now present an example with a simple information structure in which each agent's signal can be decomposed into a "private" and a "common" element.[16] Each agent i has a signal $(\theta^i, c^i) \in [0,1] \times [0,1]$, where θ^i and c^i are

[15] Our general model allows for correlation between s_i^i and s_j^i for $j \neq i$, as is shown in Example 4.

[16] Similar examples have been discussed by Maskin (1992), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Compte and Jehiel (2002).

independently distributed. Agent i's valuation for the object is given by

$$v^i = \theta^i + ac^i + b\sum_{j\neq i} c^j \quad \forall \theta^i \in [0,1], \forall c^i, c^j \in [0,1],$$

where $a, b \in \mathbb{R}_{++}$. The element θ^i is the private component of agent i's signal, as it is of interest to him only, while c^i is the common component as it is relevant to all agents. Using the notation presented in Section 2, we have $s_i^i = \theta^i + ac^i$ and $s_j^i = c^i$ for all $i \in \mathcal{I}$ and $j \neq i$. Notice that in contrast to Example 3, s_i^i and s_j^i are positively correlated and, hence, $\frac{d\bar{s}_j^i(s_i^i)}{ds_i^i} > 0$ (or $\frac{d\underline{s}_j^i(s_i^i)}{ds_i^i} > 0$) for some s_i^i and $j \neq i$. We can verify that condition (2) is satisfied if and only if $a \geq b$.[17] This is actually a single-crossing condition: the *common component* of an agent's signal has a larger marginal effect on his own valuation than on the valuations of the other agents.

3.2.2 Additivity

We now introduce the technical condition which, combined with Monotonicity, guarantees the sufficiency of local incentive compatibility:

$$\begin{aligned}
&\int_{r_i^i}^{t_i^i} \min_{F^i\in\mathcal{F}^i} \int_{S^{-i}} p^i(s^i, s^{-i}) \frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i} dF^i d\hat{s}_i^i \\
&= \min_{F^i\in\mathcal{F}^i} \int_{r_i^i}^{t_i^i} \int_{S^{-i}} p^i(s^i, s^{-i}) \frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i} dF^i d\hat{s}_i^i \quad \forall r_i^i < t_i^i, \forall s^i \in S^i, \forall i \in \mathcal{I}.
\end{aligned} \tag{3}$$

Observe that this is an additivity requirement: it requires the sum of the minimum be equal to the minimum of the sum. One immediate observation is that (3) is satisfied trivially in Bayesian environments. In the presence of ambiguity, the former

[17]To see this, observe that for every $i \in \mathcal{I}$, $j \neq i$, and $s_i^i \in [0,1]$, we have $\bar{s}_j^i(s_i^i) = 0$ and $\frac{d\bar{s}_j^i(s_i^i)}{ds_i^i} = 0$; for every $s_i^i \in (1, 1+a]$, we have $\bar{s}_j^i(s_i^i) = \frac{1}{a}(s_i^i - 1)$ and $\frac{d\bar{s}_j^i(s_i^i)}{ds_i^i} = \frac{1}{a}$. Given the change of variables, agent i's valuation can be rewritten as $v^i(s_i^i, s_i^{-i}) = s_i^i + b\sum_{j\neq i} s_i^j$, which implies $\frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} = 1$ and $\frac{\partial v^j(\bar{s}_j^i(s_i^i), s_j^{-i})}{\partial s_j^i} = b$ for all $i \in \mathcal{I}$ and $j \neq i$. Therefore, the first inequality in condition (2) is satisfied if and only if $a \geq b$. Following analogous arguments, we can show that the second inequality in condition (2) is satisfied if and only if $a \geq b$.

in general is less than the latter due to hedging against ambiguity.[18] However, we show that this condition is actually satisfied under some natural specifications of preferences. To be more precise, fix $i \in \mathcal{I}$ and $s^i \in S^i$. For every $\hat{s}^i_i \in S^i_i$, the function $p^i(s^i, \cdot)\frac{\partial v^i(\hat{s}^i_i, \cdot)}{\partial \hat{s}^i_i} : S^{-i} \to \mathbb{R}$ can be considered as an asset. Then condition (3) is satisfied if the assets $\{p^i(s^i, \cdot)\frac{\partial v^i(\hat{s}^i_i, \cdot)}{\partial \hat{s}^i_i}\}_{\hat{s}^i_i \in S^i_i}$ cannot hedge one another. There are two circumstances, besides the subjective expected utility framework, in which such hedging does not arise. One is when the assets $\{p^i(s^i, \cdot)\frac{\partial v^i(\hat{s}^i_i, \cdot)}{\partial \hat{s}^i_i}\}_{\hat{s}^i_i \in S^i_i}$ are perfectly correlated. This is because within maxmin models, no matter how the set of beliefs is specified, combining perfectly correlated assets cannot hedge against ambiguity. A sufficient condition for this case to arise is that the valuation functions are linear in a sense to be made precise in Section 4.1. The other circumstance is known as comonotonic additivity in the literature (e.g., see Schmeidler (1986)), which says combining comonotonic, weaker than perfectly correlated, assets does not reduce ambiguity if the set of beliefs has a particular shape. This case involves joint restrictions on valuation functions and beliefs which are specified in Section 4.2.

3.2.3 Sufficiency

The following lemma establishes the sufficiency result.

Lemma 3.2. *Let p be an allocation rule that satisfies Monotonicity and* (3)*. Then p is implementable if and only if $\{\mathcal{F}^i\}_{i=1}^N$ satisfies Minimal Ambiguity under p.*

The necessity of Minimal Ambiguity is given by Lemma 3.1. We now demonstrate why Monotonicity and the technical condition (3) together can guarantee the sufficiency of Minimal Ambiguity. The proof follows similar steps of the proof of Lemma 2 in Myerson (1981). To account for ambiguity aversion, we make two modifications to the proof in Myerson (1981): the first is that we construct a specific transfer scheme that implements p; the second is that we make explicit use of

[18]This feature of maxmin preferences is captured by the uncertainty aversion axiom of Gilboa and Schmeidler (1989).

the technical condition (3), which is trivially satisfied in Myerson (1981). Define a transfer scheme x_{full} as follows:

$$x^i_{full}(s^i, s^{-i}) := R^i(s^i) - p^i(s^i, s^{-i})v^i(s^i_i, s^{-i}_i), \quad \forall s \in S, \forall i \in \mathcal{I},$$

where $R^i : S^i \to \mathbb{R}$ is called the **reward function** for agent i.[19] Observe that the transfer scheme x_{full} is constructed so that if everyone reports truthfully, the ex post utility of agent i who receives signal s^i is a constant function of the other agents' reports and equal to the reward $R^i(s^i)$. Thus, agent i is fully insured against ambiguity in the interim stage. Following Bose et al. (2006), who first introduced this class of transfer schemes, x_{full} is called a **full insurance transfer scheme** and (p, x_{full}) is a **full insurance mechanism**. We now construct a specific reward function for each agent i as follows:

$$\underline{R}^i(s^i) := \int_{\tau^i_i}^{s^i_i} \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(\hat{s}^i_i), s^{-i}) \frac{\partial v^i(\hat{s}^i_i, s^{-i}_i)}{\partial \hat{s}^i_i} dF^i d\hat{s}^i_i,$$

where $\tau^i_i := \min_{t^i_i \in S^i_i} t^i_i$ for every $i \in \mathcal{I}$.[20] Let $\underline{x}_{full}$ be the full insurance transfer scheme associated with the reward functions $\underline{R}^i$ and let μ^i be the indirect utility functions associated with $(p, \underline{x}_{full})$. Notice that $\mu^i(s^i) = \underline{R}^i(s^i)$ by construction of the transfer scheme $\underline{x}_{full}$. We are going to prove that $(p, \underline{x}_{full})$ is interim incentive compatible. Fix $i \in \mathcal{I}$ and $s^i, t^i \in S^i$. To show $\mu^i(s^i) \geq u^i(t^i, s^i)$, there are two cases to consider. Suppose first that $t^i_i \leq s^i_i$. By Monotonicity and (3), we obtain

$$\begin{aligned}\mu^i(s^i) - \mu^i(t^i) = \underline{R}^i(s^i) - \underline{R}^i(t^i) &\geq \int_{t^i_i}^{s^i_i} \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(t^i_i), s^{-i}) \frac{\partial v^i(\hat{s}^i_i, s^{-i}_i)}{\partial \hat{s}^i_i} dF^i d\hat{s}^i_i \\ &= \min_{F^i \in \mathcal{F}^i} \int_{t^i_i}^{s^i_i} \int_{S^{-i}} p^i(\bar{s}^i(t^i_i), s^{-i}) \frac{\partial v^i(\hat{s}^i_i, s^{-i}_i)}{\partial \hat{s}^i_i} dF^i d\hat{s}^i_i.\end{aligned}$$

[19] Although x_{full} depends on the allocation rule p, we suppress this dependence for notational simplicity. A similar comment applies to $\underline{R}^i$ and $\underline{x}_{full}$ defined below.
[20] We show in Appendix A .3 that $\underline{R}^i$ is well defined.

Changing the order of integration and using the definition of $\bar{s}^i(t_i^i)$ yield

$$\begin{aligned}\mu^i(s^i)-\mu^i(t^i) &\geq \min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i\big(\bar{s}^i(t_i^i),s^{-i}\big)\big(v^i(s_i^i,s_i^{-i})-v^i(t_i^i,s_i^{-i})\big)dF^i \\ &\geq \min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i\big(t^i,s^{-i}\big)\big(v^i(s_i^i,s_i^{-i})-v^i(t_i^i,s_i^{-i})\big)dF^i.\end{aligned} \tag{4}$$

Similarly, if $t_i^i > s_i^i$, then

$$\begin{aligned}\mu^i(s^i)-\mu(t^i) &\geq -\int_{s_i^i}^{t_i^i}\max_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(\underline{s}^i(\hat{s}_i^i),s^{-i})\frac{\partial v^i(\hat{s}_i^i,s_i^{-i})}{\partial \hat{s}_i^i}dF^i d\hat{s}_i^i \\ &\geq -\int_{s_i^i}^{t_i^i}\max_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(\underline{s}^i(t_i^i),s^{-i})\frac{\partial v^i(\hat{s}_i^i,s_i^{-i})}{\partial \hat{s}_i^i}dF^i d\hat{s}_i^i \\ &= \min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i\big(\underline{s}^i(t_i^i),s^{-i}\big)\big(v^i(s_i^i,s_i^{-i})-v^i(t_i^i,s_i^{-i})\big)dF^i.\end{aligned}$$

The first inequality follows from Minimal Ambiguity; the second inequality follows from Monotonicity; the equality follows from (3). Combining the inequalities above and the definition of $\underline{s}^i(t_i^i)$, we can conclude that

$$\mu^i(s^i)-\mu(t^i) \geq \min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i\big(t^i,s^{-i}\big)\big(v^i(s_i^i,s_i^{-i})-v^i(t_i^i,s_i^{-i})\big)dF^i. \tag{5}$$

The combination of (4) and (5) yields

$$\mu^i(s^i) \geq \mu^i(t^i)+\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i\big(t^i,s^{-i}\big)\big(v^i(s_i^i,s_i^{-i})-v^i(t_i^i,s_i^{-i})\big)dF^i = u^i(t^i,s^i).$$

The equality follows from the construction of the transfer scheme $\underline{x}_{full}$. Since s^i and t^i were arbitrarily chosen, this shows that interim incentive compatibility is satisfied.[21]

From the proof of Lemma 3.2, we can see that the role of condition (3) is to guarantee that local incentive constraints are sufficient to imply global incentive constraints in maxmin settings, which is the key to the Myersonian approach. A combination of this condition and the maxmin preferences allows us to extract as much as possible out of the fact that agents' utilities are quasilinear in transfers.[22]

[21] There exist other transfer schemes that can implement p. Nevertheless, we can show that any interim incentive compatible mechanism is payoff equivalent to a full insurance mechanism.

[22] Another commonly used model of ambiguity in the literature is the smooth ambiguity model of Klibanoff et al. (2005). It is readily seen that if we adopt the the smooth ambiguity model, agents'

We end this section by remarking that Monotonicity is *not* necessary for an allocation rule to be implementable in an environment with maxmin agents. For example, in the extreme case of complete ambiguity, any allocation rule is implementable.

4 Sufficient Conditions

In this section, we provide two natural specifications of preferences under which the technical condition (3) is satisfied.

4.1 Linear Valuation Functions

Say that agent i's valuation v^i is **linear** if there exist functions $g^i : S_i^i \to \mathbb{R}_+$ and $f^i, h^i : S_i^{-i} \to \mathbb{R}_+$ such that

$$v^i(s_i^i, s_i^{-i}) = g^i(s_i^i)h^i(s_i^{-i}) + f^i(s_i^{-i}) \quad \forall s_i^i \in S_i^i, \forall s_i^{-i} \in S_i^{-i}. \tag{6}$$

Notice that this notion of "linearity", which is also used by Carroll (2012) and Archer and Kleinberg (2014), is very "permissive". For example, additively or multiplicatively separable valuation functions are linear in the sense of (6).[23]

Assumption 1 (Linearity). For every $i \in \mathcal{I}$, agent i's valuation function v^i is linear in the sense of (6).

We can show that under Linearity, any allocation rule satisfies (3). Then the next result follows.

Theorem 4.1. *Let p be an allocation rule that satisfies Monotonicity. Assume that Linearity holds. Then p is implementable if and only if $\{\mathcal{F}^i\}_{i=1}^N$ satisfies Minimal Ambiguity under p.*

utilities in general are not quasilinear in transfers; moreover, local incentive compatibility is typically violated. Thus, our results do not extend to the smooth ambiguity model.

[23] Agent i's valuation function is **additively separable** if h^i in (6) is a constant function; agent i's valuation function is **multiplicatively separable** if f^i is a constant function.

4.2 Nonlinear Valuation Functions

Linearity imposes a strong restriction on valuation functions so that the technical condition (3) is satisfied without restricting agents' beliefs. Now we pursue another route. Namely, we impose a weaker restriction on valuation functions but combine it with a restriction on beliefs: each agent's valuation function satisfies a suitably defined increasing differences condition and each agent's preferences satisfy the comonotonic independence axiom of Schmeidler (1989). The increasing differences condition is a familiar restriction on valuation functions in the mechanism design literature and the comonotonic independence axiom is a standard assumption in the decision theoretic literature[24]. Other than the two restrictions discussed above, the approach used in this section can only be applied to deterministic allocation rules.[25] We show that when the restrictions on beliefs and valuation functions are imposed, (3) is satisfied by all the deterministic allocation rules.

We first introduce the assumption on valuation functions. The valuation function v^i has **increasing differences** if for all $s_i^{-i}, \hat{s}_i^{-i} \in S_i^{-i}$ and all $r_i^i < s_i^i < t_i^i$,

$$v^i(t_i^i, s_i^{-i}) - v^i(t_i^i, \hat{s}_i^{-i}) > v^i(s_i^i, s_i^{-i}) - v^i(s_i^i, \hat{s}_i^{-i})$$

implies that

$$v^i(s_i^i, s_i^{-i}) - v^i(s_i^i, \hat{s}_i^{-i}) \geq v^i(r_i^i, s_i^{-i}) - v^i(r_i^i, \hat{s}_i^{-i}).$$

Observe first that if v^i is linear in the sense of (6), then v^i has increasing differences. Thus, increasing differences is a weaker restriction on valuation functions than linearity. In addition, increasing differences is quite weak when $N = 2$. For example, standard supermodular and submodular valuation functions satisfy increasing differences. However, when $N > 2$, the standard notion of supermodularity or submodularity is neither sufficient nor necessary for increasing differences.

[24]Maxmin preferences that satisfy the comonotonic independence axiom are the intersection of maxmin and Choquet expected utility model.

[25]An allocation rule p is deterministic if $p^i(s) = 1$ or 0 for all $s \in S$ and $i \in \mathcal{I}$.

Assumption 2 (**Increasing Differences**)**.** For every $i \in \mathcal{I}$, agent i's valuation function v^i has increasing differences.

We next define the comonotonic independence axiom of Schmeidler (1989). Let (Ω, Σ) be a measurable space. Two acts $f, g : \Omega \to \mathbb{R}$ are **comonotonic** if

$$\big(f(\omega) - f(\omega')\big)\big(g(\omega) - g(\omega')\big) \geq 0 \quad \forall \omega, \omega' \in \Omega.$$

Suppose that an agent's set of priors is given by $\mathcal{F}$. His preferences satisfy the **comonotonic independence axiom** if for all pairwise comonotonic acts $f, g, h : \Omega \to \mathbb{R}$, we have $\min_{F \in \mathcal{F}} \int_\Omega f dF \geq \min_{F \in \mathcal{F}} \int_\Omega g dF$ implies $\min_{F \in \mathcal{F}} \int_\Omega (f + h) dF \geq \min_{F \in \mathcal{F}} \int_\Omega (g + h) dF$. In words, this axiom says that if the agent prefers act f to act g, then combining both acts with h will not reverse his preferences, provided that f, g and h are pairwise comonotonic. Intuitively, this axiom requires that combining two comonotonic acts do not reduce ambiguity.

We now present the representation of preferences that satisfy the comonotonic independence axiom. A **capacity** is a function $\nu : \Sigma \to [0, 1]$ such that (i) $\nu(\varnothing) = 0$ and $\nu(\Omega) = 1$; (ii) $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$. A capacity is **convex** if it also satisfies

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B) \quad \forall A, B \in \Sigma.$$

The **core of a capacity** ν is

$$core(\nu) := \{\pi \in \Delta(\Omega) | \pi(A) \geq \nu(A), \forall A \in \Sigma\}.$$

Schmeidler (1989) shows that ambiguity averse preferences that satisfy the comonotonic independence axiom can be represented by maxmin expected utility with the agent's set of priors being the core of a convex capacity.[26] Thus, the desired assumption can be stated as follows.

Assumption 3 (**Comonotonic Independence**)**.** For every $i \in \mathcal{I}$, agent i's set of beliefs $\mathcal{F}^i$ is the core of a convex capacity.

[26] Shapley (1971) shows that the core of a convex capacity is not empty.

The ε-contamination model introduced in Example 1 provides a natural class of preferences where an agent's set of beliefs is a core. For any $F^i_* \in \Delta(S^{-i})$ and any $\varepsilon \in [0,1]$, it can be easily verified that $C_\varepsilon(F^i_*)$ is the core of the convex capacity

$$\nu^{F^i_*}(A) := (1-\varepsilon)F^i_*(A) \quad \forall A \in \Sigma \setminus \{\Omega\} \quad \text{and} \quad \nu^{F^i_*}(\Omega) := 1.$$

Theorem 4.2. *Let p be a deterministic allocation rule that satisfies Monotonicity. Assume that Increasing Differences and Comonotonic Independence hold. Then p is implementable if and only if $\{\mathcal{F}^i\}_{i=1}^N$ satisfies Minimal Ambiguity under p.*

5 Implementation of the Efficient Allocation Rule

We show in this section how Theorems 4.1 and 4.2 can be applied to the implementation of the efficient allocation rule. Notice that the efficient allocation rule is uniquely defined and deterministic almost everywhere. From now on, we focus exclusively on the efficient allocation rule, denoted by p_*, which resolves ties in a deterministic way.

We first demonstrate that the collection $\{S^i\}_{i=1}^N$ is complete under p_*. We show this by explicitly constructing $\bar{s}^i(s^i_i)$ and $\underline{s}^i(s^i_i)$. For every $i \in \mathcal{I}$ and every $s^i_i \in S^i_i$, let

$$\bar{s}^i_j(s^i_i) := \min_{t^i \in e(s^i_i)} t^i_j \quad \text{and} \quad \underline{s}^i_j(s^i_i) := \max_{t^i \in e(s^i_i)} t^i_j \quad \forall j \neq i.$$

Then take $\bar{s}^i(s^i_i) = \big(\bar{s}^i_1(s^i_i), ..., s^i_i, ..., \bar{s}^i_N(s^i_i)\big)$ and $\underline{s}^i(s^i_i) = \big(\underline{s}^i_1(s^i_i), ..., s^i_i, ..., \underline{s}^i_N(s^i_i)\big)$. By the assumption that S^i is a sublattice, we have $\bar{s}^i(s^i_i), \underline{s}^i(s^i_i) \in e(s^i_i)$. Since $v^j(s^i_j, s^{-i}_j)$ increases in s^i_j for every $j \neq i$, the construction of $\bar{s}^i_j(s^i_i)$ and $\underline{s}^i_j(s^i_i)$ indicates that

$$v^j(\bar{s}^i_j(s^i_i), s^{-i}_j) \leq v^j(t^i_j, s^{-i}_j) \leq v^j(\underline{s}^i_j(s^i_i), s^{-i}_j) \quad \forall t^i \in e(s^i_i), \forall s^{-i}_j \in S^{-i}_j, \forall j \neq i.$$

Therefore, efficiency implies that

$$p^i_*\big(\bar{s}^i(s^i_i), s^{-i}\big) \geq p^i_*\big(t^i, s^{-i}\big) \geq p^i_*\big(\underline{s}^i(s^i_i), s^{-i}\big) \quad \forall t^i \in e(s^i_i), \forall s^{-i} \in S^{-i}.$$

That is, the collection $\{S^i\}_{i=1}^N$ is complete under p_*. Then the following result is an immediate consequence of Theorems 4.1 and 4.2.

Corollary 5.1. *Assume that Linearity holds and the efficient allocation rule p_* satisfies Monotonicity. Then p_* is implementable if and only if $\{\mathcal{F}^i\}_{i=1}^N$ satisfies Minimal Ambiguity under p_*. The same conclusion holds if Linearity is replaced by Increasing Differences and Comonotonic Independence.*

We now explain intuitively how the presence of ambiguity aversion facilitates efficient implementation.[27] The main idea is that ambiguity aversion weakens interim incentive compatibility constraints under full insurance transfer schemes. To see this, consider a full insurance transfer scheme that implements p_*. Recall that under such a transfer scheme, the agent who is awarded the object pays his valuation conditional on all the reports and *every* agent receives a reward which is solely a function of his report. Thus, irrespective of other agents' reports, the ex post utility of agent i who receives s^i is always equal to the reward $R^i(s^i)$ as long as everyone reports truthfully—agent i is fully insured against ambiguity. In contrast, if agent i misreports, his interim utility is evaluated according to a worst-case belief: if agent i receives s^i but reports t^i, then his interim utility is

$$R^i(t^i) + \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p_*^i(t^i, s^{-i})\big(v^i(s_i^i, s_i^{-i}) - v^i(t_i^i, s_i^{-i})\big)dF^i.$$

The first term is the reward made to agent i based on his report t^i; the second term is agent i's expected gain or loss from being awarded the object. Specifically, if agent i reports a signal that results in a lower valuation, that is, $t_i^i < s_i^i$, then he attains a gain if he is awarded the object. This is because he pays for the object at a price lower than his true valuation. By ambiguity aversion, he assigns the *lowest* probability of being awarded the object. Likewise, if $t_i^i > s_i^i$, he suffers a loss if he is awarded the object, because he pays more than his true valuation. In this case, ambiguity aversion drives him to assign the *highest* probability of being awarded

[27]It should be noted that the same intuition applies to the implementation of all allocation rules. We provide the intuition in terms of the efficient allocation rule to contrast with the impossibility result of Jehiel and Moldovanu (2001).

the object. As a result, the presence of ambiguity aversion minimizes the potential gain and maximizes the potential loss—the incentive to lie is thus diminished. The final question is then how much ambiguity is needed to ensure interim incentive compatibility. This is addressed by Minimal Ambiguity: it quantifies the minimal amount of ambiguity that guarantees local incentive compatibility. Then, combined with Linearity (or Increasing Differences and Comonotonic Independence) and Monotonicity, this minimal amount of ambiguity also guarantees global incentive compatibility.[28]

5.1 Informational Size

A natural question to ask is under what conditions the efficient allocation rule p_* is implementable with an arbitrarily small amount of ambiguity. This section addresses this question by linking the required amount of ambiguity perceived by an agent to his informational size.

Our definition of informational size is a counterpart of the notion introduced in McLean and Postlewaite (2004, 2015a,b): it measures the degree to which one agent's signal can affect the valuations of other agents. Formally, define the **informational size** of agent i as

$$\gamma^i := \max_{j \neq i, s_j^i \in S_j^i, s_j^{-i} \in S_j^{-i}} \frac{\partial v^j(s_j^i, s_j^{-i})}{\partial s_j^i}.$$

Recall that s_j^i is agent i's information affecting the valuation of agent j. Thus, in a model with private values, the informational size of each agent is 0.

One final definition is required before stating the main result of this section. For every $\varepsilon \in (0,1]$, we say that a set of probability measures $\mathcal{F}$ on (Ω, Σ) **contains an ε-ball** if there exists $G \in \Delta(\Omega)$ such that $\{F \in \Delta(\Omega) | d(F,G) \leq \varepsilon\} \subseteq \mathcal{F}$, where d

[28]We want to emphasize that our results depend on agents' utilities being quasilinear in transfers. Only then, the mechanism designer can use transfers as instruments to fully insure agents against ambiguity under honest reporting but induce ambiguity otherwise.

is the Prokhorov metric.[29]

Theorem 5.1. *Assume that Linearity holds and the efficient allocation rule p_* satisfies Monotonicity. For every $\varepsilon \in (0,1]$, if each agent i's set of beliefs $\mathcal{F}^i$ contains an ε-ball, then there exists a $\delta > 0$ such that if $\gamma^i < \delta$ for every $i \in \mathcal{I}$, the efficient allocation rule p_* is implementable. The same conclusion holds if Linearity is replaced by Increasing Differences and Comonotonic Independence.*

An immediate observation from condition (2) is that if $\frac{d\bar{s}^i_j(s^i_i)}{ds^i_i}$ and $\frac{d\underline{s}^i_j(s^i_i)}{ds^i_i}$ are bounded almost everywhere, Monotonicity is automatically satisfied if the informational size of each agent is sufficiently small.

Informational smallness arises naturally when the number of agents is large. Intuitively, a single agent's private information would have a small effect on other agents' valuations in the presence of many agents. We can show that if the informational size of each agent converges to zero as the number of agents grows, then given any nontrivial amount of ambiguity, the efficient allocation rule is implementable as long as the number of agents is sufficiently large. The proof is essentially identical to that of Theorem 5.1.

6 An Example

In this section, we first present a simple example to illustrate the main insight for our results. In particular, it exhibits the conflict between Bayesian incentive compatibility and efficiency, and indicates how a certain amount of ambiguity can resolve this conflict. We further demonstrate that if the informational size of each agent converges to zero in a sequence of environments with an increasing number of agents, then the minimal amount of ambiguity that induces truth telling also converges to zero as the number of agents increases.

[29] We use Prokhorov metric to measure the distance between probability measures and the definition is provided in Appendix C .

Consider a special case of Example 3: each agent i's valuation for the object is given by

$$\theta^i + \frac{1}{N-1}\sum_{j\neq i} c^j \quad \forall \theta^i \in [0,1], \forall c^j \in [0,1],$$

where $s_i^i = \theta^i \in [0,1]$ and $s_j^i = c^i \in [0,1]$ for all $j \neq i$. Let $\theta^{-i} := \times_{j\neq i}\theta^j$ and $c^{-i} := \times_{j\neq i} c^j$. One readily sees that the efficient allocation is determined by $\theta^i - \frac{1}{N-1}c^i$. Fix i and take $(\tilde{\theta}^i, \tilde{c}^i)$ and $(\hat{\theta}^i, \hat{c}^i)$ such that

$$\tilde{\theta}^i > \hat{\theta}^i \quad \text{and} \quad \tilde{\theta}^i - \frac{1}{N-1}\tilde{c}^i < \hat{\theta}^i - \frac{1}{N-1}\hat{c}^i. \tag{7}$$

Intuitively, (7) implies that the common component swamps agent i's idiosyncratic value. Since $\tilde{\theta}^i - \frac{1}{N-1}\tilde{c}^i < \hat{\theta}^i - \frac{1}{N-1}\hat{c}^i$, efficiency requires that agent i have a higher chance of obtaining the object by reporting $(\hat{\theta}^i, \hat{c}^i)$ than $(\tilde{\theta}^i, \tilde{c}^i)$ under any belief F^i with full support, that is,

$$\int p_*^i(\hat{\theta}^i, \hat{c}^i, \theta^{-i}, c^{-i})dF^i > \int p_*^i(\tilde{\theta}^i, \tilde{c}^i, \theta^{-i}, c^{-i})dF^i. \tag{8}$$

On the other hand, simple manipulation of incentive constraints yields

$$\min_{F^i\in\mathcal{F}^i} \int p_*^i(\hat{\theta}^i, \hat{c}^i, \theta^{-i}, c^{-i})dF^i \leq \max_{F^i\in\mathcal{F}^i} \int p_*^i(\tilde{\theta}^i, \tilde{c}^i, \theta^{-i}, c^{-i})dF^i. \tag{9}$$

Observe that in a Bayesian framework, the inequality in (9) contradicts (8), showing that efficient and incentive compatible mechanisms do not exist. The basic intuition behind this impossibility result is that incentive compatibility requires agent i's probability of obtaining the object to increase in his private component θ^i; efficiency requires this probability to increase only in $\theta^i - \frac{1}{N-1}c^i$. Therefore, agent i's private incentives are not aligned with social optimality. However, if agent i perceives a nontrivial amount of ambiguity and is ambiguity averse, the requirement of incentive compatibility becomes weaker. When he is sufficiently ambiguity averse so that the inequality in (9) is satisfied, there is no conflict between agent i's personal incentives and social optimality.

Next we show heuristically that the necessary amount of ambiguity for efficient implementation converges to zero as N increases. By incentive compatibility, the inequality in (9) must hold for all pairs of signals $(\tilde{\theta}^i, \tilde{c}^i)$ and $(\hat{\theta}^i, \hat{c}^i)$ satisfying (7). One can verify that a sufficient condition for (9) to hold is

$$\min_{F^i \in \mathcal{F}^i} \max_{c^i \in [0,1]} \int p^i_*(\theta^i, c^i, \theta^{-i}, c^{-i}) dF^i \leq \max_{F^i \in \mathcal{F}^i} \min_{c^i \in [0,1]} \int p^i_*(\theta^i, c^i, \theta^{-i}, c^{-i}) dF^i \quad \forall \theta^i \in [0,1],$$

which is equivalent to the following condition:

$$\begin{aligned} &\min_{F^i \in \mathcal{F}^i} F^i(\{(\theta^{-i}, c^{-i}) | \max_{j \neq i} \theta^j - \frac{1}{N-1} c^j \leq \theta^i\}) \\ &\quad \leq \max_{F^i \in \mathcal{F}^i} F^i(\{(\theta^{-i}, c^{-i}) | \max_{j \neq i} \theta^j - \frac{1}{N-1} c^j \leq \theta^i - \frac{1}{N-1}\}) \quad \forall \theta^i \in [0,1]. \end{aligned} \tag{10}$$

The inequality in (10) imposes a minimum requirement on the size of $\mathcal{F}^i$, which can be interpreted as the required amount of ambiguity for efficient implementation.[30] Observe that as $N \to \infty$, this required amount of ambiguity goes to zero. It is straightforward to verify that (10) is the Minimal Ambiguity condition in this example. Since the valuation functions are linear and moreover, the efficient allocation rule satisfies Monotonicity as is shown in Example 3, our sufficiency result Theorem 4.1 applies: given any nontrivial amount of ambiguity, the efficient allocation rule is implementable if the number of agents is sufficiently large.

7 Discussion and Related Literature

Independence assumption. We assume that each agent's set of beliefs does not depend on the realization of his signal, which is an analogue of the "independence of signals" assumption from Bayesian settings. In the literature on efficient mechanism design, a correlated information condition proposed by Cremer and

[30]To see this, suppose that all agents' signals are identically and independently distributed. Let $\hat{F}^i$ be a cumulative density function of $\theta^j - \frac{1}{N-1} c^j$ for all $j \neq i$. Suppose that agent i's set of beliefs about $\theta^j - \frac{1}{N-1} c^j$ for all $j \neq i$ is $C_\varepsilon(\hat{F}^i)$. Then the inequality in (10) reduces to $\varepsilon \geq \max_{\theta^i \in [0,1]} \frac{\hat{F}^i(\theta^i) - \hat{F}^i(\theta^i - \frac{1}{N-1})}{1 + \hat{F}^i(\theta^i) - \hat{F}^i(\theta^i - \frac{1}{N-1})} > 0$.

McLean (1985, 1988) is used extensively to bypass the impossibility result.[31] They show that efficient implementation is possible if the conditional distribution on other agents' signals varies with the realized signal of an agent exogenously; our model suggests that even if there is no a priori heterogeneity among beliefs given different realized signals, such heterogeneity can emerge endogenously due to ambiguity aversion. More importantly, the belief used to evaluate a misreport is the one that makes this misreport the least profitable. Furthermore, the efficient mechanism from Cremer and McLean (1985, 1988) relies on the construction of lotteries whereas our full insurance mechanism does not.

If an agent's set of beliefs can depend on his type, an immediate consequence of Cremer and McLean (1985, 1988) is that ambiguity is not necessary for efficient implementation. Our necessity result hence fails in settings with correlated information.

Yet we are able to extend our sufficiency result to a special case of correlated information, namely affiliation. We can generalize the Minimal Ambiguity condition and full insurance transfer schemes to a setting with affiliated signals, and show that under some standard conditions, Minimal Ambiguity is still sufficient for efficient implementation. Details are provided in Appendix D . Notice that a straightforward application of the lottery mechanism from Cremer and McLean (1985, 1988) has its limitations in maxmin settings (e.g., see Renou (2015)): when agents are ambiguity averse, the belief used to evaluate a lottery is endogenously determined and, hence, it is difficult to construct a lottery for each type with the desired property.

Common knowledge assumption. In Section 2, we assume that each agent's set of beliefs is common knowledge. However, all our results continue to hold under a weaker condition: it is common knowledge that each agent's set of beliefs

[31] Cremer and McLean (1985, 1988) assume finite signal space. McAfee and Reny (1992) and Miller et al. (2007) extend their results to allow for infinite signal space.

contains a certain set.[32] Formally, we assume it is common knowledge that a set of probability measures $\mathcal{F}^i_* \subseteq \Delta(S^{-i})$ is contained in agent i's set of beliefs for all $i \in \mathcal{I}$.[33] This common knowledge condition is clearly weaker than the one in Section 2. Furthermore, if we replace $\mathcal{F}^i$ with $\mathcal{F}^i_*$ in the definition of Minimal Ambiguity, the conclusions of Theorems 4.1 and 4.2 will continue to hold.[34] Also, Theorem 5.1 can be strengthened: as long as it is common knowledge that for every $i \in \mathcal{I}$, there exists $G^i \in \Delta(S^{-i})$ such that agent i's set of beliefs contains an ε-ball around G^i, the conclusion of Theorem 5.1 continues to hold. Notice that this statement does not require the mechanism designer to acquire full knowledge of each agent's beliefs.

Efficiency. There are three notions of efficiency for settings with incomplete information: ex ante efficiency, interim efficiency, and ex post efficiency. In a Bayesian setting with quasilinear utilities, if all agents and the mechanism designer share the same ex ante belief, the three notions coincide.[35] However, they generally differ in a setting with maxmin agents. The notion we use in this paper is ex post efficiency, which is not affected by the presence of ambiguity aversion. In Appendix E , we show that our results remain valid if we adopt the notion of interim efficiency. However, there exist situations in which ex ante efficiency cannot be attained.

Mechanism design with maxmin preferences. Bose et al. (2006), Bodoh-Creed (2012), and Carroll (2017) study revenue maximization with maxmin agents.[36] By

[32] I am very grateful to Pietro Ortoleva for suggesting this weakening of the common knowledge assumption.

[33] Since the closed convex hull of a set of beliefs and that set generate the identical preference, an equivalent assumption is that it is common knowledge that $\mathcal{F}^i_*$ is contained in the closed convex hull of agent i's set of beliefs for all $i \in \mathcal{I}$.

[34] To see why, consider a full insurance transfer scheme that implements the allocation rule if each agent i's set of beliefs were $\mathcal{F}^i_*$. Suppose now that agent i becomes more ambiguity averse, that is, his true set of beliefs contains $\mathcal{F}^i_*$. His interim utility when he reports truthfully remains the same, but his interim utility when he misreports is lower under a larger set of beliefs. Since truthful revelation is optimal when agent i's set of beliefs were $\mathcal{F}^i_*$, it remains optimal when agent i is in fact more ambiguity averse.

[35] See Laffont (1985).

[36] Lopomo et al. (2014) study mechanism design problems where agents have incomplete prefer-

comparison, our paper focuses on efficient implementation. Also, those papers adopt a setting with private valuations, whereas we focus on a setting with interdependent valuations. Bose and Daripa (2009) show that a descending auction can extract almost all surplus. However, their approach does not extend to settings with interdependent valuations and multidimensional signals. In our setting, full surplus extraction is impossible except in the case of complete ambiguity.

Wolitzky (2016) studies the bilateral trade problem of Myerson and Satterthwaite (1983) with each agent's set of priors taking a particular form. In contrast, we fix an object allocation problem and identify the minimal requirement on agents' sets of priors for efficient implementation. Moreover, we focus on efficient mechanisms without imposing balanced budget constraint.[37]

Bose and Renou (2014) study situations in which the mechanism designer can create ambiguity deliberately through an ambiguous communication device.[38] Consequently, the allocation rules that are not implementable with respect to the priors become implementable. Our paper complements their paper in the sense that we show precisely how much ambiguity is sufficient for implementing an efficient allocation rule and their result suggests that this amount of ambiguity can be generated through an ambiguous communication device. We provide an example in Appendix F to illustrate how to generate the required amount of ambiguity for efficient implementation through an ambiguous communication device.

Mechanism design with approximate implementation. McLean and Postlewaite (2015b) show that a generalized VCG mechanism[39] is approximately incentive compatible if agents are informationally small. Roughly speaking, a mecha-

ences as in Bewley (2002).

[37]If the mechanism designer employs a full insurance mechanism, there exist realizations of types such that the mechanism designer runs a deficit ex post. However, whether the mechanism designer can achieve an ex ante surplus depends crucially on his ex ante beliefs. Generally speaking, if the agents face more ambiguity than the mechanism designer, then the mechanism designer can generate positive revenue ex ante.

[38]Di Tillio et al. (2016) and Guo (2017) study the effects of introducing ambiguity in mechanisms.

[39]See Clarke (1971), Groves (1973) and Vickrey (1961).

nism is approximately incentive compatible if an agent would not misreport when there is a small utility gain. The literature does not provide an explicit reason why an agent would forgo a small utility gain. Alternatively, we adopt a different class of transfer schemes, full insurance transfer schemes, and show that the realization of weaker interim incentive compatibility constraints arises endogenously as a result of ambiguity aversion. We should point out that our result is not an immediate extension of the result of McLean and Postlewaite (2015b). This is because the generalized VCG mechanism fails to take into account the ambiguity aversion of the agents and, hence, is not incentive compatible except in the case of complete ambiguity, regardless of the informational size of the agents. In contrast, full insurance mechanisms exploit the ambiguity aversion of agents so as to create correct incentives for the agents. Even in a Bayesian environment with private values, the full insurance mechanism does not reduce to the standard VCG mechanism.

Appendix

A Appendix for Section 3

A .1 Preliminary Lemmas

Lemma A 1. *Suppose that the allocation rule p is implementable with associated indirect utility functions $\{\mu^i\}_{i=1}^N$. For every $i \in \mathcal{I}$ and $s^i, t^i \in S^i$, we have*

$$\begin{aligned}\max_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(t^i,s^{-i})\big(v^i(t^i_i,s^{-i}_i)-v^i(s^i_i,s^{-i}_i)\big)dF^i &\geq \mu^i(t^i)-\mu^i(s^i)\\ &\geq \min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(s^i,s^{-i})\big(v^i(t^i_i,s^{-i}_i)-v^i(s^i_i,s^{-i}_i)\big)dF^i;\end{aligned} \tag{A1}$$

in particular, if $s^i_i = t^i_i$, then $\mu^i(s^i)=\mu^i(t^i)$.

Proof. By definition, for every $i \in \mathcal{I}$ and $s^i, t^i \in S^i$,

$$\begin{aligned}u^i(t^i,s^i) &= \min_{F^i\in\mathcal{F}^i}\int_{S^{-i}}\Big(p^i(t^i,s^{-i})v^i(s^i_i,s^{-i}_i)+x^i(t^i,s^{-i})\Big)dF^i\\ &\geq \mu^i(t^i)+\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(t^i,s^{-i})\big(v^i(s^i_i,s^{-i}_i)-v^i(t^i_i,s^{-i}_i)\big)dF^i.\end{aligned}$$

Thus, the interim incentive compatibility constraint $\mu^i(s^i) \geq u^i(t^i, s^i)$ implies

$$\mu^i(s^i) \geq \mu^i(t^i) + \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(t^i, s^{-i})\big(v^i(s^i_i, s^{-i}_i) - v^i(t^i_i, s^{-i}_i)\big) dF^i. \tag{A2}$$

Reversing the roles of s^i and t^i, we obtain

$$\mu^i(t^i) \geq \mu^i(s^i) + \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(s^i, s^{-i})\big(v^i(t^i_i, s^{-i}_i) - v^i(s^i_i, s^{-i}_i)\big) dF^i. \tag{A3}$$

The desired inequalities in (A1) follow by combining (A2) and (A3).

If $s^i_i = t^i_i$, then $v^i(s^i_i, s^{-i}_i) = v^i(t^i_i, s^{-i}_i)$ for every $s^{-i}_i \in S^{-i}_i$ and $\mu^i(s^i) = \mu^i(t^i)$ is an immediate consequence of (A1). □

Lemma A 2. *For every $i \in \mathcal{I}$, there exists $M > 0$ such that*

$$|v^i(t^i_i, s^{-i}_i) - v^i(s^i_i, s^{-i}_i)| \leq M|t^i_i - s^i_i| \quad \forall s^i_i, t^i_i \in S^i_i, \forall s^{-i}_i \in S^{-i}_i.$$

Proof. Fix $i \in \mathcal{I}$. For every $s^i_i, t^i_i \in S^i_i$ and $s^{-i}_i \in S^{-i}_i$, the Mean Value Theorem allows us to write $|v^i(s^i_i, s^{-i}_i) - v^i(t^i_i, s^{-i}_i)| = |\frac{\partial v^i(\hat{s}^i_i, s^{-i}_i)}{\partial \hat{s}^i_i}(t^i_i - s^i_i)|$ for some $\hat{s}^i_i$ between s^i_i and t^i_i. Since v^i is continuously differentiable, the compactness of the signal spaces implies there exists $M > 0$ such that $|\frac{\partial v^i(\hat{s}^i_i, s^{-i}_i)}{\partial \hat{s}^i_i}| < M$ for all $\hat{s}^i_i$ and s^{-i}_i. □

Lemma A 3. *If the allocation rule p is implementable, the associated indirect utility function μ^i is Lipschitz continuous on S^i_i for every $i \in \mathcal{I}$.*

Proof. Fix $i \in \mathcal{I}$. Lemmas A 1 and A 2 imply that there exists $M > 0$ such that for every $s^i, t^i \in S^i$ with $\mu^i(s^i) - \mu^i(t^i) \geq 0$, we have

$$\mu^i(s^i) - \mu^i(t^i) \leq \max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(s^i, s^{-i})\big(v^i(s^i_i, s^{-i}_i) - v^i(t^i_i, s^{-i}_i)\big) dF^i \leq M|s^i_i - t^i_i|.$$

Similarly, for every $s^i, t^i \in S^i$ with $\mu^i(s^i) - \mu^i(t^i) \leq 0$, we have

$$\mu^i(t^i) - \mu^i(s^i) \leq \max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(t^i, s^{-i})\big(v^i(t^i_i, s^{-i}_i) - v^i(s^i_i, s^{-i}_i)\big) dF^i \leq M|s^i_i - t^i_i|.$$

Combining the two inequalities above, we can conclude that μ^i is Lipschitz continuous. □

Lemma A 4. *For every $i \in \mathcal{I}$, every $r^i \in S^i$, every $s_i^i \in S_i^i$, and every $F^i \in \mathcal{F}^i$,*

$$\lim_{t_i^i \to s_i^i} \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} dF^i = \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i.$$

Proof. Fix $i \in \mathcal{I}$, $r^i \in S^i$, $s_i^i \in S_i^i$, and $F^i \in \mathcal{F}^i$. For every $t_i^i \in S_i^i$, the Mean Value Theorem allows us to write $\frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} = \frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i}$ for some $\hat{s}_i^i$ between s_i^i and t_i^i. Then the equicontinuity condition implies

$$\begin{aligned}
&|\lim_{t_i^i \to s_i^i} \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} dF^i - \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i| \\
&\leq \lim_{\hat{s}_i^i \to s_i^i} \int_{S^{-i}} p^i(r^i, s^{-i}) |\frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i} - \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i}| dF^i = 0.
\end{aligned}$$

□

Lemma A 5. *For every $i \in \mathcal{I}$, every $r^i \in S^i$, and every $s_i^i \in S_i^i$,*

$$\lim_{t_i^i \to s_i^i} \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} dF^i = \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i.$$

Proof. [40]Fix $i \in \mathcal{I}$, $r^i \in S^i$, and $s_i^i \in S_i^i$. It follows from Lemma A 4 that for all $\tilde{F}^i \in \mathcal{F}^i$,

$$\begin{aligned}
&\lim_{t_i^i \to s_i^i} \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} dF^i \\
&\leq \lim_{t_i^i \to s_i^i} \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} d\tilde{F}^i = \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} d\tilde{F}^i.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{t_i^i \to s_i^i} \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} dF^i \\
&\qquad \leq \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i.
\end{aligned} \tag{A4}$$

On the other hand, for every $t_i^i \neq s_i^i$, let $\underline{F}^i(t_i^i) \in \mathcal{F}^i$ be such that

$$\int_{S^{-i}} p^i(r^i, s^{-i}) \frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} d\underline{F}^i(t_i^i) = \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(r^i, s^{-i}) \frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} dF^i.$$

[40]This proof follows similar arguments as in the proof of Proposition 2 in Bose et al. (2006).

Such $\underline{F}^i(t_i^i)$ exists as $\mathcal{F}^i$ is weak* compact. By passing to a subsequence, $\underline{F}^i(t_i^i)$ converges to $\underline{F}^i \in \mathcal{F}^i$ as $t_i^i \to s_i^i$. Observe that

$$
\begin{aligned}
&|\int_{S^{-i}} p^i(r^i,s^{-i})\frac{v^i(t_i^i,s_i^{-i})-v^i(s_i^i,s_i^{-i})}{t_i^i-s_i^i}d\underline{F}^i(t_i^i) - \int_{S^{-i}} p^i(r^i,s^{-i})\frac{\partial v^i(s_i^i,s_i^{-i})}{\partial s_i^i}d\underline{F}^i| \\
\leq &|\int_{S^{-i}} p^i(r^i,s^{-i})\frac{v^i(t_i^i,s_i^{-i})-v^i(s_i^i,s_i^{-i})}{t_i^i-s_i^i}d\underline{F}^i(t_i^i) - \int_{S^{-i}} p^i(r^i,s^{-i})\frac{v^i(t_i^i,s_i^{-i})-v^i(s_i^i,s_i^{-i})}{t_i^i-s_i^i}d\underline{F}^i| \\
&+|\int_{S^{-i}} p^i(r^i,s^{-i})\frac{v^i(t_i^i,s_i^{-i})-v^i(s_i^i,s_i^{-i})}{t_i^i-s_i^i}d\underline{F}^i - \int_{S^{-i}} p^i(r^i,s^{-i})\frac{\partial v^i(s_i^i,s_i^{-i})}{\partial s_i^i}d\underline{F}^i|.
\end{aligned}
$$

The first term approaches 0 as $t_i^i \to s_i^i$ since $\underline{F}^i(t_i^i) \to \underline{F}^i$ and $\frac{v^i(t_i^i,s_i^{-i})-v^i(s_i^i,s_i^{-i})}{t_i^i-s_i^i}$ is continuous and uniformly bounded; from Lemma A 4, the second term approaches 0 as well. Hence,

$$
\lim_{t_i^i\to s_i^i}\int_{S^{-i}} p^i(r^i,s^{-i})\frac{v^i(t_i^i,s_i^{-i})-v^i(s_i^i,s_i^{-i})}{t_i^i-s_i^i}d\underline{F}^i(t_i^i) = \int_{S^{-i}} p^i(r^i,s^{-i})\frac{\partial v^i(s_i^i,s_i^{-i})}{\partial s_i^i}d\underline{F}^i.
$$

By the definition of $\underline{F}^i(t_i^i)$, we obtain

$$
\lim_{t_i^i\to s_i^i}\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(r^i,s^{-i})\frac{v^i(t_i^i,s_i^{-i})-v^i(s_i^i,s_i^{-i})}{t_i^i-s_i^i}dF^i = \int_{S^{-i}} p^i(r^i,s^{-i})\frac{\partial v^i(s_i^i,s_i^{-i})}{\partial s_i^i}d\underline{F}^i,
$$

which implies that

$$
\lim_{t_i^i\to s_i^i}\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(r^i,s^{-i})\frac{v^i(t_i^i,s_i^{-i})-v^i(s_i^i,s_i^{-i})}{t_i^i-s_i^i}dF^i \geq \min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(r^i,s^{-i})\frac{\partial v^i(s_i^i,s_i^{-i})}{\partial s_i^i}dF^i.
$$

The previous inequality together with (A4) completes the proof. □

A .2 Proof of Lemma 3.1

The Lipschitz continuity of μ^i is given by Lemma A 3. We are going to show that $\frac{\partial \mu^i(s^i)}{\partial s_i^i}$ lies in the specified interval. Fix $i \in \mathcal{I}$ and $s^i \in S^i$. For every $\hat{s}^i \in e(s_i^i)$ and every $t^i \notin e(s_i^i)$, Lemma A 1 implies

$$
\mu^i(t^i) - \mu^i(\hat{s}^i) \geq \min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(\hat{s}^i,s^{-i})\big(v^i(t_i^i,s_i^{-i}) - v^i(s_i^i,s_i^{-i})\big)dF^i.
$$

For every $\hat{s}^i \in e(s_i^i)$, Lemma A 1 implies $\mu^i(s^i) = \mu^i(\hat{s}^i)$. Thus,

$$
\mu^i(t^i) - \mu^i(s^i) \geq \min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(\hat{s}^i,s^{-i})\big(v^i(t_i^i,s_i^{-i}) - v^i(s_i^i,s_i^{-i})\big)dF^i. \tag{A5}
$$

If $t_i^i > s_i^i$, it follows from the above expression that

$$\frac{\mu^i(t^i) - \mu^i(s^i)}{t_i^i - s_i^i} \geq \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\hat{s}^i, s^{-i}) \frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} dF^i.$$

By Lemma A 3, μ^i is differentiable a.e. in S_i^i. Thus, if μ^i is differentiable at s_i^i, then taking the lower limit as $t_i^i \downarrow s_i^i$ and applying Lemma A 5 yield

$$\frac{\partial \mu^i(s^i)}{\partial s_i^i} \geq \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\hat{s}^i, s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i.$$

Since $\hat{s}^i \in e(s_i^i)$ was arbitrarily chosen, the above expression holds for $\overline{s}^i(s_i^i)$, that is,

$$\frac{\partial \mu^i(s^i)}{\partial s_i^i} \geq \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\overline{s}^i(s_i^i), s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i.$$

Similarly, if $t_i^i < s_i^i$, it follows from the inequality in (A5) that

$$\frac{\mu^i(t^i) - \mu^i(s^i)}{t_i^i - s_i^i} \leq \max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\hat{s}^i, s^{-i}) \frac{v^i(t_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{t_i^i - s_i^i} dF^i.$$

If μ^i is differentiable at s_i^i, taking the upper limit as $t_i^i \uparrow s_i^i$ and applying Lemma A 5 yield

$$\frac{\partial \mu^i(s^i)}{\partial s_i^i} \leq \max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\hat{s}^i, s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i.$$

Since $\hat{s}^i \in e(s_i^i)$ was arbitrarily chosen, the above inequality holds for $\underline{s}^i(s_i^i)$, that is,

$$\frac{\partial \mu^i(s^i)}{\partial s_i^i} \leq \max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\underline{s}^i(s_i^i), s^{-i}) \frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i.$$

A .3 Appendix for Section 3.2

We next show that $\underline{R}^i$ is well defined. Recall that $\tau_i^i := \min_{t_i^i \in S_i^i} t_i^i$. Let $\hat{\tau}_i^i := \max_{t_i^i \in S_i^i} t_i^i$. Let $\{a_0, ..., a_n\}$ be a partition of S_i^i, that is, $\tau_i^i = a_0 < ... < a_n = \hat{\tau}_i^i$. Given a partition $\{a_0, ..., a_n\}$, let $|P| := \max_{k \in \{1,...,n\}} a_k - a_{k-1}$ and let

$$m_k(s_i^{-i}) := \min_{\hat{s}_i^i \in [a_{k-1}, a_k]} \frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i} \quad \text{and} \quad M_k(s_i^{-i}) := \max_{\hat{s}_i^i \in [a_{k-1}, a_k]} \frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i}.$$

Lemma A 6. *Let p be an allocation rule that satisfies Monotonicity. Then the function $\min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\overline{s}^i(\hat{s}_i^i), s^{-i}) \frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i} dF^i$ is Riemann integrable on S_i^i.*

Proof. Take $\varepsilon > 0$. Since $\frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i}$ is equicontinuous, there exists a $\delta > 0$ such that

$$|\frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i} - \frac{\partial v^i(\tilde{s}_i^i, s_i^{-i})}{\partial \tilde{s}_i^i}| < \frac{\varepsilon}{2(\hat{\tau}_i^i - \tau_i^i)} \quad \forall |\hat{s}_i^i - \tilde{s}_i^i| < \delta, \forall s_i^{-i} \in S_i^{-i}. \tag{A6}$$

Choose a partition that contains n intervals with equal length $|P| = \frac{\hat{\tau}_i^i - \tau_i^i}{n}$ with $n > \max\{\frac{2(\hat{\tau}_i^i - \tau_i^i)}{\delta}, \frac{2(\hat{\tau}_i^i - \tau_i^i)\overline{M}}{\varepsilon}\}$, where $\overline{M} := \max_{\hat{s}_i^i \in S_i^i, s_i^{-i} \in S_i^{-i}} \frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i}$. Since $\frac{2(\hat{\tau}_i^i - \tau_i^i)}{n} < \delta$, (A6) implies

$$M_{k-1}(s_i^{-i}) - m_k(s_i^{-i}) < \frac{\varepsilon}{2(\hat{\tau}_i^i - \tau_i^i)} \quad \forall k = 1, .., n, \forall s_i^{-i} \in S_i^{-i}. \tag{A7}$$

Also, Monotonicity implies that for every $k = 1, ..., n$, every $\hat{s}_i^i \in [a_{k-1}, a_k]$, and every $s^{-i} \in S^{-i}$, we have $p^i(\bar{s}^i(a_{k-1}), s^{-i}) \leq p^i(\bar{s}^i(\hat{s}_i^i), s^{-i}) \leq p^i(\bar{s}^i(a_k), s^{-i})$. Then, the difference between the upper sum and the lower sum is[41]

$$\begin{aligned}
&\sum_{k=1}^{n} \left(\min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(a_k), s^{-i}) M_k(s_i^{-i}) dF^i - \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(a_{k-1}), s^{-i}) m_k(s_i^{-i}) dF^i \right) |P| \\
&= \sum_{k=1}^{n} \left(\min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(a_k), s^{-i}) M_k(s_i^{-i}) dF^i - \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(a_{k-1}), s^{-i}) M_{k-1}(s_i^{-i}) dF^i \right. \\
&\left. + \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(a_{k-1}), s^{-i}) M_{k-1}(s_i^{-i}) dF^i - \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(a_{k-1}), s^{-i}) m_k(s_i^{-i}) dF^i \right) |P| \\
&\leq |P| \left(\min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(\hat{\tau}_i^i), s^{-i}) M_n(s_i^{-i}) dF^i - \min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(\tau_i^i), s^{-i}) M_0(s_i^{-i}) dF^i \right) \\
&\qquad + \sum_{k=1}^{n} |P| \max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(a_{k-1}), s^{-i}) (M_{k-1}(s_i^{-i}) - m_k(s_i^{-i})) dF^i \\
&\leq \frac{\hat{\tau}_i^i - \tau_i^i}{n} \overline{M} + \sum_{k=1}^{n} \frac{\hat{\tau}_i^i - \tau_i^i}{n} \frac{\varepsilon}{2(\hat{\tau}_i^i - \tau_i^i)} < \varepsilon.
\end{aligned}$$

The second inequality follows from the definition of $\overline{M}$ and the inequality in (A7); the last inequality follows from the choice of n. By Theorem 11.30 in Aliprantis and Border (2006), $\min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i(\bar{s}^i(\hat{s}_i^i), s^{-i}) \frac{\partial v^i(\hat{s}_i^i, s_i^{-i})}{\partial \hat{s}_i^i} dF^i$ is Riemann integrable. $\square$

[41] The lower sum of a function $f : [a, b] \to \mathbb{R}$ relative to a partition $\{a_0, ..., a_n\}$ is defined by $\sum_{k=1}^{n} m_k(a_k - a_{k-1})$, where $m_k := \inf_{y \in [a_{k-1}, a_k]} f(y)$. Analogously, the upper sum is $\sum_{k=1}^{n} M_k(a_k - a_{k-1})$, where $M_k := \sup_{y \in [a_{k-1}, a_k]} f(y)$. Relevant concepts and results on Riemann integral can be found in Section 11.7 in Aliprantis and Border (2006).

B Appendix for Section 4

To prove Theorem 4.1, our first step is to show that under Linearity, any allocation rule p satisfies the following additivity condition:

$$\begin{aligned}&\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(s^i,s^{-i})\frac{\partial v^i(r_i^i,s_i^{-i})}{\partial r_i^i}dF^i+\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(s^i,s^{-i})\frac{\partial v^i(t_i^i,s_i^{-i})}{\partial t_i^i}dF^i\\&=\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(s^i,s^{-i})\Big(\frac{\partial v^i(r_i^i,s_i^{-i})}{\partial r_i^i}+\frac{\partial v^i(t_i^i,s_i^{-i})}{\partial t_i^i}\Big)dF^i\quad \forall r_i^i,t_i^i\in S_i^i,\forall s^i\in S^i,\forall i\in\mathcal{I}.\end{aligned}\tag{B8}$$

The next step is then to show that the condition above further implies (3). Then applying Lemma 3.2 completes the proof of Theorem 4.1.

Lemma B 7. *Assume Linearity. Then any allocation rule p satisfies (B8).*

Proof. Fix $i\in\mathcal{I}$, $r_i^i,t_i^i\in S^i$, and $s^i\in S^i$. By Linearity, for every $s_i^{-i}\in S_i^{-i}$,

$$\frac{\partial v^i(r_i^i,s_i^{-i})}{\partial r_i^i}=\frac{dg^i(r_i^i)}{dr_i^i}h^i(s_i^{-i})\quad\text{and}\quad\frac{\partial v^i(t_i^i,s_i^{-i})}{\partial t_i^i}=\frac{dg^i(t_i^i)}{dt_i^i}h^i(s_i^{-i}).\tag{B9}$$

Plugging (B9) into (B8) establishes the desired equality. □

Lemma B 8. *Let p be an allocation rule that satisfies (B8). Then p satisfies (3).*

Proof. Fix $i\in\mathcal{I}$, $r_i^i<t_i^i\in S_i^i$, and $s^i\in S^i$. Let $\{a_0,...,a_n\}$ be a partition of $[r_i^i,t_i^i]$ that contains n intervals with equal length. Then, using the definition of Riemann integral and condition (B8), we obtain

$$\begin{aligned}&\int_{r_i^i}^{t_i^i}\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(s^i,s^{-i})\frac{\partial v^i(\hat{s}_i^i,s_i^{-i})}{\partial\hat{s}_i^i}dF^i d\hat{s}_i^i\\&=\lim_{n\to\infty}\sum_{k=1}^n\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(s^i,s^{-i})\frac{\partial v^i\left(a_k,s_i^{-i}\right)}{\partial a_k}\frac{t_i^i-r_i^i}{n}dF^i\\&=\lim_{n\to\infty}\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(s^i,s^{-i})\sum_{k=1}^n\frac{\partial v^i\left(a_k,s_i^{-i}\right)}{\partial a_k}\frac{t_i^i-r_i^i}{n}dF^i.\end{aligned}$$

Applying similar proofs to Lemma A 5 and the definition of Riemann integral, the last term above is equal to

$$\min_{F^i\in\mathcal{F}^i}\int_{S^{-i}} p^i(s^i,s^{-i})\int_{r_i^i}^{t_i^i}\frac{\partial v^i(\hat{s}_i^i,s_i^{-i})}{\partial\hat{s}_i^i}d\hat{s}_i^i dF^i=\min_{F^i\in\mathcal{F}^i}\int_{r_i^i}^{t_i^i}\int_{S^{-i}} p^i(s^i,s^{-i})\frac{\partial v^i(\hat{s}_i^i,s_i^{-i})}{\partial\hat{s}_i^i}dF^i d\hat{s}_i^i,$$

as desired. $\square$

The proof of Theorem 4.2 follows analogous steps: we first show that under Comonotonic Independence and Increasing Differences, any deterministic allocation rule p satisfies (B8). Then applying Lemmas B 8 and 3.2 completes the proof.

Lemma B 9. *If the valuation function v^i has increasing differences, then $\frac{\partial v^i(s_i^i,\cdot)}{\partial s_i^i}$ and $\frac{\partial v^i(t_i^i,\cdot)}{\partial t_i^i}$ are comonotonic for all $s_i^i, t_i^i \in S_i^i$.*

Proof. Suppose that v^i has increasing differences. Take $s_i^i \in S_i^i$ and $s_i^{-i}, \hat{s}_i^{-i} \in S_i^{-i}$. Without loss of generality, assume $\frac{\partial v^i(s_i^i,s_i^{-i})}{\partial s_i^i} > \frac{\partial v^i(s_i^i,\hat{s}_i^{-i})}{\partial s_i^i}$. We are going to show that $\frac{\partial v^i(t_i^i,s_i^{-i})}{\partial t_i^i} \geq \frac{\partial v^i(t_i^i,\hat{s}_i^{-i})}{\partial t_i^i}$ for all $t_i^i \in S_i^i$. The proof is by contradiction. Suppose that there exists t_i^i such that $\frac{\partial v^i(t_i^i,s_i^{-i})}{\partial t_i^i} < \frac{\partial v^i(t_i^i,\hat{s}_i^{-i})}{\partial t_i^i}$. If $t_i^i > s_i^i$, there exists $l > 0$ such that $t_i^i - l > s_i^i$ and

$$v^i(t_i^i, s_i^{-i}) - v^i(t_i^i - l, s_i^{-i}) < v^i(t_i^i, \hat{s}_i^{-i}) - v^i(t_i^i - l, \hat{s}_i^{-i}),$$

which is equivalent to

$$v^i(t_i^i - l, \hat{s}_i^{-i}) - v^i(t_i^i - l, s_i^{-i}) < v^i(t_i^i, \hat{s}_i^{-i}) - v^i(t_i^i, s_i^{-i}).$$

Since v^i has increasing differences, for every $s_i^i < \hat{s}_i^i < t_i^i - l$, we have

$$v^i(s_i^i, \hat{s}_i^{-i}) - v^i(s_i^i, s_i^{-i}) \leq v^i(\hat{s}_i^i, \hat{s}_i^{-i}) - v^i(\hat{s}_i^i, s_i^{-i}).$$

Rearranging the inequality above yields

$$v^i(\hat{s}_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i}) \leq v^i(\hat{s}_i^i, \hat{s}_i^{-i}) - v^i(s_i^i, \hat{s}_i^{-i}).$$

Dividing both sides by $\hat{s}_i^i - s_i^i$ and taking the limit $\hat{s}_i^i \downarrow s_i^i$ yield

$$\lim_{\hat{s}_i^i \downarrow s_i^i} \frac{v^i(\hat{s}_i^i, s_i^{-i}) - v^i(s_i^i, s_i^{-i})}{\hat{s}_i^i - s_i^i} \leq \lim_{\hat{s}_i^i \downarrow s_i^i} \frac{v^i(\hat{s}_i^i, \hat{s}_i^{-i}) - v^i(s_i^i, \hat{s}_i^{-i})}{\hat{s}_i^i - s_i^i}.$$

By the assumption that v^i is differentiable, the above inequality is equivalent to $\frac{\partial v^i(s_i^i,s_i^{-i})}{\partial s_i^i} \leq \frac{\partial v^i(s_i^i,\hat{s}_i^{-i})}{\partial s_i^i}$, a contradiction. The case in which $t_i^i < s_i^i$ can be handled analogously. $\square$

Lemma B 10. *Assume Comonotonic Independence and Increasing Differences. Then any deterministic allocation rule p satisfies* (B8).

Proof. Fix $i \in \mathcal{I}$, $r_i^i, t_i^i \in S_i^i$, and $s^i \in S^i$. By Lemma B 9, Increasing Differences implies that $\frac{\partial v^i(r_i^i,\cdot)}{\partial r_i^i}$ and $\frac{\partial v^i(t_i^i,\cdot)}{\partial t_i^i}$ are comonotonic functions. Since p is deterministic, the functions $p^i(s^i,\cdot)\frac{\partial v^i(r_i^i,\cdot)}{\partial r_i^i}$ and $p^i(s^i,\cdot)\frac{\partial v^i(t_i^i,\cdot)}{\partial t_i^i}$ are comonotonic as well. Combining this observation with Comonotonic Independence, Theorem and Proposition 3 in Schmeidler (1986) together imply the equality in (B8). $\square$

C Appendix for Section 5

For any two probability measures $F, G \in \Delta(\Omega)$, the Prokhorov metric is

$$d(F,G) := \inf\{\varepsilon > 0 | F(A) \leq G(A^\varepsilon) + \varepsilon, \forall A \in \Sigma\},$$

where $A^\varepsilon := \{\omega \in \Omega | \inf_{\omega' \in A} d_\infty(\omega,\omega') \leq \varepsilon\}$ and d_∞ denotes the uniform metric in Ω.

Recall that $\bar{s}_j^i(s_i^i) := \min_{t^i \in e(s_i^i)} t_j^i$ and $\underline{s}_j^i(s_i^i) := \max_{t^i \in e(s_i^i)} t_j^i$ for every $i \in \mathcal{I}, j \neq i$, and $s_i^i \in S_i^i$.

Lemma C 11. *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\gamma^i < \delta$ for every $i \in \mathcal{I}$,*

$$\max_{j \neq i} v^j\left(\bar{s}_j^i(s_i^i), s_j^{-i}\right) > \max_{j \neq i} v^j\left(\underline{s}_j^i(s_i^i), s_j^{-i}\right) - \varepsilon \quad \forall i \in \mathcal{I}, s_i^i \in S_i^i, s^{-i} \in S^{-i}.$$

Proof. Take $\varepsilon > 0$ and

$$\delta := \frac{\varepsilon}{\max_{i \in \mathcal{I}, j \neq i, s_i^i \in S_i^i}\left(\underline{s}_j^i(s_i^i) - \bar{s}_j^i(s_i^i)\right)}.$$

Suppose that $\gamma^i < \delta$ for every $i \in \mathcal{I}$. By construction, for every $i \in \mathcal{I}$, $s_i^i \in S_i^i$, and $s^{-i} \in S^{-i}$, we obtain

$$\begin{aligned}\max_{j \neq i} v^j\left(\underline{s}_j^i(s_i^i), s_j^{-i}\right) - \max_{j \neq i} v^j\left(\bar{s}_j^i(s_i^i), s_j^{-i}\right) &\leq \max_{j \neq i}\left(v^j\left(\underline{s}_j^i(s_i^i), s_j^{-i}\right) - v^j\left(\bar{s}_j^i(s_i^i), s_j^{-i}\right)\right) \\ &\leq \max_{j \neq i} \gamma^i\left(\underline{s}_j^i(s_i^i) - \bar{s}_j^i(s_i^i)\right) < \varepsilon.\end{aligned}$$

$\square$

Let $\eta := \min_{i\in\mathcal{I}, s^i_i \in S^i_i, s^{-i}_i \in S^{-i}_i} \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i}$. By assumption, $\eta > 0$. For every $i \in \mathcal{I}$ and $s^i \in S^i$, define

$$A_{s^i} := \{s^{-i} \in S^{-i} | v^i(s^i_i, s^{-i}_i) \geq \max_{j\neq i} v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right)\},$$

$$B_{s^i} := \{s^{-i} \in S^{-i} | v^i(s^i_i, s^{-i}_i) \geq \max_{j\neq i} v^j\left(\bar{s}^i_j(s^i_i), s^{-i}_j\right)\}.$$

Since $\max_{j\neq i} v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right) \geq \max_{j\neq i} v^j\left(\bar{s}^i_j(s^i_i), s^{-i}_j\right)$, we have $A_{s^i} \subseteq B_{s^i}$.

Lemma C 12. *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\gamma^i < \delta$ for every $i \in \mathcal{I}$, we have $B_{s^i} \subseteq A^{\varepsilon}_{s^i}$ for all $i \in \mathcal{I}$ and $s^i \in S^i$.*

Proof. Take $\varepsilon > 0$. By Lemma C 11, there exists a $\delta > 0$ such that if $\gamma^i < \delta$ for every $i \in \mathcal{I}$, we have

$$\max_{j\neq i} v^j\left(\bar{s}^i_j(s^i_i), s^{-i}_j\right) > \max_{j\neq i} v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right) - \eta\varepsilon \quad \forall i \in \mathcal{I}, s^i_i \in S^i_i, s^{-i} \in S^{-i}. \tag{C10}$$

Assume $\gamma^i < \delta$ for every $i \in \mathcal{I}$. Fix $i \in \mathcal{I}$ and $s^i \in S^i$. Since $A_{s^i} \subseteq B_{s^i}$, we only need to show that $\inf_{\hat{s}^{-i}\in A_{s^i}} d_\infty(s^{-i}, \hat{s}^{-i}) \leq \varepsilon$ for every $s^{-i} \in B_{s^i} \setminus A_{s^i}$. Take any $s^{-i} \in B_{s^i} \setminus A_{s^i}$ and construct one $\hat{s}^{-i}$ as follows:

$$\hat{s}^j_j = s^j_j - \varepsilon \quad \text{and} \quad \hat{s}^j_k = s^j_k \quad \forall j \neq i, \forall k \neq j.$$

By construction, $d_\infty(s^{-i}, \hat{s}^{-i}) = \varepsilon$. We are going to show that $\hat{s}^{-i} \in A_{s^i}$.[42] Let $l \in \text{argmax}_{j\neq i} v^j\left(\underline{s}^i_j(s^i_i), \hat{s}^{-i}_j\right)$. By construction, $v^i(s^i_i, \hat{s}^{-i}_i) = v^i(s^i_i, s^{-i}_i)$. Thus,

$$\begin{aligned}
&v^i(s^i_i, \hat{s}^{-i}_i) - \max_{j\neq i} v^j\left(\underline{s}^i_j(s^i_i), \hat{s}^{-i}_j\right) = v^i(s^i_i, s^{-i}_i) - v^l\left(\underline{s}^i_l(s^i_i), \hat{s}^{-i}_l\right)\\
\geq &\max_{j\neq i} v^j\left(\bar{s}^i_j(s^i_i), s^{-i}_j\right) - v^l\left(\underline{s}^i_l(s^i_i), \hat{s}^{-i}_l\right) \geq \max_{j\neq i} v^j\left(\underline{s}^i_j(s^i_i), s^{-i}_j\right) - \eta\varepsilon - v^l\left(\underline{s}^i_l(s^i_i), \hat{s}^{-i}_l\right)\\
\geq &v^l\left(\underline{s}^i_l(s^i_i), s^{-i}_l\right) - \eta\varepsilon - v^l\left(\underline{s}^i_l(s^i_i), \hat{s}^{-i}_l\right) \geq \eta\varepsilon - \eta\varepsilon = 0.
\end{aligned}$$

The first inequality follows from $s^{-i} \in B_{s^i}$; the second inequality follows from

[42] If $\hat{s}^{-i} \notin S^{-i}$, we can enlarge the set of signals to include $\hat{s}^{-i}$. To see this, define $D := \{\tilde{s}^{-i} | \tilde{s}^j_j = s^j_j - \varepsilon, \tilde{s}^j_k = s^j_k, \forall j \neq i, \forall k \neq j, \forall s^{-i} \in B_{s^i} \setminus A_{s^i}\}$ and $A_+ := A_{s^i} \cup D$. Extend agent i's beliefs to this larger domain $S^{-i} \cup D$ such that $F^i(D) = 0$ for all $F^i \in \mathcal{F}^i$. Lemma C 12 then becomes $B_{s^i} \subseteq A^{\varepsilon}_+$ and the proof of Theorem 5.1 remains valid.

(C10); the last inequality follows from the construction of $\hat{s}_l^l$ and the definition of η. $\square$

Proof of Theorem 5.1. Take $\varepsilon > 0$. Assume that each agent i's set of beliefs $\mathcal{F}^i$ contains an ε-ball. By Lemma C 12, there exists a $\delta > 0$ such that if $\gamma^i < \delta$ for every $i \in \mathcal{I}$, $B_{s^i} \subseteq A^{\varepsilon}_{s^i}$ for all $i \in \mathcal{I}$ and $s^i \in S^i$. Assume that $\gamma^i < \delta$ for every $i \in \mathcal{I}$. Fix $i \in \mathcal{I}$ and $s^i \in S^i$. By the definition of Prokhorov metric, $B_{s^i} \subseteq A^{\varepsilon}_{s^i}$ implies that there exists a probability measure $\hat{G}^i \in \mathcal{F}^i$ such that $\hat{G}^i(B_{s^i} \setminus A_{s^i}) = 0$. Therefore,

$$\min_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i_*(\bar{s}^i(s^i_i), s^{-i}) \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} dF^i = \min_{F^i \in \mathcal{F}^i} \int_{B_{s^i}} \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} dF^i \leq \int_{B_{s^i}} \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} d\hat{G}^i$$
$$= \int_{A_{s^i}} \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} d\hat{G}^i \leq \max_{F^i \in \mathcal{F}^i} \int_{A_{s^i}} \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} dF^i = \max_{F^i \in \mathcal{F}^i} \int_{S^{-i}} p^i_*(\underline{s}^i(s^i_i), s^{-i}) \frac{\partial v^i(s^i_i, s^{-i}_i)}{\partial s^i_i} dF^i.$$

That is, Minimal Ambiguity is satisfied. Applying Corollary 5.1 completes the proof. $\square$

D Affiliated signals

In this section, we show that our result can be generalized to environments with some correlation of signals. In particular, we consider a special form of correlation: the variables $s^1_1, ..., s^N_N$ are affiliated and s^i_i and s^j_k are independently distributed for all $i, j, k \in \mathcal{I}$ and $j \neq k$.[43] Let $\mathcal{F}$ be the set of prior distributions of $s^1, ..., s^N$ and for each $F \in \mathcal{F}$, let $F^i(s^i_i)$ be the distribution of s^{-i} conditional on s^i_i that is consistent with F. We use $\mathcal{F}^i(s^i_i)$ to denote agent i's set of beliefs about other agents' signals conditional his realized signal $s^i \in e(s^i_i)$.

We next generalize Minimal Ambiguity to this setting.

Definition 4. For every $i \in \mathcal{I}$ and every $s^i_i \in S^i_i$, the set $\mathcal{F}^i(s^i_i)$ satisfies **Minimal**

[43]Suppose the random variables $Z^1, ..., Z^N$ have joint density F. Then the random variables are affiliated if and only if $F(z \vee \hat{z})F(z \wedge \hat{z}) \geq F(z)F(\hat{z})$ for all $z, \hat{z} \in \mathbb{R}^N$, where $z \vee \hat{z}$ denotes the component-wise maximum of z and $\hat{z}$, and $z \wedge \hat{z}$ denotes the component-wise minimum.

Ambiguity under the allocation rule p if

$$\min_{F^i(s_i^i)\in\mathcal{F}^i(s_i^i)} \int_{S^{-i}} p^i\left(\bar{s}^i(s_i^i), s^{-i}\right)\frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i(s_i^i)$$
$$\leq \max_{F^i(s_i^i)\in\mathcal{F}^i(s_i^i)} \int_{S^{-i}} p^i\left(\underline{s}^i(s_i^i), s^{-i}\right)\frac{\partial v^i(s_i^i, s_i^{-i})}{\partial s_i^i} dF^i(s_i^i).$$

Proposition D 1. *Assume that Linearity holds. Then p_* is implementable if for every $i \in \mathcal{I}$ and every $s_i^i \in S_i^i$, the set $\mathcal{F}^i(s_i^i)$ satisfies Minimal Ambiguity under p_*.*

Proof. For every $i \in \mathcal{I}$, Linearity implies that there exist functions $g^i : S_i^i \to \mathbb{R}_+$ and $f^i, h^i : S_i^{-i} \to \mathbb{R}_+$ such that

$$v^i(s_i^i, s_i^{-i}) = g^i(s_i^i)h^i(s_i^{-i}) + f^i(s_i^{-i}) \quad \forall s_i^i \in S_i^i, \forall s_i^{-i} \in S_i^{-i}.$$

Define a transfer scheme x_{full} as follows:

$$\underline{x}_{full}^i(s^i, s^{-i}) := \underline{R}^i(s^i) - p_*^i(s^i, s^{-i})v^i(s_i^i, s_i^{-i}), \quad \forall s \in S, \forall i \in \mathcal{I},$$

where

$$\underline{R}^i(s^i) := \int_{\tau_i^i}^{s_i^i} \min_{F^i(\hat{s}_i^i)\in\mathcal{F}^i(\hat{s}_i^i)} \int_{S^{-i}} p_*^i(\bar{s}^i(\hat{s}_i^i), s^{-i})\frac{dg^i(\hat{s}_i^i)}{d\hat{s}_i^i} h^i(s_i^{-i})dF^i(\hat{s}_i^i)d\hat{s}_i^i.$$

Let μ^i be the indirect utility functions associated with $(p_*, \underline{x}_{full})$. Notice that $\mu^i(s^i) = \underline{R}^i(s^i)$ by construction of the transfer scheme $\underline{x}_{full}$. We are going to prove that $(p_*, \underline{x}_{full})$ is interim incentive compatible. Fix $i \in \mathcal{I}$ and $s^i, t^i \in S^i$. To show $\mu^i(s^i) \geq u^i(t^i, s^i)$, there are two cases to consider. Suppose that $t_i^i \leq s_i^i$. By the assumption that s_i^i and s_j^i are independently distributed for all $j \neq i$, the efficient allocation rule p_* satisfies Monotonicity as is shown in Example 3. Therefore,

$$\begin{aligned}\mu^i(s^i) - \mu^i(t^i) &= \underline{R}^i(s^i) - \underline{R}^i(t^i)\\ &\geq \int_{t_i^i}^{s_i^i} \min_{F^i(\hat{s}_i^i)\in\mathcal{F}^i(\hat{s}_i^i)} \int_{S^{-i}} p_*^i(\bar{s}^i(t_i^i), s^{-i})\frac{dg^i(\hat{s}_i^i)}{d\hat{s}_i^i} h^i(s_i^{-i})dF^i(\hat{s}_i^i)d\hat{s}_i^i.\end{aligned} \tag{D11}$$

For every $\hat{s}_i^i \in S_i^i$, let

$$H^i(\hat{s}_i^i) \in \operatorname*{argmin}_{F^i(\hat{s}_i^i)\in\mathcal{F}^i(\hat{s}_i^i)} \int_{S^{-i}} p_*^i(\bar{s}^i(t_i^i), s^{-i})\frac{dg^i(\hat{s}_i^i)}{d\hat{s}_i^i} h^i(s_i^{-i})dF^i(\hat{s}_i^i).$$

Let $H_{\hat{s}_i^i} \in \mathcal{F}$ be the corresponding unconditional prior and $H^i_{\hat{s}_i^i}(\tilde{s}_i^i) \in \mathcal{F}^i(\tilde{s}_i^i)$ be the conditional distribution function of s^{-i} that is consistent of $H_{\hat{s}_i^i}$. Then for every $\tilde{s}_i^i > \hat{s}_i^i$,

$$\begin{aligned}
&\min_{F^i(\hat{s}_i^i)\in\mathcal{F}^i(\hat{s}_i^i)} \int_{S^{-i}} p_*^i(\bar{s}^i(t_i^i), s^{-i}) \frac{dg^i(\hat{s}_i^i)}{d\hat{s}_i^i} h^i(s_i^{-i}) dF^i(\hat{s}_i^i) \\
&= \int_{S^{-i}} p_*^i(\bar{s}^i(t_i^i), s^{-i}) \frac{dg^i(\hat{s}_i^i)}{d\hat{s}_i^i} h^i(s_i^{-i}) dH^i(\hat{s}_i^i) \\
&\geq \int_{S^{-i}} p_*^i(\bar{s}^i(t_i^i), s^{-i}) \frac{dg^i(\hat{s}_i^i)}{d\hat{s}_i^i} h^i(s_i^{-i}) dH^i_{\hat{s}_i^i}(\tilde{s}_i^i) \\
&\geq \min_{F^i(\tilde{s}_i^i)\in\mathcal{F}^i(\tilde{s}_i^i)} \int_{S^{-i}} p_*^i(\bar{s}^i(t_i^i), s^{-i}) \frac{dg^i(\hat{s}_i^i)}{d\hat{s}_i^i} h^i(s_i^{-i}) dF^i(\tilde{s}_i^i).
\end{aligned}$$

The first equality follows from the definition of $H^i(\hat{s}_i^i)$; the first inequality follows from the observation that $p_*^i(\bar{s}^i(t_i^i), s^{-i})$ decreases in s_j^j for all $j \neq i$, the assumption that s_i^i and s_j^j are affiliated for all $j \neq i$, the assumption that s_i^i and s_i^j are independently distributed for all $j \neq i$, and Theorem 5.4.5 in Milgrom (2004). Combining the inequalities above and the inequalities in (D11), we obtain

$$\begin{aligned}
\mu^i(s^i) &\geq \mu^i(t^i) + \int_{t_i^i}^{s_i^i} \min_{F^i(s_i^i)\in\mathcal{F}^i(s_i^i)} \int_{S^{-i}} p_*^i(\bar{s}^i(t_i^i), s^{-i}) \frac{dg^i(\hat{s}_i^i)}{d\hat{s}_i^i} h^i(s_i^{-i}) dF^i(s_i^i) d\hat{s}_i^i \\
&= \mu^i(t^i) + \min_{F^i(s_i^i)\in\mathcal{F}^i(s_i^i)} \int_{t_i^i}^{s_i^i} \int_{S^{-i}} p_*^i(\bar{s}^i(t_i^i), s^{-i}) \frac{dg^i(\hat{s}_i^i)}{d\hat{s}_i^i} h^i(s_i^{-i}) dF^i(s_i^i) d\hat{s}_i^i \\
&\geq \mu^i(t^i) + \min_{F^i(s_i^i)\in\mathcal{F}^i(s_i^i)} \int_{S^{-i}} p_*^i(t^i, s^{-i}) (v^i(s_i^i, s_i^{-i}) - v^i(t_i^i, s_i^{-i})) dF^i(s_i^i) \\
&= u^i(t^i, s^i),
\end{aligned}$$

where the second inequality follows from the definition of $\bar{s}^i(t_i^i)$ and the last equality follows from the construction of the transfer scheme $\underline{x}_{full}$. The proof for the case in which $t_i^i > s_i^i$ follows from analogous arguments. Since s^i and t^i were arbitrarily chosen, this shows that interim incentive compatibility is satisfied. □

E Ex ante, Interim, and Ex post Efficient Mechanisms

This section formalizes different notions of efficiency and illustrate their relationships in our setting. For any signal $s \in S$, we assume that the mechanism designer's ex post utility is $-\sum_i x^i(s)$, where $x^i(s)$ is the transfer to agent i. The mechanism designer's ex ante preferences are represented by maxmin expected utility where $\mathcal{G}_i^M$ denotes his set of ex ante beliefs about agent i's signals. Further, we assume that the mechanism designer believes that all the signals are independently distributed and $\mathcal{G}^M := \{\times_i G_i^M | G_i^M \in \mathcal{G}_i^M\}$ denotes his set of ex ante beliefs about all agents' signals. Also, define $\mathcal{G}_{-i}^M := \{\times_{j \neq i} G_j^M | G_j^M \in \mathcal{G}_j^M\}$. In the ex ante stage, agents have not observed their signals. Each agent's ex ante preference is represented by maxmin expected utility with $\mathcal{G}^i$ being the set of ex ante beliefs of agent i. In the interim stage, each agent has observed his own signal, but not the signals of the others. Recall that $\mathcal{F}^i$ denotes the set of interim beliefs of agent i. Finally, in the ex post stage, all the signals are publicly revealed.

A mechanism (p, x) is **ex ante efficient** if there is no other mechanism $(\hat{p}, \hat{x})$ that yields a higher ex ante utility to some agent or the mechanism designer, without lowering the ex ante utilities of the others. Interim and ex post efficient mechanisms can be defined analogously. Let E_A, E_I, and E_P denote the sets of mechanisms that are respectively ex ante, interim, and ex post efficient.

In a Bayesian setting with quasilinear utilities, if all agents and the mechanism designer share the same ex ante belief, the three notions coincide: $E_A = E_I = E_P = \{(p, x) | p \in P_*\}$, where $P_* := \{p | p^i(s) > 0 \Rightarrow i \in \text{argmax}_j\, v^j(s_j^1, ..., s_j^N), \forall s \in S\}$. In our setting, the three notions generally differ. Obviously, the set of ex post efficient mechanisms remains the same. We now examine how the sets of ex ante and interim efficient mechanisms are affected by the presence of ambiguity aversion. Define a transfer scheme x_C as follows: $x_C^i(s) := A^i - p^i(s) v^i(s_i^1, ..., s_i^N)$ for all $s \in S$, all $i \in \mathcal{I}$, where $A^i \in \mathbb{R}$. Each agent is fully insured against ambiguity in the ex ante stage under x_C. Denote the set of all such transfer schemes by X_C and denote the set of all full insurance transfer schemes by X_{full}.

Proposition E 2. *If $\mathcal{G}^M \subseteq \mathcal{G}^i$ for all $i \in \mathcal{I}$, then $\{(p, x_C) | p \in P_*, x_C \in X_C\} \subseteq E_A$;If $\mathcal{G}^M_{-i} \subseteq \mathcal{F}^i$ for all $i \in \mathcal{I}$, then $\{(p, x_{full}) | p \in P_*, x_{full} \in X_{full}\} \subseteq E_I$.*

The proof of Proposition E 2 follows similar lines as that of Proposition 1 in Bose et al. (2006). The intuition is simple: if the mechanism designer faces less ambiguity than the agents, he can improve social welfare by fully insuring the agents.

An implication of Proposition E 2 is that the full insurance mechanisms we construct are interim efficient. Thus, our results will remain the same if we use the notion of interim efficiency. However, ex ante efficient and interim incentive compatible mechanisms may not exist—there exist situations in which $\{(p, x_C) | p \in P_*, x_C \in X_C\} = E_A$.[44] Since each agent's indirect utility is independent of the realization of his signal under x_C, ex ante efficient mechanisms are not interim incentive compatible except in the case of complete ambiguity.

F Ambiguous Communication Devices

The central idea of Bose and Renou (2014) is that the mechanism designer can create ambiguity through an ambiguous communication device. To illustrate how the use of such device can facilitate efficient implementation, we explicitly construct one using the example from Section 6. We assume that there is no prior ambiguity, that is, each agent has a single prior over the other agents' types. We also assume that agents adopt full Bayesian updating. For simplicity, let $N = 2$ and let agent i's prior distribution of $\theta^j - c^j$ be uniform on $[-1, 1]$ for every $i \in \{1, 2\}$ and $j \neq i$. The ambiguous communication device is constructed as follows: Before the standard allocation mechanism is carried out, each agent i can send a message $(\tilde{\theta}^i, \tilde{c}^i) \in [0, 1] \times [0, 1]$ to the mechanism designer and can receive message a

[44]For example, as shown in Proposition 3 in Bose et al. (2006), this is the case when the mechanism designer has a single ex ante belief $F^M \in \Delta(S)$ with full support and there exists $\varepsilon \in (0, 1)$ such that $C_\varepsilon(F^M) \subseteq \mathcal{G}^i$ for every $i \in \mathcal{I}$.

or b from the mechanism designer. All messages are confidential. The messages sent to agents are drawn according to one of the two probability systems φ and $\hat{\varphi}$: Denote $\varphi^i(m^i|\tilde{\theta}^j, \tilde{c}^j)$ the probability that the mechanism designer sends message $m^i \in \{a, b\}$ to agent i conditional on receiving the message $(\tilde{\theta}^j, \tilde{c}^j)$ from agent $j \neq i$ and let $\varphi\big((m^1, m^2)|(\tilde{\theta}^1, \tilde{c}^1), (\tilde{\theta}^2, \tilde{c}^2)\big) := \varphi^1(m^1|\tilde{\theta}^2, \tilde{c}^2)\varphi^2(m^2|\tilde{\theta}^1, \tilde{c}^1)$ for all (m^1, m^2), $(\tilde{\theta}^1, \tilde{c}^1)$, and $(\tilde{\theta}^2, \tilde{c}^2)$. Similarly, we can define the other probability system $\hat{\varphi}$. For every $i \in \{1, 2\}$ and $j \neq i$, let

$$\varphi^i(a|\tilde{\theta}^j, \tilde{c}^j) = \begin{cases} 1 & \text{if } \tilde{\theta}^j - \tilde{c}^j \in [-1, 0], \\ 0 & \text{otherwise}; \end{cases} \qquad \hat{\varphi}^i(a|\tilde{\theta}^j, \tilde{c}^j) = \begin{cases} 0 & \text{if } \tilde{\theta}^j - \tilde{c}^j \in [-1, 0], \\ 1 & \text{otherwise}. \end{cases}$$

Since agents are ambiguous about the probability system that has been used by the mechanism designer, they perceive ambiguity after one round of communication.

The mechanism designer uses the following two-stage mechanism to implement the efficient allocation rule p_*. In the first stage, agents communicate with the mechanism designer through the ambiguous communication device constructed above. In the second stage, agents report their signals and the mechanism designer carries out the mechanism $(p_*, \underline{x}_{full})$, where $\underline{x}_{full}$ is constructed according to posterior beliefs. We can verify that each agent i reporting (θ^i, c^i) truthfully in both stages is an equilibrium. Thus, the efficient allocation rule is indeed implementable.

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