Stable Price Dispersion

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Abstract. We study the pricing of homogeneous products sold to customers who consider different sets of suppliers. We seek prices that are stable in the sense that no firm wishes to undercut any rival or to raise its price when rivals have a subsequent opportunity to undercut it. We identify stable and dispersed prices that emerge from both collective choice and non-cooperative pricing games, and derive predictions for prices across several price-consideration specifications. We show how the implications for firms and customers compare to those generated by conventional approaches.

Seemingly identical products are regularly offered at different yet persistent prices. We offer a theory of price formation centered on the idea that stable pricing positions are robust to threats of undercuts from rivals. We use our approach to predict stable and dispersed prices.

Strategic accounts of price-setting date back to canonical Bertrand models of competition, where the core incentive of a firm is to lower its price sufficiently to "undercut" rivals. The simplest case is typically understood to lead to marginal-cost pricing. Such prices are stable in the sense that no firm would undercut any rival further, but of course they are not dispersed.

An established approach extends this setting by studying a situation with heterogenous price consideration: different buyers evaluate different prices. A price-setting firm faces a trade-off: undercut rivals to sell to buyers who compare many prices, or elevate price to profit from those who do not. Modeling price-setting in the standard way (a non-cooperative single-stage game) results in equilibrium prices that are set randomly via mixed strategies. The realized prices are dispersed, but lack stability: upon seeing cheaper firms' prices, a higher-priced competitor regrets its choice and wishes to undercut a rival.

We provide a theory that delivers prices that are both stable and dispersed. Our approach recognizes the reality that sellers are often able to slash prices or to offer sales easily so that dropping a price is easier or can be executed more quickly than pushing it upwards. It is exactly that threat that disciplines the prices that we predict: if a firm were to allow its price to creep up then it would trigger a rival to undercut it.

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The demand-side context is a distribution over consideration sets. Such a set is a list of firms; the distribution specifies the mass of buyers who consider the prices of those firms.

We seek "stable" prices that satisfy two criteria. Firstly, they are "undercut proof" so that no firm can profitably undercut a cheaper competitor. Secondly, we require "creep resistance" which means that no firm can profit from raising its price by a little given that, were it to do so, firms would have a subsequent chance to cut prices in response.

Undercut-proof profiles of prices are (under mild regularity conditions) entirely dispersed: each firm charges a different price. The key profiles (one for each ordering of firms) are "maximal" in the sense that no price can be raised without violating undercut-proofness. At least one maximal profile is optimal for the industry and, in several settings, it is unique.

An undercut-proof profile of prices is resistant to creep if no firm can increase its profit by raising its price slightly if others can profitably respond with a lower price. Most obviously, prices must be maximal if they satisfy this criterion. We must then investigate a deviant firm that allows its price to creep further upward from a maximal undercut-proof profile. This requires us to study games in which firms are offered the opportunity to cut their prices in response. We construct equilibria of such price-cutting games and check to see if the deviant firm gains from the creep upward in its price. From this work we find a sufficient condition on the distribution over consideration sets for the existence of a stable price profile: this is a maximal undercut-proof profile of prices that satisfies creep resistance. Our condition is satisfied by a wide set of specifications, including those we later study in depth.

We then ask: do stable prices emerge from non-cooperative interaction? One simple implementation of a price-formation game that is consistent with our approach has two stages of play: firms simultaneously set initial price positions and then have a simultaneous opportunity to lower their prices, prior to purchases. The second stage is readily interpreted as the opportunity for firms to engage in "flash sales" relative to their regular prices; the first stage allows those regular prices to be established non-cooperatively. We look for prices played as pure strategies on the path of a subgame-perfect equilibrium of such a game. In several settings we show that stable, disperse, and industry-optimal prices emerge from such play.

In two broad classes of consideration-set specifications we sharpen our results. In one of our settings the mass of customers who consider each set of prices depends only on the number of prices, and so firms are "exchangeable" across similarly sized consideration sets. We allow, in addition, for different sizes of firms' captive audiences as well as retaining flexibility for the relative importance of different consideration-set sizes. We identify a unique industry-optimal stable price profile in which firms with larger captive bases set higher prices. In the second of our settings we allow for greater asymmetry, so that some firms are more likely to lie within any size of consideration set and so benefit from higher awareness. There, however, we specify consideration across firms to be independent, as in some popular models of informative advertising. We find that in an industry-optimal stable price profile the highest price is charged by the firm that enjoys the greatest awareness amongst customers.

We also expose the complexities of the environment. We do that via three triopoly examples. The first example finds industry-optimal stable prices from which one firm can profitably deviate with a non-local increase (a jump rather than a creep) in its price; the second example shows that multiple industry-optimal profiles can be supported by non-cooperative pure-strategy play; and the third shows that stable prices are not always ordered by natural notions of firm size.

A primitive for our work is the distribution of consideration sets. This can in principle be influenced by the actions of firms and customers, and so we also illustrate how our framework can be used to study such actions, via two duopoly examples. Firstly, we allow firms to influence awareness via costly advertising. We find that symmetric firms make different advertising decisions, resulting in a marked dispersion of their stable prices. Secondly, we allow costly buyer search. Search decisions are strategic complements (search by others increases dispersion, raising the benefit of another quotation), which readily generates multiple equilibria. A (non-knife-edge) equilibrium with high search exhibits substantial stable price dispersion.

Related Literature. The environment of price-setting firms, homogeneous goods, and heterogeneous buyer consideration is a classic one. The focal modeling items are buyers' "consideration sets," a term originating in the marketing literature and used by many in economics including (for example) Eliaz and Spiegler (2011) and Manzini and Mariotti (2014).

The workhorse model of price-setting in this environment is a single-stage non-cooperative game in which all firms simultaneously set prices, where founding contributions include Varian (1980), Rosenthal (1980), Narasimhan (1988), and Baye, Kovenock, and de Vries (1992). Armstrong and Vickers (2022) made substantial progress and solved that game under several consideration structures, interpreting the equilibrium strategies as the "patterns of competitive interaction."

The consideration-set demand structure famously leaves a single-stage pricing game with no pure-strategy Nash equilibrium, and so price dispersion is typically interpreted as the outcome from the realizations of mixed strategies.[2] Such predictions are unstable in the sense that realized prices are not best replies ex post. A literal interpretation of repeated play of the game implies rapidly fluctuating prices. Instead, we harness firms' ability (and credible threat) to cut quickly or discount a price, to arrive at stable outcomes. The prices we predict are both undercut-proof and resistant to price increases that can subsequently be undercut. When stable prices appear on a game's equilibrium path, they necessarily do so in pure strategies.

One way to connect our predictions to those of the traditional approach is to lean on an interpretation (inspired by Varian, 1980 and Baye, Kovenock, and de Vries, 1992) of mixed-strategy realizations being discounts from a higher "regular" price. Rather than describing any such flash sales behavior, our approach can explain how underlying and stable regular prices originate as a function of buyer consideration, and that they themselves are dispersed.

[2]To illustrate, consider the structure of best replies in a duopoly: firms undercut each other to capture buyers who consider both prices and so walk down a staircase of prices; but at a sufficiently low price one firm abandons the shoppers and elevates its price back up to exploit buyers who only consider their price. This "Edgeworth cycle" logic (Maskin and Tirole, 1988a,b) rules out pure-strategy equilibria.

Our approach can also be viewed as a weakening of a commitment assumption. A firm is fully committed to a price point when it cannot adjust it after setting it, at all, upwards or downwards. In that case, a mixed-strategy prediction can rationalize persistent prices: firms want to undercut rivals' realized prices but cannot. In contrast, we assume (at most) one-sided commitment. Firms are always free to lower their prices to undercut a rival at any time, but they face less upward flexibility: any such move gives rivals a further opportunity to undercut.

Empirical Considerations. Empirical studies have identified extensive price dispersion. For example, Kaplan and Menzio (2015) used the large Kilts-Nielsen panel of 50,000 households to show that the standard deviation (relative to the mean) of prices at brick-and-mortar stores ranges from 19% (when products are defined narrowly) to 36% (when defined broadly).

Despite both cross-sectional and inter-temporal variation in price,[3] prices are well-known to be sticky or persistent in many markets. Using data from the Bureau of Labor Statistics, Nakamura and Steinsson (2008) estimated the median duration of a (regular) price in the US to be between 8–11 months. The European Central Bank (ECB, 2005) found the "median firm changes its price once a year." Using Norwegian retail data, Wulfsberg (2016) and Moen, Wulfsberg, and Aas (2020) found a high persistence of price dispersion: prices on average last 6–16 months depending on the product category and macro environment, and stores charging prices in a particular quartile of the distribution stay there with high probability (0.83–0.93) month-to-month. Remarkably, Gorodnichenko, Sheremirov, and Talavera (2018) examined daily online pricing data and reported (pp. 1764–1766) that "although online prices change more frequently than offline prices, they nevertheless exhibit relatively long spells of fixed prices." Specifically, prices are fixed for long spells of 7–20 weeks and "do not adjust every instant." They concluded that prices tend to vary in the cross section rather than over time.[4]

In principle, long price spells could be because of a paucity of opportunity for firms to change their prices. However, some evidence suggests firms do not change their prices every time they can. For example, 43% of Euro-area firms reviewed prices at least four times a year, but only 14% changed price that often (ECB, 2005, see also ECB, 2019).

These applied considerations motivate a theory of stable price dispersion. Our approach posits that firms are free to cut price and undercut rivals at any point, but that it can be harder to raise a price once it is set or to prevent price-cutting responses. This gels with an intuition that retailers can typically slash prices, by even large amounts, at short notice.

[3]For example, Kaplan and Menzio (2015) decomposed the variation and found the intertemporal component (their "transaction" component) accounted for a substantial fraction, but less than half. Also with Kilts-Niesen data, Kaplan, Menzio, Rudanko, and Trachter (2019) reported that "a sizable fraction of the variance of prices for the same good is caused by persistent differences in the price that different stores set for that good [...]."

[4]Many industry-specific studies are also consistent with this summary, e.g., those on prescription drugs (Sorensen, 2000), illicit drugs (Galenianos, Pacula, and Persico, 2012), memory chips (Moraga-González and Wildenbeest, 2008), and textbooks (Hong and Shum, 2010). A contrasting conclusion was provided by Lach (2002), who emphasized that (p. 433) "stores move up and down the cross-sectional price distribution." And in some industries, gasoline for example, substantial dynamic price movements have been documented (see, e.g., Chandra and Tappata, 2011; Pennerstorfer, Schmidt-Dengler, Schutz, Weiss, and Yontcheva, 2020).

One reason for this could be that customers may see attempts to charge above some initial price as unfair or socially unacceptable. It may be that an initial price sets a reference point for loss-aversion arguments (Kahneman and Tversky, 1979), which suggests a higher elasticity of demand above the initial price than below (Ahrens, Pirschel, and Snower, 2017).[5] The role of fairness concerns is central to the work of Kahneman, Knetsch, and Thaler (1986). Relating their ideas to Okun (1981), they reported "the hostile reaction of customers to price increases that are not justified by increased costs . . . "[6] Firms are not ignorant of these concerns: the ECB reports cited above found that a firm's "implicit contract" with their customers (that their prices will not rise) was a primary reason behind the observed price stickiness.[7] Additionally, in some markets direct legal constraints force a firm to meet any published offer or to limit price rises. For example, in a study of dispersed prices for prescription drugs, Sorensen (2000, p. 837) reported that "price-posting legislation dictates that any posted price must be honored at the request of the consumer."[8] In the gasoline market, Obradovits (2014) documented regulations that prohibited price rises (except once a day at noon), while price cuts were freely permitted.[9]

Plan of the Paper. We now illustrate our ideas in a duopoly (Section 1). We then develop our theory (Sections 2 and 3) and apply it to two classes of consideration: exchangeability and independent awareness (Sections 4 and 5). We study limitations in triopoly settings (Section 6), before closing with two consideration-endogenizing applications (Section 7). Proofs and extensions appear in main (Appendix A) and supporting (Appendices B and C) supplements.

1. A Duopoly Illustration

Consider a homogeneous-good, zero-cost duopoly. The price of firm $i \in \{1, 2\}$ is $p_i \in [0, v]$. Each customer is interested in obtaining a single unit and is willing to pay at most $v > 0$ for it. A mass $\lambda_i > 0$ of customers consider only i and so are "captive" to that firm, while $\lambda_S > 0$ "shoppers" consider both prices and buy from a cheapest firm.

We begin by deriving the set of "undercut proof" prices. Trivially, of course, two zero prices are undercut-proof. Otherwise, for strictly positive prices to be undercut proof we need them to differ and to be sufficiently far apart. For the alternative rankings of the two prices, the corresponding "undercut proofness" constraints are

$$\underbrace{p_1\lambda_1 \geq p_2(\lambda_1 + \lambda_S)}_{\text{if } p_1 > p_2} \quad \text{or} \quad \underbrace{p_2\lambda_2 \geq p_1(\lambda_2 + \lambda_S)}_{\text{if } p_2 > p_1}. \tag{1}$$

[5]Relatedly, marketing research documents how "advertised reference prices" set value perceptions and purchase intentions, e.g., Urbany, Bearden, and Weilbaker (1988); Lichtenstein, Burton, and Karson (1991); Grewal, Monroe, and Krishnan (1998); Alford and Engelland (2000); Kan, Lichtenstein, Grant, and Janiszewski (2013).

[6]The importance of fairness considerations in pricing is central to many marketing studies (e.g., Campbell, 1999, 2007; Bolton, Warlop, and Alba, 2003; Xia, Monroe, and Cox, 2004).

[7]Of course prices sometimes rise, but typically with inflation, unlike cuts (e.g., Nakamura and Steinsson, 2008).

[8]Charging "overs" at the point of sale can also fall under definitions of deceptive pricing. For example, the UK's Advertising Standards Authority advises that a product should be available at its listed price.

[9]Price-control and profit-control laws (which are often temporary measures) also typically impose frictions, explicitly or implicitly, on upwards but not downwards price movements.

The left-hand side of each inequality is earned by the higher-price firm from its captive customers; the right-side reflects the extra sales to shoppers by undercutting a cheaper competitor.

Pairs of prices that satisfy one of the (mutually exclusive) inequalities are undercut proof. But what if a firm allows its price to creep upward? If there is slack in the relevant no-undercutting constraint the cheaper firm can raise its price (locally) and neither firm would wish to cut price in response. Similarly, if the higher price firm prices strictly below v then its price can also creep up without prompting an undercut. For "creep resistance" we need undercut-proof prices to be as high as possible or "maximal" and so

$$\underbrace{p_1 = v \,\text{and}\, p_2 = \frac{v\lambda_1}{\lambda_1 + \lambda_S}}_{\text{if } p_1 > p_2} \quad \text{or} \quad \underbrace{p_1 = \frac{v\lambda_2}{\lambda_2 + \lambda_S} \,\text{and}\, p_2 = v.}_{\text{if } p_1 < p_2} \tag{2}$$

If $\lambda_1 > \lambda_2$ then the first pair of prices in eq. (2) generates strictly more profit for each firm than the second, and so is an "industry optimal" pair amongst undercut-proof prices.

Restricting attention to the two maximal price pairs reported in eq. (2), we now ask: can the lower-price firm deviate and do better by creeping its price further upward (and so break the no-undercutting constraint) given that its rival has a subsequent opportunity to cut its price? If the rival is uniquely able to respond then (of course) that rival will take that opportunity to undercut and so the deviant firm will lose the sales of shoppers.[10] If the creep upward in price is sufficiently small then this loss in sales to shoppers ensures that the deviant firm performs strictly worse. In this sense, both maximal price pairs are creep-resistant.

Suppose instead that both firms are able to engage in price cuts. Here we must study a simultaneous-move price-cutting game. Consider the first pair of maximal undercut-proof prices from eq. (2) and a move upward of $\Delta > 0$ in the second (and cheaper) firm's price. The two firms choose final prices $\tilde{p}_1 \in [0, v]$ and $\tilde{p}_2 \in [0, p_2 + \Delta]$. If $\Delta > 0$ is not too large then this game has a unique mixed-strategy Nash equilibrium in which both firms mix continuously over the interval $[p_2, p_2 + \Delta)$ with residual atoms at v and $p_2 + \Delta$ respectively, giving firms expected profits precisely equal to those obtained from the original undercut-proof prices. We conclude that maximal undercut-proof prices are creep resistant, and so (as we define it) stable.

If $\lambda_1 \geq \lambda_2$ then the argument above holds for Δ of any size. This implies a stronger result: the "industry optimal" prices in eq. (2) are played as pure strategies on the path of a subgame perfect equilibrium of a two-stage game in which firms adopt regular price positions (in the first stage) and then are free to engage in price cuts (in the second stage). Notably, that logic is not true of $\lambda_1 < \lambda_2$: the second (and cheaper) firm with the larger captive audience would prefer to raise its regular price to v, despite the subsequent undercutting. We conclude (for $\lambda_1 \neq \lambda_2$) that the pair of industry-optimal, dispersed and stable prices in eq. (2) is also the unique one supported via the pure-strategy play of a subgame-perfect equilibrium.

[10]Precisely, the unique best reply is to match the deviator's new price with the tie broken in the responder's favor. The endogenous settlement of the tie-break in the spirit of Simon and Zame (1990) solves the standard best-reply existence issue in Bertrand games, by making an "undercut" well-defined.

2. A Model of Heterogeneous Price Consideration

The Economic Environment. On the supply side, $n > 1$ firms indexed by $i \in \{1, \ldots, n\}$ produce a homogeneous product with the same constant marginal cost which, without (further) loss of generality, we set to zero. A firm's profit is its price multiplied by its sales.

On the demand side, each customer is willing to pay at most $v > 0$ for a single unit. A customer's *consideration set* lists those firms from whom they may buy. Each customer buys from the cheapest firm in their consideration set; ties can be broken in any interior way.

We write $\lambda(B) : 2^N \mapsto \mathcal{R}^+$ for the mass of customers considering firms within $B \subseteq \{1, \ldots, n\}$, and we write $B_i = \mathcal{I}[i \in B] \in \{0, 1\}$ for the indicator of whether firm i is a member of B. We also use the shorthand $\lambda_i = \lambda(\{i\})$ for the mass of those who are "captive" to a single firm i. To set aside uninteresting cases, we assume that each firm i has some captive customers ($\lambda_i > 0$) and that a positive mass consider i together with at least one other firm $j \neq i$.[11]

In a classic "captive and shopper" model a mass of $\lambda_S \equiv \lambda(\{1, \ldots, n\})$ customers are "shoppers" who consider every firm; all other non-singleton consideration sets have zero mass. The latter feature guarantees that a single-stage pricing game with symmetric firms has (infinitely) many equilibria.[12] In their analysis of a game in which consideration sets are formed randomly and symmetrically, Johnen and Ronayne (2021) noted the property of "twoness" meaning that all consideration pairs have positive mass: $\lambda(\{i, j\}) > 0$ for $i \neq j$. A symmetric single-stage pricing game has a unique equilibrium if and only if this holds. Our exposition simplifies appreciably if we maintain this property as an assumption, and so (for convenience) we do so. (Our claims hold without it, but our statements can become tedious.) Of course, the "twoness" property does not hold for a strict captive-and-shopper world (in which shoppers' comparisons are made between all firms) with $n > 2$ but we do, nevertheless, also handle that case (in Section 4).

Solution Concept (i): Stable Prices. Our novel approach is to seek prices that satisfy two criteria. Firstly, prices should be "undercut proof" so that no firm strictly gains by undercutting a cheaper competitor. Secondly, prices should be "creep resistant" so that no firm profits from a small move upward (a "creep" in our terminology of price cuts and creeps) in its price given that all firms subsequently enjoy an opportunity to revise their prices downward. These two criteria capture the ideas that prices can be easier to lower than to raise and that a firm can readily respond to competitors with a discount or special offer.

When a profile of prices (one price per firm) satisfy these criteria, we call those prices "stable."

[11] If i is not considered with another firm, then i's audience is entirely captive and it sets the monopoly price. If there are no captive customers, then Bertrand competition forces all profits down to zero.
[12] Under asymmetry, two (smallest) firms mix, while others charge v (Baye, Kovenock, and de Vries, 1992).

Definition (Stable Prices). *Prices (one per firm) are* stable *if they satisfy both (1) and (2):*

(1) Undercut-proofness*. Equivalently, in a game in which firms have a simultaneous opportunity to lower prices there is a Nash equilibrium in which no firm does so.*

(2) Creep resistance*. No firm gains from a creep upward in its price, given that all firms would enjoy a subsequent opportunity to cut their prices. Formally, following a sufficiently small increase in the price of a firm i, consider a price-cutting game in which firms simultaneous choose whether to lower prices. That game has a Nash equilibrium that gives a weakly lower expected profit to firm i, relative to the equilibrium in (1).*

Both criteria refer to a simultaneous-move game in which firms may lower their prices from some initial price positions. Later we find a unique Nash equilibrium for each relevant game.

Requirement (1) is weaker than pure-strategy Nash equilibrium, and so many price profiles are undercut-proof. Amongst this set, we might seek the highest prices that are best for firms' profits. Such prices are (as we will confirm) candidates to meet requirement (2). Given suitable conditions, we also show (in our results) something stronger: a further movement upwards in a firm's price (to a profile that is no longer undercut-proof) will be followed by a Nash equilibrium of a simultaneous-move price-cutting game in which the relevant firm does not gain.

Solution Concept (ii): Non-cooperative Price Formation. Our notion of resistance to "creep" in a firm's price is limited to a small price rise. We might also investigate larger price rises. Relatedly, we left open the nature of price formation: what if initial prices are chosen non-cooperatively? We can model the non-cooperative formation and adjustment of prices using several different games.[13] The key feature is that the final adjustment opportunity available to firms is a downward price cut. This is most simply captured via a two-stage pricing game:

(i) firms simultaneously choose their initial price positions; and then
(ii) firms simultaneously choose whether to revise downward their prices.

As usual, firms' payoffs in this game are profits and they are assumed to be risk neutral.

The second stage allows firms to offer "sales" in the spirit of Varian (1980). Such sales are readily interpreted as special offers relative to a firm's regular price. However, in the classic model of sales the determination of firms' regular prices is not addressed; those prices are implicitly assumed to be equal to the monopoly price v that fully exploits any captive buyers.

A natural solution concept is subgame-perfect equilibrium. Such equilibria can include the (both on-path and off-path) play of mixed strategies. However, we seek (and will tend to find) equilibria in which a profile of prices is played via pure strategies on the equilibrium path.

[13] Others have studied related price-then-cut games. For example, Anderson, Baik, and Larson (2023) modeled targeted price discrimination with mixed-strategy personalized prices and Gill and Thanassoulis (2016) studied a duopoly in which some customers see list prices while others access to second-stage discounts.

Definition (Pure-Strategy Play). *A profile of prices is* supported by the equilibrium play of pure strategies *if there is a subgame-perfect equilibrium of the two-stage pricing game in which that profile is played on the equilibrium path in both the first and second stage.*

If a price profile satisfies this definition then (straightforwardly) it must be undercut proof, for if it were not then a firm would exploit a profitable second-stage deviation. Another possible deviation for a firm is to raise its price in the first stage, and so prices supported by pure-strategy play must also be resistant to price creep. In summary, such prices must be stable. However, the definition of pure-strategy play can rule out certain stable price profiles. For example, our consideration of duopoly (in Section 1) identified a unique profile of prices supported by the equilibrium play of pure strategies even though there are two stable profiles.

3. Characterizing Stable Prices

In this section we characterize the candidate profiles that can satisfy our price-stability definition by identifying a unique such profile for each possible ordering of firms' prices. We then derive a simple sufficient condition for the existence of a stable price profile.

Maximal Undercut-Proof Prices. We begin by identifying undercut-proof prices.

One such profile is the trivial profile of zero prices. On the other hand, some profiles are never undercut-proof: if there are ties of positive prices then they are pairwise compared (given the "twoness" assumption) which generates an incentive to undercut.[14] We conclude that any strictly positive undercut-proof prices must be entirely distinct.

Let us now (without loss of generality) label firms in decreasing order of initial price, so that $p_1 \geq \cdots \geq p_n$, and (from the argument above) the inequality $p_i \geq p_{i+1}$ is strict if $p_i > 0$.

If firm j charges $p_j > 0$ then it wins all price comparisons that involve only firms from the j most expensive. These are the consideration sets $B \subseteq \{1, \ldots, j\}$. We need to include only those in which firm j is considered, which is achieved via the indicator variable $B_j \in \{0, 1\}$. Hence firm j earns $p_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \lambda(B)$. If firm j undercuts a cheaper firm $i > j$ then it wins all price comparisons which involve the i most expensive firms. To avoid a profitable undercut we need $p_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \lambda(B) \geq p_i \sum_{B \subseteq \{1,\ldots,i\}} B_j \lambda(B)$, or equivalently

$$p_i \leq \frac{p_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \lambda(B)}{\sum_{B \subseteq \{1,\ldots,i\}} B_j \lambda(B)}. \tag{3}$$

This must hold for every $j < i$, which gives the characterization of eq. (4) in Lemma 1 below.

If a profile of undercut-proof prices is established then the absence of undercutting opportunities means it is the only prediction of a game in which firms can only cut their prices.

[14] More generally, if a "k-ness" property holds, so that any consideration set containing $k > 1$ firms has positive mass (formally: $|B| = k \Rightarrow \lambda(B) > 0$) then at most $k - 1$ strictly positive undercut-proof prices can be tied.

Lemma 1 (Basic Properties of Undercut-Proof Price Profiles). *Without loss of generality, label firms in order of their prices from highest to lowest so that* $p_1 \geq \cdots \geq p_n$.

(i) Any profile of strictly positive undercut-proof prices is strictly ordered: $p_1 > \cdots > p_n > 0$.

(ii) A profile of prices is undercut-proof if and only if

$$p_i \leq \min_{j \in \{1,\ldots,i-1\}} \left\{ \frac{p_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B)}{\sum_{B \subseteq \{1,\ldots,i\}} B_j \, \lambda(B)} \right\} \quad \textit{for all } i \in \{2,\ldots,n\}, \tag{4}$$

(iii) For any such profile, a simultaneous-move price-cutting game in which firm i chooses $\tilde{p}_i \in [0, p_i]$ *is dominance solvable, and the unique Nash equilibrium has* $\tilde{p}_i = p_i$ *for all* i.

(For results that follow, any proof beyond the argument in the text is reported in Appendix A.)

There are many possibilities for undercut-proof prices. We pause here to ask: to which undercut-proof prices would firms collectively agree? Presumably, they would wish to raise prices as high as possible, and looking across all such profiles, would prefer those with (Pareto) superior profits from the perspective of firms. We use these definitions.

Definition (Maximal and Industry-Optimal Profiles). *The* maximal undercut-proof prices *for an ordering of firms are those that are higher than all other undercut-proof profiles that place firms' prices in the same order. An undercut-proof price profile is* industry optimal *if there is no other (Pareto) superior (in terms of firms' profits) undercut-proof profile.*

Maximal undercut-proof prices are readily identified. Strictly positive undercut-proof prices retain this peroperty if we raise the highest price to $p_1 = v$. We then iteratively raise each successively lower price so that eq. (4) binds. Each price rise is a Pareto improvement for the firms: a price goes up, and the allocation of sales to firms is preserved. Lemma 2 summarizes.

Lemma 2 (Properties of Maximal Prices). *Maximal undercut-proof prices satisfy*

$$p_1 = v, \textit{ and iteratively, } p_i = \min_{j \in \{1,\ldots,i-1\}} \left\{ \frac{p_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B)}{\sum_{B \subseteq \{1,\ldots,i\}} B_j \, \lambda(B)} \right\} \textit{ for all } i \in \{2,\ldots,n\}. \tag{5}$$

Industry-optimal profiles are a (non-empty) subset of the $n!$ *maximal profiles defined via* (5).

This lemma allows us to construct $n!$ profiles (one for each ordering of firms' prices) that are candidates for industry optimality. For some consideration set specifications (Section 4) we find a unique such profile; and in others (Sections 5 and 6) we find multiple.

Stable Prices. Stable prices are undercut-proof, by definition. If firms play a simultaneous-move price-cutting game relative to those "regular" prices then (from claim (iii) of Lemma 1) there is a unique Nash equilibrium with no price cuts. Mirroring our discussion of industry profitability, suppose that the undercut-proof prices are not maximal. This means that there is at least one that can creep upwards while preserving undercut-proofness and so maintaining the "no price cuts" equilibrium in the price-cutting game. This necessarily generates a strict gain for the creeping firm. This argument focuses our attention on maximal prices.

There are $n!$ maximal undercut-proof price profiles; one for each ordering of the firms. Fixing such an ordering, those maximal prices satisfy the binding no-undercutting constraints of eq. (5). Each constraint checks which firm $j < i$ is most tempted to undercut firm i. If one of the binding "temptation" constraints is generated by the firm immediately above, so that firm $i-1$ is (one of the) most tempted to undercut firm i, then we are able to show that such prices are resistant to upward price creep. This provides a sufficient condition for stable prices.

Proposition 1 (Necessary and Sufficient Conditions for Stable Prices).

(i) A stable profile comprises maximal undercut-proof prices. There are at most $n!$ such profiles.

(ii) Fix an ordering of firms, and consider the unique maximal undercut-proof prices. If

$$i-1 \in \arg\min_{j\in\{1,\dots,i-1\}} \left\{ \frac{p_j \sum_{B\subseteq\{1,\dots,j\}} B_j\, \lambda\,(B)}{\sum_{B\subseteq\{1,\dots,i\}} B_j\, \lambda\,(B)} \right\} \quad \textit{for all } i \in \{2,\dots,n\}, \tag{6}$$

so that firm $i-1$ is one of the most tempted amongst $\{1,\dots,i-1\}$ to undercut firm i, then that profile of maximal prices is resistant to price creep, and so is a stable price profile.

To prove this result (in Appendix A) we construct an equilibrium of a pricing subgame following a upward adjustment in initial price by some firm $i > 1$, where the expected profit of firm i matches what it earns when firms charge their initial prices. If i does deviate upwards, then a lower-indexed (and so higher priced) firm is tempted to undercut i's new (higher) price. Equation (6) says that the non-undercutting constraint binds for firm $i-1$. Our approach is to construct a mixed-strategy equilibrium in which the firms i and $i-1$ mix continuously, or "tango" (terminology from Baye, Kovenock, and de Vries, 1992), over an interval including p_i.

The condition of eq. (6) is immediately satisfied for the duopoly case of $n = 2$, given that there is only one other higher-priced firm that can be tempted to undercut a lower-priced competitor.

Corollary (Duopoly). *In a duopoly, both maximal undercut-proof price profiles are stable.*

Discussion. To satisfy our notion of stability we may focus on the maximal prices associated with the $n!$ possible ordering of firms. We note, however, three potential limitations.

Firstly, not all maximal prices are industry-optimal. The duopoly case illustrates this: placing the firm with fewer captives at the high-price position generates lower profits for both firms; they would prefer to switch positions. Nevertheless, the maximal prices from this sub-optimal firm order satisfy our stability definition. Secondly, our notion of "creep resistance" is limited to local price movements. We might ask whether a firm benefits from a larger price rise. Thirdly, our definition of stability leaves open the source of the initial prices. One possibility is for such prices to be chosen collectively. Industry optimal prices are maximal and so can be stable (if, for example, the condition of eq. (6) in Proposition 1 is satisfied). A second possibility is that prices form solely through non-cooperative actions.

To address the latter two points, our second solution concepts asks whether prices are supported by the pure-strategy play of non-cooperative price-formation games. Most parsimoniously, our two-stage pricing game allows firms to choose their initial (or "regular") prices and then (following the observation of them) offers an opportunity to lower prices, prior to purchases.

To do this, we need to consider the possibility of subgames in which a firm raises its price non-locally, including all the way to the monopoly price v. If all firms made that largest move, then we would enter a subgame in which firms play a standard simultaneous-move pricing game with an arbitrary distribution over (the 2^n distinct) consideration sets.[15] As such, we inherit here the substantial analytic complexity found in the classic approach.

The established literature does not contain a solution of the classic pricing game for a general distribution over consideration sets. Narasimhan (1988) solved the duopoly case, but only recently have we seen a full analysis of triopoly from Armstrong and Vickers (2022); the step from $n = 2$ to $n = 3$ involves substantial intricacies. For asymmetric consideration distributions they considered $n > 3$ only for some special cases such as the nested consideration of firms, but studied (in their Section 3) general symmetric consideration specifications (see also Johnen and Ronayne, 2021). For asymmetric cases, other papers have made progress by specifying that potential buyers are independently aware of each firm (Ireland, 1993; McAfee, 1994). Other developments in the literature have varied the captive-and-shopper specification (Baye, Kovenock, and de Vries, 1992).[16] Our own subgames add extra complications to the classic game owing to the fact that firms have different initial prices.

Our foundational result (Proposition 1) imposes little structure on the distribution over consideration sets. We now impose more structure to develop our theory fully. The first broad class of consideration distributions (in Section 4) allows for complete heterogeneity in the distribution of how many prices buyers compare, while assuming firms are equally and symmetrically considered in the aggregate. We also go further by presenting a generalized version allowing additionally for asymmetric captive bases.[17] The second broad class (in Section 5) allows for firms to be considered by asymmetric shares of buyers, but also assumes that buyers' considerations of different sellers are independent. For each, we apply the apparatus introduced above: we derive maximal and industry-optimal undercut-proof price profiles, and we show that eq. (6) holds so that such prices are stable. We also demonstrate situations in which those prices are supported by pure-strategy play, and identify those in which the only prices supported by such play are industry-optimal and undercut-proof. Overall, the level of generality that we reach through these two classes of buyer consideration broadly matches that seen in the classic study of simultaneous-move pricing games.

[15] We can instead make assumptions that permit fewer firms to react, which can reduce complexity, as we demonstrated in the duopoly of Section 1. As we explained there, that led to a strict loss from a creep in price.

[16] An early extension of the captive-shopper model allowed for asymmetric marginal costs (as well as asymmetric captive audiences) for duopoly (Golding and Slutsky, 2000), but only recently (in our own work, Myatt and Ronayne, 2024b) has it been extended to asymmetric marginal costs for $n > 2$.

[17] This also allows us to study the (general, asymmetric version of the) classic captive-shopper configuration.

4. Exchangeability

Here we allow differently sized consideration sets to have arbitrary masses, but all consideration sets of the same size have the same mass. We call this property *exchangeability*.

Exchangeable Consideration Sets. In addition to captive customers, $\lambda_i > 0$ for each i, $I_m \geq 0$ customers know $m \in \{2, \ldots, n\}$ prices. Such consideration is random and symmetric across firms so that the mass I_m comprises equal shares of every combination of m firms: consideration sets of the same size have equal mass. For $B \subseteq \{1, \ldots, n\}$ with $|B| \geq 2$ members,

$$\lambda(B) = I_{|B|} \Big/ \binom{n}{|B|} \tag{7}$$

Firms differ by the size of their captive audiences, but are otherwise "exchangeable". We use the term "fully exchangeable" when $\lambda_i = \lambda_j$ for all $i \neq j$.[18]

An interpretation is that non-singleton consideration sets arise from shoppers who obtain quotations via a search technology that does not bias toward any firm. On the other hand, the singleton consideration sets include some (possibly different in mass) local, loyal, or non-shopper customers who are exogenously locked in to a specific supplier. We retrieve the classic captive-shopper setting with $\lambda_S \equiv I_n$ and $I_m = 0$ for $1 < m < n$. However exposition is smoother if we abstract (for now) from zero masses of comparison shoppers and set $I_m > 0$ for all m.[19]

Maximal Undercut-Proof Prices. Setting $I_m > 0$ for all m implies that $I_2 > 0$, the "twoness" property. We know, therefore, that any profile of maximal undercut-proof prices contains n distinct prices. We label the firms so that: $p_1 > \cdots > p_n > 0$. It is convenient to denote by X_i, the mass of customers (excluding captives) buying from $i \geq 2$:

$$X_i \equiv \sum_{m=2}^{i} I_m \left[\binom{i-1}{m-1} \Big/ \binom{n}{m} \right] \quad \text{and so using this notation} \quad \sum_{B \subseteq \{1,\ldots,i\}} B_i \lambda(B) = \lambda_i + X_i. \tag{8}$$

The term X_i sums over the relevant consideration-set sizes (no sale is made if $m > i$ because then m is cheaper than i). For each m, there are $\binom{n}{m}$ equally-sized consideration sets. Firm m makes a sale only if compared to $m-1$ others from the $i-1$ competitors with higher prices. There are $\binom{i-1}{m-1}$ such sets. We define $X_1 = 0$ for completeness.

To find the maximal undercut-proof prices for this ordering of firms, we can apply Lemma 2:

$$p_1 = v \quad \text{and} \quad p_i = \min_{j<i} \left\{ p_j \frac{\lambda_j + X_j}{\lambda_j + X_i} \right\} \quad \text{for } i > 1. \tag{9}$$

Because cheaper firms have more sales ($X_j < X_i$), the term $(\lambda + X_j)/(\lambda + X_i)$ increases in λ. This means that a firm with fewer captives has a greater incentive to undercut. To keep prices high, therefore, it is helpful to place larger firms (with more captives) higher in the ladder

[18] The full exchangeability setting was analysed with a single-stage game by many (e.g. Burdett and Judd, 1983; Johnen and Ronayne, 2021; Lach and Moraga-González, 2017; Nermuth, Pasini, Pin, and Weidenholzer, 2013).

[19] Those zero-mass cases are cumbersome to carry, but our results extend naturally (continuously) to them.

of prices. This also pushes captive customers to higher prices. This suggests that industry optimality will order firms so that $\lambda_1 \geq \cdots \geq \lambda_n$: firms with more captives are more expensive.

Our proof of this claim works by contradiction. We proceed down the list of firms until we find the lowest k where $\lambda_k < \lambda_{k+1}$. We can then show that switching the positions of those two firms results in a (Pareto) superior profile for firms. Once this is established, we can also show that the binding "no undercutting" constraint for each price p_i is the one corresponding to firm $i-1$ undercutting firm i. We can then use eq. (9) to solve recursively for prices.

Proposition 2 (Industry Optimal Prices under Exchangeability). *For the exchangeability setting, and in an undercut-proof industry-optimal profile: prices are distinct, higher prices are charged by firms with more captive customers, and those prices are given by*

$$p_i = \begin{cases} v & \text{if } i = 1 \\ v \displaystyle\prod_{j=2}^{i} \frac{\lambda_{j-1} + X_{j-1}}{\lambda_{j-1} + X_j} & \text{if } i \geq 2. \end{cases} \tag{10}$$

This profile is unique if firms have differently sized captive audiences so that $\lambda_1 > \ldots > \lambda_n$.

In Appendix C we supplement this with Proposition C3, which shows that the prices identified by Proposition 2 are upper bounds for any maximal prices, and so for any stable prices.

Stable Prices. Suppose that captive masses are distinct and label firms so that $\lambda_1 > \cdots > \lambda_n$. Proposition 2 maps firms to prices, uniquely: firms with more captives are more expensive. With the strict ordering of captive audiences, firms' binding undercut-proofness constraints are always local (from the next most expensive firm, whereas non-local constraints have slack). These are the conditions that we need to apply Proposition 1.

Specifically, suppose that firm $k > 1$ creeps up to $p_k + \Delta \in (p_k, p_{k-1}]$. Just as we did when developing Proposition 1, we construct the following (asymmetric) mixed-strategy profile:

$$F_{k-1}(p) = \frac{(p - p_k)(\lambda_k + X_k)}{p(X_k - X_{k-1})} \quad \text{and} \quad F_k(p) = \frac{(p - p_k)(\lambda_{k-1} + X_k)}{p(X_k - X_{k-1})}, \tag{11}$$

which generates on-path expected profits. The firms place any remaining mass at their initial prices. Each firm $i < k-1$ has more captive customers than $k-1$ and k, and does not have a profitable deviation into the interval in which $k-1$ and k mix. This leads us to Proposition 3.

Proposition 3 (Stable Prices under Exchangeability). *Under exchangeability the (unique, if captive-audience sizes differ) undercut-proof industry-optimal profile of prices is stable.*

Pure-Strategy Play. We can say more when firms are fully exchangeable. Let $\lambda_i \equiv \lambda > 0$ for all i. Proposition 2 applies, and, because firms are symmetric, there is only one maximal profile (albeit we cannot predict which firm charges which price). We build an equilibrium of our two-stage price-formation game in which the firms charge the prices of Proposition 2 on the equilibrium path. In the second stage, we have $\tilde{p}_i = p_i$ from claim (iii) of Lemma 1.

Consider a first-stage deviation by firm $k > 1$ (there is no opportunity for an upward deviation by the highest price firm) and write $\hat{p}_k > p_k$ for that deviant initial price. This deviation means that $\hat{p}_k \in (p_{i+1}, p_i]$ for some higher-priced competitor $i < k$. In the subgame that follows there is no pure-strategy Nash equilibrium. We illustrate the strategies of a Nash equilibrium of that subgame below and place details of the proof of Lemma 3 in Appendix A.

The most straightforward case is when $\hat{p}_k \in (p_k, p_{k-1}]$. In this case, the deviation by firm k does not (strictly) upset the ordering of prices. We can construct an equilibrium in which firms k and $k-1$ continuously (and symmetrically) mix over the interval $[p_k, \hat{p}_k)$ via

$$F_k(p) = F_{k-1}(p) = \frac{(\lambda + X_k)(p - p_k)}{p(X_k - X_{k-1})}, \tag{12}$$

which (by construction) gives the firms the same expected profit as on the equilibrium path. If $\hat{p}_k = p_{k-1}$, the solution is continuous up to the common upper bound, and satisfies $F_k(p_{k-1}) = F_{k-1}(p_{k-1}) = 1$. If $\hat{p}_k < p_{k-1}$ then the firms place residual mass at their initial prices.

A more complex case is when k deviates further upward. We discuss an example here: suppose $\hat{p}_k = p_{k-2}$. To cope with this, we construct an equilibrium in which the three firms $k-2$, $k-1$, and k all mix (symmetrically) over the interval $[p_k, p_{k-1})$. Firm $k-1$ then places an atom with remaining mass at its constraining initial price p_{k-1}. Firms k and $k-2$ then begin mixing again at some price $p^\dagger \in (p_{k-1}, p_{k-2})$, and the construction continues. We can repeat this process similarly for higher deviations. Doing so, the proof of the next lemma shows that profits from the constructed Nash equilibrium match those from on-path play. (The complexity of the relevant strategies varies with the deviation, $\hat{p}_k$; the proof contains a complete treatment.)

Lemma 3 (Second-Stage Subgames: Full Exchangeability). *In the full exchangeability setting, consider the subgame following the first-stage prices reported in Proposition 2, with the exception of firm $k > 1$, which deviates to an initial price $\hat{p}_k > p_k$. There is a Nash equilibrium of that price-cutting subgame in which each firm i earns a profit of $p_i(\lambda + X_i) = v\lambda$.*

To complete a subgame-perfect equilibrium's specification, we allow any equilibrium to be played in second-stage subgames that are further off-path. We then arrive at Proposition 4.

Proposition 4 (Pure-Strategy Play under Full Exchangeability). *Under full exchangeability, the unique industry-optimal undercut-proof profile of prices (reported in Proposition 2) is the same as the unique profile of prices supported by the equilibrium play of pure strategies.*

To summarize, the unique profile of industry-optimal prices is both dispersed and stable. Under full exchangeability, those prices also emerge when firms non-cooperatively set initial prices. Does that result extend to asymmetric captive audiences? In Section 6 we respond and pin down (in a triopoly) conditions for a positive answer; but that answer can also be negative.

Models of Sales. The exchangeable specification can encompass the model of sales (Varian, 1980). However, we simplified exposition by setting $I_m > 0$ for all m. Nevertheless, we can apply our results to specifications that are arbitrarily close to the classic model.

Fix a model of sales with asymmetric captive audiences: $\lambda_1 > \cdots > \lambda_n$. We specify exchangeable models indexed by $\varepsilon > 0$ where, using obvious notation, (i) $\lambda_i^\varepsilon = \lambda_i$; (ii) $\lambda_S^\varepsilon = \lambda_S$; and (iii) $0 < I_m^\varepsilon \leq \varepsilon$ for $m \in \{2, \ldots, n-1\}$. This converges to a model of sales as $\varepsilon \to 0$.

Recall from eq. (8) that X_i is the mass of non-captive customers who buy from firm i. Inspecting that expression, notice that $\lim_{\varepsilon \to 0} X_i^\varepsilon = 0$ for all $i \in \{2, \ldots, n-1\}$, and so in the model-of-sales limit ($\varepsilon \to 0$) only firm n serves non-captive customers. Similarly, an inspection of eq. (10) from Proposition 2 shows that for industry-optimal undercut-proof prices

$$\lim_{\varepsilon \to 0} p_i^\varepsilon = v \quad \text{for} \quad i < n \quad \text{and} \quad \lim_{\varepsilon \to 0} p_n^\varepsilon = p_{n-1}^\dagger \quad \text{where} \quad p_i^\dagger \equiv \frac{v\lambda_i}{\lambda_i + \lambda_s}. \tag{13}$$

Here $p_i^\dagger$ is the lowest undominated price for firm i: by setting this price and serving all shoppers, it earns the same as it does from exploiting its captive audience at the monopoly v. Equation (13) says that the n distinct prices collapse to two points: the smallest firm (in terms of captive audience) charges a distinct lowest price, while all other firms serve only their captives.

Our key results extend (continuously) to an exact ($\varepsilon = 0$) model-of-sales specification. We omit the details here, relegating them to Appendix A where we prove the following result.

Proposition 5 (Model of Sales). *In a model of sales, there is a unique industry-optimal undercut-proof profile in which firm n, with fewest captives, sets $p_n = p_{n-1}^\dagger$ and others price at v. Each firm $i < n$ earns its "captive only" profit $v\lambda_i$. Firm n earns more: $p_{n-1}^\dagger(\lambda_n + \lambda_S) > v\lambda_n$.*

This is the unique price profile supported by pure-strategy play of our price-formation game.

Any other profile of maximal undercut-proof prices is stable.

The mixed strategies of Varian (1980) were intended to capture "sales". Realized prices may be either close or far apart, and the identity of the cheapest firm is uncertain.[20] In contrast, we predict $n-1$ "regular" prices alongside one starkly-lower "on-sale" price from the firm with fewest captive customers. Of course, over time there can be changes in which firm has the fewest, which would flip the identity of the on-sale firm. Such shifts can be more frequent if the sizes of captive audiences are relatively close, and so their order can change more easily. In that sense, we retain the spirit of sales, while providing new insights and predictions.

[20]With asymmetric captive shares, this is true for the two firms who "tango" (meaning: mix over more than one price) while $n-2$ others maintain the "regular" monopoly price (Baye, Kovenock, and de Vries, 1992, Section V). In particular, $i \in \{1, \ldots, n-2\}$ charge $p_i = v$ while n and $n-1$ mix over the interval $[p_{n-1}^\dagger, v)$ with firm $n-1$ placing an atom at v. When marginal costs and captive shares are asymmetric, analysis is substantially complicated. There, we found there is (generically) a unique Nash equilibrium in which the (cost and captive) parameters dictate how many firms mix, which can be anything between 2 and n (Myatt and Ronayne, 2024b).

5. Independent Awareness

Exchangeability allows variation in the mass of customers who consider a particular number of firms. It also allows consideration to be correlated: in the captive-shopper case, a customer who sees more than one firm sees them all. However, it does impose substantial symmetry.

Here we restrict the correlation of firms in consideration sets, but allow for much greater generality in asymmetries. We build upon prior work including Butters (1977), Grossman and Shapiro (1984), Ireland (1993), McAfee (1994), and Eaton, MacDonald, and Meriluoto (2010): each price is exposed to an independent (but asymmetric) fraction of potential customers.

Consideration Sets. On the demand side, a fraction $\alpha_i \in (0,1)$ of customers is *independently aware* of firm i.[21] The mass of customers who consider firms $B \subseteq \{1,\ldots,n\}$ is

$$\lambda(B) = \left(\prod\nolimits_{i\in B}\alpha_i\right)\left(\prod\nolimits_{i\notin B}(1-\alpha_i)\right) = \prod_{i=1}^{n} \alpha_i^{B_i}(1-\alpha_i)^{1-B_i}. \tag{14}$$

We say that firm i is larger than firm j if it enjoys greater awareness, so that $\alpha_i \geq \alpha_j$.[22]

Maximal Undercut-Proof Prices. Here, the "twoness" property holds because $\lambda(\{i,j\}) = \alpha_i\alpha_j \prod_{k\notin\{i,j\}}(1-\alpha_k) > 0$, which means that maximal undercut-proof prices are distinct. We label firms so that $p_1 > \cdots > p_n$ and apply Lemma 2 to produce Lemma 4.

Lemma 4 (Maximal Prices for Independent Awareness). *Under independent awareness, maximal undercut-proof prices are* $p_1 = v$ *and* $p_i = (1-\alpha_i)p_{i-1} = v\prod_{j=2}^{i}(1-\alpha_j)$ *for* $i > 1$. *Under these prices every firm* j *is indifferent to undercutting any other firm* $i > j$. *The profit of each firm* $i \in \{1,\ldots,n\}$ *from this price profile is* $\pi_i = v\alpha_i\prod_{j=2}^{n}(1-\alpha_j)$.

All no-undercutting constraints bind simultaneously. To see why, note that firm j receives p_j from customers aware of it, true with probability α_j, and who are aware of no cheaper firm, true with probability $\prod_{k>j}(1-\alpha_k)$, giving profit $p_j\alpha_j\prod_{k>j}(1-\alpha_k)$. If j undercuts $i > j$ then it gets p_i (or close to it) from customers who consider j and no firm cheaper than i. That gives j a profit of (or arbitrarily close to) $p_i\alpha_j\prod_{k>i}(1-\alpha_k)$. The no-undercutting constraint is then

$$p_i\alpha_j\prod_{k>i}(1-\alpha_k) \leq p_j\alpha_j\prod_{k>j}(1-\alpha_k) \quad\Leftrightarrow\quad p_i \leq p_j \prod_{k=j+1}^{i}(1-\alpha_k). \tag{15}$$

A special case is the local constraint: $p_i \leq p_{i-1}(1-\alpha_i)$, which does not involve the awareness of firm j, α_j. Note that undercutting constraints do not depend on the type of the firm considering the undercut; it is the awareness of the firm being undercut (firm i) that is relevant.[23]

[21] If two or more firms enjoy complete awareness, $\alpha_i = 1$, then the Bertrand (zero profit) outcome follows. Allowing (at most) one firm to be known to all customers does not affect our results (and is relevant to some results under endogenous advertising). But for smoother exposition, we carry $\alpha_i \in (0,1)\,\forall i$ forward in the text.
[22] The special case of symmetry ($\alpha_i = \alpha_j$ for all i,j) falls within the full exchangeability specification of Section 4.
[23] This contrasts with our exchangeability specification under which it is the type of the firm contemplating the undercut that matters: local no-undercutting constraints take the form $p_i \leq p_{i-1}(\lambda_{i-1}+X_{i-1})/(\lambda_{i-1}+X_i)$. Note that this depends on the type (λ_{i-1}) of firm $i-1$, which is the firm that contemplates the undercut.

The profit expressions reported in Lemma 4 are also of interest. Note that

$$\pi_i = v\alpha_i \prod_{j=2}^{n}(1-\alpha_j) = \frac{v\alpha_i}{1-\alpha_1}\prod_{j=1}^{n}(1-\alpha_j). \tag{16}$$

A firm's profit depends on its own awareness α_i in a natural way. The product term in the second expression ranges over all firms, and does not depend on the order of them. That order influences firm i's profit only via the denominator term $1-\alpha_1$, which depends upon the awareness of the firm at the top of the pricing ladder. The profit of firm i (and of every firm) is increasing in α_1; and indeed this is the only way in which the order of firms influences the profits obtained from maximal undercut-proof prices, giving Proposition 6.

Proposition 6 (Industry-Optimal Prices under Independent Awareness). *Under independent awareness, the industry-optimal undercut-proof profiles order firms' prices so that the largest firm charges the monopoly price, v. Other firms' prices, for any order of these firms, are given by Lemma 4. All generate the same profits for each firm.*

We can contrast this result with Proposition 2 from our exchangeability specification. There, we found (at least for strictly asymmetric firms) a unique industry-optimal undercut-proof profile of prices. Here, however, we identify $(n-1)!$ such profiles (all of which are profit equivalent).

Stable Prices and Pure-Strategy Play. For any profile identified by Proposition 6 and price p_i for $i>1$, we know the no-undercutting constraint of $i-1$ binds. Proposition 1 applies: any industry-optimal undercut-proof profile is stable.

However, it is instructive to look at equilibrium strategies following an adjustment. Suppose that firm $i>1$ allows its price to creep upward by $\Delta < p_{i-1}-p_i$. In the associated subgame, we construct an equilibrium in which firms $j \notin \{i-1, i\}$ continue with their on-path play by choosing $p_j = p_j$, while firms $i-1$ and i continuously mix over $[p_i, p_i+\Delta)$ with distributions

$$F_j(p) = \frac{1}{\alpha_j}\left(1-\frac{p_i}{p}\right) \quad \text{for } j \in \{i-1, i\}, \tag{17}$$

and place remaining mass at p_{i-1} and $p_i+\Delta$ respectively, earning their on-path profits, π_j.

This argument is enough to establish that for a small adjustment (so that Δ is not too large) we can construct an equilibrium which removes the incentive for any firm to deviate. In other words: any profile of maximal undercut-proof prices is robust to upward adjustments.

For larger deviations, this "tango" between firms $i-1$ and i can fail. To see why, suppose that firm i deviates all of the way up to the initial price of firm $i-1$. (This is an upward deviation of $\Delta = p_{i-1}-p_i$.) For our distributions to be valid, we need

$$\max_{j\in\{i-1,i\}} F_j(p_{i-1}) = \frac{1}{\min\{\alpha_{i-1},\alpha_i\}}\left(1-\frac{p_i}{p_{i-1}}\right) = \frac{\alpha_i}{\min\{\alpha_{i-1},\alpha_i\}} \leq 1 \quad \Leftrightarrow \quad \alpha_i \leq \alpha_{i-1}. \tag{18}$$

This says that the two firms must be in awareness order, with the larger firm at the higher price position. If they are out of order, so that $\alpha_i > \alpha_{i-1}$, then this construction fails.

To resolve the problem of deviations to higher prices, we bring another firm on to the "dance floor." For example, in this case, and if $i > 2$, we can construct an equilibrium in which firms $i-2$, $i-1$, and i mix. The construction is quite complex, and becomes more so for higher initial prices by the deviant firm i that move above the initial prices of other (lower indexed) firms. What is crucial for the construction is to find some more expensive firm that is larger than the deviant firm. For example, if $i = 2$ then we must have $\alpha_1 \geq \alpha_2$ if the construction of the equilibrium is to work. This holds for all possible deviations by all possible deviants if we place the largest firm at the top of the price sequence, so that $\alpha_1 \geq \alpha_i$ for all $i \in \{2, \ldots, n\}$.

More generally, if we do not place the largest firm at the highest price position then this firm will have a profitable deviation in the first stage. To see why, suppose that $\alpha_i > \alpha_1$ so that firm i is strictly larger than the highest-priced firm. Using Lemma 4, i's profit is

$$\frac{v\alpha_i \prod_{j=1}^{n}(1-\alpha_j)}{1-\alpha_1} < \frac{v\alpha_i \prod_{j=1}^{n}(1-\alpha_j)}{1-\alpha_i} = v\alpha_i \prod_{j \neq i}(1-\alpha_j). \tag{19}$$

This last expression is the profit that i achieves by setting $p_i = p_i = v$ and selling only to captive customers. From this, we conclude that we must order firms with the largest awareness at the top. If we order firms completely (from the largest to the smallest as we move down the sequence of prices) then we can construct an equilibrium with on-path payoffs in any subgame when a firm deviates upward by any amount in the first stage.

Lemma 5 (Price-Cutting Subgames under Independent Awareness). *In the independent awareness setting, consider the price-cutting subgame following maximal prices when firms are ordered by size except that firm k deviates upward. There is a Nash equilibrium of that subgame in which each firm i earns its on-path profit, $\pi_i = v\alpha_i \prod_{j=2}^{n}(1-\alpha_j)$.*

Note that this lemma asks the firms to be completely ordered by size: $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. In the next section we obtain the same result in a triopoly where $\alpha_1 \geq \alpha_3 > \alpha_2$, so that firms are not completely ordered (but nevertheless the largest firm is at the top).

We assemble our observations into the following proposition.

Proposition 7 (Stability and Pure-Strategy Play under Independent Awareness). *If a profile is supported by the equilibrium play of pure strategies then those undercut-proof prices are industry optimal, and so the most expensive firm is a largest firm: $\alpha_1 \geq \alpha_i$ for $i > 1$.*

The undercut-proof industry-optimal profile for size-order firms, $\alpha_1 \geq \cdots \geq \alpha_n$, is supported by the equilibrium play of pure strategies. Any other maximal undercut-proof profile is stable.

This leaves open the possibility that multiple industry-optimal profiles are supported by the equilibrium play of pure strategies. However, the expressions for firms' expected profits are the same for all such profiles, and also match those reported by Ireland (1993) and McAfee (1994). They considered the classic single stage of pricing and characterized its mixed-strategy Nash equilibria. A message here is that we can establish stable price dispersion (entirely distinct prices chosen as pure strategies) without impacting firms' expected profits.

6. Limitations

We do not offer unambiguous results for all consideration-set specifications. Such results (as we now show) are not always possible. Here we use a trio of triopoly models to expose the limits of our results and to illustrate the environment's complexities.

We show that (i) industry-optimal undercut-proof prices are not always supported by pure-strategy play; (ii) multiple profiles can sometimes be supported by pure-strategy play; and (iii) the order of firms' prices does not always correspond to a natural ordering of their sizes.

Pure-Strategy Play under Exchangeability. Industry-optimal prices are stable under exchangeability (Proposition 3). However, they are not always robust to larger deviations: here we find such prices that are not supported by the equilibrium play of our price-formation game.

Consider exchangeability with $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and so industry-optimal undercut-proof prices

$$p_1 = v, \quad p_2 = \frac{v\lambda_1}{\lambda_1 + X_2}, \quad \text{and} \quad p_3 = \frac{v\lambda_1}{\lambda_1 + X_2}\frac{\lambda_2 + X_2}{\lambda_2 + X_3}. \tag{20}$$

We can handle any upward deviation in the first-stage price of firm 2 (where only two firms mix). Similarly, if 3 deviates upward to $\hat{p}_3 \in (p_3, p_2]$ then, just as we did for creep in prices, we can construct the following mixed-strategy profile for firms 2 and 3:

$$F_2(p) = \frac{(\lambda_3 + X_3)(p - p_3)}{p(X_3 - X_2)} \quad \text{and} \quad F_3(p) = \frac{(\lambda_2 + X_3)(p - p_3)}{p(X_3 - X_2)}, \tag{21}$$

which generates on-path expected profits for both firms across this range. These functions are increasing from $F_2(p_3) = F_3(p_3) = 0$, and satisfy $F_3(p) > F_2(p)$ for higher prices. Furthermore,

$$F_3(p) \leq F_3(p_2) = \frac{(\lambda_2 + X_3)(p_2 - p_3)}{p_2(X_3 - X_2)} = 1, \tag{22}$$

and so we have valid distribution functions, which are completed by placing any remaining mass at firms' initial prices. Now suppose that firm 3 deviates to $\hat{p}_3 \in (p_2, p_1]$. We can build an equilibrium in which firms 2 and 3 mix over $[p_3, p_2)$ and where firm 2 places remaining mass at p_2. We have noted, however, that firm 3's distribution satisfies $F_3(p_2) = 1$. Nevertheless, the higher initial price means that firm 3 is able to price higher than p_2. Indeed, if it chooses $p > p_2$ (sacrificing the capture of the atom played by firm 2) then it will move all the way up $p = \hat{p}_3$. It earns this price on λ_3 captive customers and X_2 customers with consideration set $\{1, 3\}$, and so earns an expected profit of $\hat{p}_3(\lambda_3 + X_2)$. For our construction to work, this must be less than $p_3(\lambda_3 + X_3)$. The deviant first-stage price can be as high as $p_1 = v$, and so we need

$$v(\lambda_3 + X_2) \leq p_3(\lambda_3 + X_3) \quad \Leftrightarrow \quad \frac{\lambda_3 + X_2}{\lambda_3 + X_3} \leq \frac{\lambda_1}{\lambda_1 + X_2}\frac{\lambda_2 + X_2}{\lambda_2 + X_3}. \tag{23}$$

This holds if X_2 is sufficiently small. If X_2 approaches zero then this exchangeable triopoly becomes a model of sales: if we are close to a captive-shopper model then industry-optimal prices are supported by pure-strategy play. However, eq. (23) fails if λ_2 and λ_3 are close. (It strictly fails if $\lambda_2 = \lambda_3$.) This means that we can find circumstances (e.g., $\lambda_1 > \lambda_2 \approx \lambda_3$) in which the industry-optimal profile is not supported by the equilibrium play of pure strategies.

Our discussion relies upon a "tango" danced by firms 2 and 3. For our fully exchangeable specification, we dealt with higher initial price deviations by constructing equilibria in which higher priced firms joined in. Here, this cannot work: firm 1 (with the largest captive audience) is strictly unwilling to mix down to p_3. The proof of the result that follows confirms that if eq. (23) fails then we cannot construct a suitable equilibrium of the subgame; we construct another equilibrium (of the subgame) which is strictly better for the deviant.

Proposition 8 (An Exchangeable Triopoly). *In an exchangeable triopoly, the industry-optimal undercut-proof prices are supported by equilibrium pure-strategy play if and only if*

$$\frac{\lambda_3 + X_2}{\lambda_3 + X_3} \leq \frac{\lambda_1}{\lambda_1 + X_2}\frac{\lambda_2 + X_2}{\lambda_2 + X_3}. \tag{24}$$

This holds if the mass of customers who conduct pairwise comparisons is sufficiently small, but it fails if the masses of captive customers for the second and third firms are sufficiently similar.

Multiple Equilibria under Independent Awareness. For independent awareness (Section 5), industry-optimal prices place the largest (most widely known) firm at the highest price ($p_1 = v$). This does not pin down the other firms: there are $(n-1)!$ orderings with the same profits. The price profile in which firms are ordered by size (firms with more awareness charge more) is supported by the equilibrium play of pure strategies (Proposition 7). We left open the possibility that other industry-optimal profiles are supported. We take up that issue here.

Order three firms with parameters $\alpha_i \in (0,1)$: $\alpha_1 > \alpha_3 > \alpha_2$ so that $p_1 > p_2 > p_3$. This means that the second and third firms are not in size order. The associated prices are

$$p_1 = v, \;\; p_2 = v(1-\alpha_2), \;\text{ and } p_3 = v(1-\alpha_2)(1-\alpha_3). \tag{25}$$

Now consider our two-stage pricing game and an upward deviation by firm 3. Our previous approach was to build an equilibrium in the subgame in which firms 2 and 3 mix according to

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{p_3}{p}\right) \quad \text{for } j \in \{2,3\}. \tag{26}$$

The problem with this is that for p sufficiently high, $F_2(p)$ is not a valid distribution:

$$F_2(p_2) = \frac{1}{\alpha_2}\left(1 - \frac{p_3}{p_2}\right) = \frac{\alpha_3}{\alpha_2} > 1. \tag{27}$$

This means that if firm 3 deviates upward (even while remaining below p_2) then, in the subgame, we need all three firms to mix. In the proof Proposition 9 we build an equilibrium in which all three firms mix up until a specific price at which firm 1 places all remaining mass at its initial price while the remaining two firms continue to "tango". The construction is conceptually straightforward, but nevertheless detailed and delicate, and underpins our next result.

Proposition 9 (Pure-Strategy Play of an Awareness Triopoly). *In a triopoly under (strictly asymmetric) independent awareness, there are two industry-optimal undercut-proof price profiles, both of which assign the highest price to the largest firm, but which differ by the order of the other two firms. Both are supported by the equilibrium play of pure strategies.*

A Prominence Setting. Here we find two industry-optimal profiles with different profits. Only one is supported by pure-strategy play, and it does not place firms in "size" order.

Firm $i = 1$ is "prominent" and so is known to all customers. Customers see at most one of firms $i \in \{2, 3\}$, but never see all three. Summarizing, the three positive-mass consideration sets are $\{1\}$, $\{1, 2\}$, and $\{1, 3\}$. We use the following notation:

$$\phi_1 = \lambda(\{1\}),\ \phi_2 = \lambda(\{1, 2\}),\ \text{and } \phi_3 = \lambda(\{1, 3\}). \tag{28}$$

An interpretation is that the prominent firm is a national sales channel, whereas other firms are local suppliers. Each local firm $i \in \{2, 3\}$ has access customers who see i's price. Additionally, all such customers are informed of firm 1's price.[24] We let $\phi_2 \geq \phi_3$.

Only a customer with consideration set $\{1\}$ is truly captive: $\lambda_1 = \phi_1 > 0$ but $\lambda_2 = \lambda_3 = 0$. This does not fit the regularity condition (in Section 2) which says that all firms have captive customers. Similarly, "twoness" fails: there are no pairwise comparisons of firms 2 and 3.

If prices are undercut-proof and strictly positive, then they must place the prominent firm at the top.[25] (If the prominent firm charges strictly less than a local firm, then that local firm would undercut the prominent firm.) If prices are maximal, then of course $p_1 = v$.

Turning to no-undercutting constraints, we need only to check that the prominent firm does not wish to undercut the local firms. If local firms are ordered so that $p_2 \geq p_3$, then the relevant constraints are $v\phi_1 \geq p_2(\phi_1 + \phi_2)$ and $v\phi_1 \geq p_3(\phi_1 + \phi_2 + \phi_3)$. For prices to be maximal, these bind. Let $j > 1$ set the lowest price, p_j. It is undercut-proof if $p_j \leq v\phi_1/(\phi_1 + \phi_2 + \phi_3)$. For efficiency this also must bind, and so the lowest price is independent of which firm sets it. Notice also that if both local firms set this price, then either could raise it slightly without provoking an undercut by firm 1, so such prices are not maximal: firms set distinct prices.

Let $i \neq j$ be the local firm that sets the higher price. Firm 1 does not undercut p_i if $p_i \leq v\phi_1/(\phi_1 + \phi_i)$. For efficiency this must also bind: p_i depends on how many customers consider i's price, unlike p_j. Given $p_1 = v$, each local supplier prefers that they charge p_i and the other charges p_j, than vice versa. Thus, there are two industry-optimal undercut-proof profiles: one in which $i = 2$ and $j = 3$, and one in which $i = 3$ and $j = 2$ (these coincide if $\phi_2 = \phi_3$

We proceed to investigate unilateral deviations in our price-formation game. It is straightforward to construct an equilibrium in the subgame following a deviation upward by a non-prominent firm that preserve the order of firms' initial prices: the deviator and the prominent firm mix in the interval up to the deviant initial price, while the other firm maintains its initial price; the deviator's profit is unchanged.

[24] Inderst (2002) considered a related single-stage model, but did not fully characterize equilibrium; we characterize an equilibrium for one of his cases in Appendix B. Armstrong and Vickers (2022, Section 4) solved the single-stage game in a closely related setting, interpreting firms as a chain store with local rivals. In their setting any comparison involving a local firm involves the chain store, but local rivals also have captive audiences.

[25] The claim holds for the ordered-search model of Arbatskaya (2007) and in the search-and-prominence duopoly model of Moraga-González, Sándor, and Wildenbeest (2021). In contrast, Armstrong, Vickers, and Zhou (2009) used a sequential-search model to predict that a prominent firm offers the lowest price.

It remains to consider a deviation by the cheapest firm j to an initial price $\hat{p}_j \in (p_i, p_1]$. In Appendix C we show that when $\phi_2 > \phi_3$ and $j = 3$, such a deviation leads to a subgame in which any Nash equilibrium gives firm 3 strictly greater profit. The reason is that the larger non-prominent firm 2 charges a low intermediate price to prevent the prominent firm undercutting it ($p_2 = v\phi_1/(\phi_1 + \phi_2)$). This leaves an interval of prices, $(p_2, v\phi_1/(\phi_1 + \phi_3))$, which are dominated for the prominent firm (by v), and so are safe for firm 3 to deviate to and yield it a profit strictly greater than $p_3\phi_3$. (If $j = 2$, then a deviation of this sort is unavailable.)

Proposition 10 (Stable Prices in a Prominence Setting). *In the prominence triopoly, there are two industry-optimal undercut-proof profiles, described by:*

$$p_1 = v, \quad p_i = \frac{v\phi_1}{\phi_1 + \phi_i}, \quad \textit{and} \quad p_j = \frac{v\phi_1}{\phi_1 + \phi_2 + \phi_3} \quad \textit{for} \quad i, j \in \{2, 3\} \quad \textit{and} \quad i \neq j, \tag{29}$$

Each firm $k \in \{1, 2, 3\}$ makes a profit equal to $p_k\phi_k$. Both of these profiles are stable. There is a unique profile supported by the equilibrium play of pure strategies, in which the larger non-prominent firm is the cheapest, i.e., $i = 3$ and $j = 2$.

With n symmetrically-sized firms ($\phi_i = \phi$ for all $i \in \{1, \ldots, n\}$), there is a unique undercut-proof profile of prices in which a firm's price declines inversely to its position in the sequence:

$$p_i = \frac{v}{i} \quad \textit{for all} \quad i \in \{1, \ldots, n\}. \tag{30}$$

This profile is supported by the equilibrium play of pure strategies.

A (non-prominent) firm with a larger audience is cheaper. It remains the case that the profit of a non-prominent firm is increasing in its own size. However, the larger non-prominent firm can make a smaller profit than the other. This is true whenever their sizes are sufficiently close.

As the prominent firm's position strengthens (greater ϕ_1) price cuts hurt it more and so its rivals can set higher prices without being undercut. This implies non-prominent firms' prices and profits are increasing in ϕ_1 and that customers are worse off with a larger prominent firm.[26]

7. Stable Prices as Components of Deeper Models

Consideration sets themselves might respond to the actions by firms (such as advertising choices) and customers (such as price discovery). Naturally, a full study is beyond the scope of (at least the main body of) this paper. However, here we use duopoly analyses to illustrate the likely impact of firms' and customers' actions on the consideration-set environment.

Duopoly, Revisited. The duopoly specification of Section 1 fits within the frameworks (exchangeability and independent awareness) considered in Sections 4 and 5. Recall that λ_i customers are captive to firm $i \in \{1, 2\}$ and $\lambda_S \equiv \lambda(\{1, 2\})$ customers compare both prices. For

[26] In Appendix B we develop this model of prominence by adding an earlier stage to consider the incentives of a "prominence provider" which brings one of many local firms to national prominence. We find that this provider makes a prominence offer (which is accepted) to the local firm with the largest local customer base, which is the worst choice for customers because it amplifies the asymmetry between firms.

$\lambda_1 > \lambda_2$ the industry-optimal undercut-proof prices and corresponding profits are

$$p_1 = v \quad \text{and} \quad p_2 = \frac{v\lambda_1}{\lambda_1 + \lambda_S} \quad \Longrightarrow \quad \pi_1 = v\lambda_1 \quad \text{and} \quad \pi_2 = \frac{v\lambda_1(\lambda_2 + \lambda_S)}{\lambda_1 + \lambda_S}. \tag{31}$$

The profits match those from single-stage pricing models. Notice that (the larger, in terms of awareness) firm 1 cares solely about expanding its captive audience. The incentives of (the smaller) firm 2 are nuanced. For example, firm 2 benefits from an expansion in firm 1's captives.

Advertising. We now build upon the independent awareness model of Section 5, where the awareness α_i of a firm is a consequence of its advertising activities. This maps to the general duopoly model via $\lambda_1 = \alpha_1(1 - \alpha_2)$, $\lambda_2 = \alpha_2(1 - \alpha_1)$, and finally $\lambda_S = \alpha_1\alpha_2$. For $\alpha_1 > \alpha_2$,

$$\pi_1 = v\alpha_1(1 - \alpha_2) \quad \text{and} \quad \pi_2 = v\alpha_2(1 - \alpha_2). \tag{32}$$

We see that the two firms have very different incentives. The larger and so more expensive firm sets a price of $p_1 = v$ that is not limited by any "no undercutting" constraint. For a given $\alpha_2 < 1$, its profits are linearly increasing in its advertising intensity.

The smaller and cheaper firm, however, sets $p_2 = v(1 - \alpha_2)$, preventing an undercut by firm 1. The more it advertises, the more attractive such an undercut becomes, and so the lower its price must be to keep firm 1 at bay. This leads to a trade off for firm 2 when choosing how much to advertise, reflected by the non-monotonicity of π_2 in α_2. In particular (and putting aside advertising costs, for now) firm 2 (if it smaller) will always prefer $\alpha_2 \leq \frac{1}{2}$.

We can readily embed the profits of (32) into an advertising-choice game. (More fully, we can imagine a three-stage game in which firms: (i) choose awareness parameters; (ii) form their regular prices; and, finally, (iii) can offer price cuts.) For example, if advertising is free and awareness is chosen from $\alpha_i \in [0, \bar{\alpha}]$ for some $\bar{\alpha} \in \left(\frac{1}{2}, 1\right)$, then a pure-strategy Nash equilibrium of an advertising game will take the form $\alpha_1 = \bar{\alpha}$ and $\alpha_2 = \frac{1}{2}$. The associated (stable) prices

$$p_1 = v \quad \text{and} \quad p_2 = \frac{v\lambda_1}{\lambda_1 + \lambda_S} = v(1 - \alpha_2) = \frac{v}{2} \tag{33}$$

are dispersed. One firm maximizes its exposure to customers and sets the monopoly price, while the other limits its exposure to half of customers and charges half the monopoly price.

In Appendix B we provide a full treatment with n firms and asymmetric advertising cost functions. One firm charges v and advertises distinctly more than all the others, who each advertise to a minority of customers and set mutually distinct lower prices. When advertising is costless, adding extra competitors adds additional lower prices (while retaining existing price positions) and increases the range of dispersed prices. With costly adverting, a fall in costs increases the awareness of each firm and the dispersed prices of the firms become further apart.

A conclusion here is that our pricing approach matches established predictions for advertising (Ireland, 1993; McAfee, 1994) which use conventional pricing games, owing to the fact firms' expected profits are the same. (Our actual pricing predictions differ, of course.)

Search. Retaining the duopoly framework, suppose that a potential customer uses fixed-sample search technology à la Burdett and Judd (1983): this customer seeks (without replacement) either zero, one, or two price quotations. Searching once finds each firm with equal probability. If the quotation is from the high-price firm v then there is no benefit; but if the low-price firm is found (probability $\frac{1}{2}$) then the customer gains $v - p_2$. A second search is guaranteed to find the cheaper firm, but this is beneficial only if the first search did not already do so. As such, the second search also generates a gain of $v - p_2$ with probability $\frac{1}{2}$. Summarizing,

$$\text{E[benefit of 1st search]} = \text{E[benefit of 2nd search]} = \frac{v - p_2}{2} = \frac{v\lambda_S}{2(\lambda_1 + \lambda_S)}. \tag{34}$$

Now adopt the classic constant-returns search technology so that gathering each quotation costs $\kappa \in \left(0, \frac{v}{2}\right)$. A customer finds it strictly optimal to obtain two quotations if and only if

$$\kappa < \frac{v\lambda_S}{2(\lambda_1 + \lambda_S)} \quad \Leftrightarrow \quad \lambda_S > \frac{2\kappa\lambda_1}{v - 2\kappa}, \tag{35}$$

will not search at all if the opposite strict inequality holds, and will be indifferent between all search strategies if there is an equality. This inequality reveals a strategic complementarity: if many others seek out both price quotations (so that the mass of shoppers λ_S is large) then there is greater price dispersion, and this increases the incentive of a customer to search.

This strategic complementarity suggests there may be multiple equilibria with endogenous search.[27] To sketch a model of this, let us suppose that a mass $\bar{\lambda}_i$ of customers are exogenously captive to firm i, a mass $\bar{\lambda}_S$ are exogenously shoppers, and mass μ decide whether to search once, twice, or not at all. Writing μ_L and μ_H for the masses of buyers searching once and twice (we choose these subscripts to avoid confusion with firm labels) we have $\mu_L + \mu_H \leq \mu$, and

$$\lambda_i = \bar{\lambda}_i + \mu_L \quad \text{for } i \in \{1, 2\} \text{ and } \quad \lambda_S = \bar{\lambda}_S + \mu_H. \tag{36}$$

There are multiple equilibria if the various parameters here satisfy

$$\bar{\lambda}_S + \mu > \frac{2\kappa\bar{\lambda}_1}{v - 2\kappa} > \bar{\lambda}_S. \tag{37}$$

We can construct a "high search" equilibrium (of a fully defined game, with an appropriate solution concept) in which all endogenous searchers seek out two quotations ($\mu_L = \mu$) and become shoppers. In this equilibrium prices are more dispersed:

$$p_1 = v \quad \text{and} \quad p_2 = \frac{v\bar{\lambda}_1}{\bar{\lambda}_1 + \bar{\lambda}_S + \mu} \tag{38}$$

In a second "low search" equilibrium endogenous searchers stay home ($\mu_L = \mu_H = 0$). In this equilibrium the two price points are closer together and total search is limited to $\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_S$.[28] In a model with a conventional pricing game (Burdett and Judd, 1983) the "high search" equilibrium has incomplete search, and so our approach is consequential for search behavior.

[27] We provide a full treatment in related work (Myatt and Ronayne, 2024c); see Appendix B for some discussion.

[28] There can be an interior equilibrium satisfying $\bar{\lambda}_S + \mu_H = \kappa(2\bar{\lambda}_1 + \mu_L)/(v - 2\kappa)$. This is unstable in the sense of Fershtman and Fishman (1992): shifting extra customers to search twice (and letting firms' prices adjust) the benefit of a second search increases and so all customers wish to search twice. In contrast, a single-stage duopoly analysis yields an equilibrium with search with (using present notation) $\mu_L + \mu_H = 1$ and $\mu_L, \mu_H > 0$.

8. Concluding Discussion

The economic interest in price competition with heterogeneous consideration arises in many areas, including strategic clearing-houses such as comparison websites (Baye and Morgan, 2001, 2009; Moraga-González and Wildenbeest, 2012; Ronayne, 2021; Shelegia and Wilson, 2021), more general platform models (Bergemann and Bonatti, 2024; Hagiu and Wright, 2024), price discrimination (Armstrong and Vickers, 2019; Fabra and Reguant, 2020), product substitutability (Inderst, 2002), consumer search (Stahl, 1989), and boundedly-rational consumers (Carlin, 2009; Chioveanu and Zhou, 2013; Heidhues, Johnen, and Kőszegi, 2021; Inderst and Obradovits, 2020; Piccione and Spiegler, 2012). Our work shows that predictions of prices that are stable and dispersed can be recovered within this environment.[29]

Our construction of stable dispersed prices has (if all firms can use price cuts) the feature that a cheaper firm is indifferent between maintaining an undercut-proof price and raising it. The equality of expected profits suggests a concern: a firm is indifferent to creeping its price up and so its choice is a weak, rather than strict, best reply. Nevertheless, additional assumptions readily make equilibria strict in the sense that such an upward deviation strictly hurts.

One such approach is to consider the asymmetric ability of firms to respond with price cuts. Suppose, for example, that a deviant firm that nudges upward its price is unable to adjust its price again; and yet its competitors are then free to implement price cuts. In our duopoly analysis (of Section 1) we found that makes a creep upward in price strictly harmful.[30]

Another natural case is when a decision-maker is averse to risk. In the pure-strategy play of our price-formation games, a firm's profit is fully determined when the equilibrium path is followed. A first-stage upward deviation leads to the same expected profit, but makes the eventual realization of profit uncertain. Suppose now that the initial pricing decision is made by a risk-averse manager (who maximizes the expected utility of profit), but that the final price is set by a risk-neutral operational pricing agent (who maximizes expected profit).[31] The manager now has a strict incentive not to increase the firm's initial price away from an industry-optimal undercut-proof profile and trigger a mixed-strategy equilibrium in the ensuing subgame.

[29] Other (very different) environments can generate dispersed pure-strategy prices. Reinganum (1979) offered a version of Diamond (1971) in which firms have different costs and set different monopoly prices. Anderson and De Palma (2005) studied customers who (exogenously) consider firms in a random order, but without any meaningful price comparison. Arnold (2000) studied capacity-constrained firms and single-search customers who see prices but not whether a firm is stocked out. Firms trade off price for the (endogenous) number of buyers that buy from them. For csome valuations, there is an equilibrium in which firms choose different prices.

[30] Relatedly, the first stage of our price-formation game gives each firm the opportunity to take a de facto Stackelberg leadership position. In Appendix B we study a sequential-move captive-and-shopper game. There, and as in Deneckere, Kovenock, and Lee (1992), the firm most keen to set low prices (with the fewest captive customers) acts as a Stackelberg leader, setting a price just low enough to ensure that no other firm undercuts it in the second stage. The same effect occurs in a fully specified multi-stage game in which all firms are given one full commitment opportunity to reach shoppers (Myatt and Ronayne, 2024b, Section 3).

[31] We cover this in brief in Appendix B and in more detail elsewhere (Myatt and Ronayne, 2024b, Appendix C).

We conclude by comparing our results and predictions to those from conventional approaches. We identify three advantages. Firstly, we predict that disperse prices can be stable rather than randomized. Secondly, we provide a clear process to solve for those stable prices. Thirdly, those prices are expressed via relatively simple closed-form analytic solutions.

Turning to predictions, we can identify (in some, but not all, circumstances) consequences for firms and aspects of their behavior that coincide with the conventional approach. For buyers, however, we identify distinctly different implications for their incentives to search.

For firms, and for the major settings that we covered, expected profits match those earned in the equilibrium of a single-stage game.[32] This means that researchers analyzing settings with a single-stage model in a subgame (for example, the platform analysis by Hagiu and Wright (2024) or the endogenous advertising in our Section 7) can supplement or replace it with our approach.[33] The profit equivalence (in many cases) means there is no disruption to earlier stages (at least with risk-neutral players) and so we do not expect substantial changes in firm-related actions such as advertising. An illustrative exception is the case of the prominence triopoly in Section 6. Firms' profits from the pure-strategy play of our price-formation game strictly exceed, for one firm, the expected profit earned from an equilibrium of a single-stage game.[34]

From the perspective of buyers, however, things are markedly different. Consider the duopoly search model (of Section 7) so that "search" effectively refers to buyers retrieving both firms' prices. Stable prices diverge as buyers search more, increasing the payoff from doing so, revealing strategic complementarities to search. In contrast, single-stage mixed-strategy pricing (Burdett and Judd, 1983, with a finite number of firms) predicts that firms use prices drawn randomly from the same distribution. Crucially, there is strategic substitutability as search becomes sufficiently strong: the incentive for an individual buyer to search falls if search amongst others is higher. As search becomes complete (so that almost all buyers obtain two price quotations) the firms' (symmetric) mixing distribution collapses to marginal cost. This, of course, makes a second search redundant; and so this rules out an equilibrium in which search is complete.

We pursue a full analysis of stable prices in the presence of such costly buyer search elsewhere (Myatt and Ronayne, 2024c). We identify novel effects: search (in a stable equilibrium equilibrium with positive search) is higher than predicted by the conventional approach; an increase in the number of firms lowers the intensity of search; and such entry to the industry raises (rather than lowers) aggregate profit in the industry. The duopoly sketches we provided in this paper serve to demonstrate how stable prices can generate novel applied insights.

[32]In the captive-shopper setting, the profits from Proposition 5 match those of Baye, Kovenock, and de Vries (1992); for full exchangeability, profits from Proposition 2 match those of Johnen and Ronayne (2021); for independent awareness, the profits from Proposition 6 match those of Ireland (1993) and McAfee (1994).

[33]The analysis of Hagiu and Wright (2024) includes a fee-setting plaform, requiring them to consider a captive-and-shopper subgame with asymmetric marginal costs, and for that they use results from our analysis of asymmetric models of sales (Myatt and Ronayne, 2024b). Our theory of stable price dispersion extends readily to a captive-and-shopper model of sales with asymmetric marginal costs.

[34]For completeness, we characterize that equilibrium in Appendix B.

Appendix A. Omitted Proofs

The proofs of Lemma 2 and Propositions 3, 6 and 7 follow arguments in the main text. The proofs of Propositions 8 to 10 are contained in our supplemental Appendix C.

Proof of Lemma 1. Claims (i) and (ii) follow from the main text. (See also Appendix C.)

For claim (iii) it is without loss to focus on strictly positive prices. For undercut-proof prices $p_1 > \cdots > p_n > 0$ write $\pi_i = p_i \sum_{B\subseteq\{1,\dots,i\}} B_i\,\lambda(B)$ for the profit of firm i.

By charging $\tilde{p}_1 = p_1$, firm 1 achieves the profit π_1, independent of the choices of other firms. For any other price it may charge, $\tilde{p}_1 < p_1$, its profit is highest if all others maintain their initial prices. The profile of initial prices is undercut-proof, and firm 1 earns strictly less than π_1 by strictly undercutting any other firm. If it matches another firm, then (given that ties are broken in an interior way) it also earns strictly less. Therefore, all $\tilde{p}_1 < p_1$ are strictly dominated for firm 1. We conclude (as an induction basis) that firm 1 must charge $\tilde{p}_1 = p_1$.

For $i > 1$, suppose that $\tilde{p}_j = p_j$ for all $j < i$. Firm i can guarantee a profit π_i by charging p_i. Recycling the argument above, even if others maintain their initial prices (so maximizing the profit of firm i) then firm i earns strictly less from $\tilde{p}_i < p_i$. We conclude that $\tilde{p}_i = p_i$. By the principle of induction, this holds for all $i \in \{1,\dots,n\}$. □

The following lemma is used in proofs that follow, including that of Proposition 1.

Lemma A1. *Consider a strategy profile in a pricing game in which firms i and j mix (continuously) over an interval $[p_L, p_H]$ not intersecting the support of any other firm, and where i and j are both indifferent (as they are in mixed-strategy Nash equilibrium) across that interval. The expected profit of any other firm from deviating to a price $p \in [p_L, p_H]$ is convex in p.*

Proof. We note that $F_l(p)$ is constant for $p \in [p_L, p_H]$ and $l \notin \{i, j\}$; we write F_l for this constant. Varying the prices of i and j within the interval $[p_L, p_H]$ has no effect on their sales when there is no comparison between them. We write Y_i and Y_j for such sales:

$$Y_i = \sum_{B\subseteq\{1,\dots,n\}} \lambda(B)B_i(1-B_j)\prod_{l\notin\{i,j\}}(1-B_lF_l) \tag{A1}$$

$$Y_j = \sum_{B\subseteq\{1,\dots,n\}} \lambda(B)B_j(1-B_i)\prod_{l\notin\{i,j\}}(1-B_lF_l) \tag{A2}$$

We also write Z for the sales made by the cheaper of i and j when they are compared:

$$Z = \sum_{B\subseteq\{1,\dots,n\}} \lambda(B)B_iB_j\prod_{l\notin\{i,j\}}(1-B_lF_l). \tag{A3}$$

With this notation in hand, the firms' expected profits from any price $p \in [p_L, p_H]$ are

$$\pi_i(p) = p\left(Y_i + Z(1-F_j(p))\right) \quad \text{and} \quad \pi_j(p) = p\left(Y_j + Z(1-F_i(p))\right). \tag{A4}$$

These profits are constant across this interval and so

$$1 - F_j(p) = \frac{\pi_i - pY_i}{pZ} \quad \text{and} \quad 1 - F_i(p) = \frac{\pi_j - pY_j}{pZ}. \tag{A5}$$

Now consider the profit of some firm $k \notin \{i, j\}$ deviating to a price in this interval. We write Y_k for the sales made when there is no comparison between k and either (or both) of i and j:

$$Y_k = \sum_{B \subseteq \{1,\ldots,n\}} \lambda(B) B_k (1 - B_i)(1 - B_j) \prod_{l \notin \{i,j,k\}} (1 - B_l F_l), \tag{A6}$$

where these expected sales are guaranteed for any price in $[p_L, p_H]$. Other possible sales involve comparisons of k with i, with j, or with both i and j. Possible sales for these three cases are

$$Z_{ik} = \sum_{B \subseteq \{1,\ldots,n\}} \lambda(B) B_k B_i (1 - B_j) \prod_{l \notin \{i,j,k\}} (1 - B_l F_l), \tag{A7}$$

$$Z_{jk} = \sum_{B \subseteq \{1,\ldots,n\}} \lambda(B) B_k (1 - B_i) B_j \prod_{l \notin \{i,j,k\}} (1 - B_l F_l), \tag{A8}$$

$$Z_{ijk} = \sum_{B \subseteq \{1,\ldots,n\}} \lambda(B) B_k (1 - B_i)(1 - B_j) \prod_{l \notin \{i,j,k\}} (1 - B_l F_l). \tag{A9}$$

The expected profit of firm k from charging price $p \in [p_L, p_H]$ is

$$\begin{aligned}
\pi_k(p) &= p\left[Y_k + Z_{ik}(1 - F_i(p)) + Z_{jk}(1 - F_j(p)) + Z_{ijk}(1 - F_i(p))(1 - F_j(p))\right] \\
&= p\left[Y_k + Z_{ik}\frac{\pi_j - pY_j}{pZ} + Z_{jk}\frac{\pi_i - pY_i}{pZ} + Z_{ijk}\frac{\pi_j - pY_j}{pZ}\frac{\pi_i - pY_i}{pZ}\right] \\
&= pY_k + Z_{ik}\frac{\pi_j - pY_j}{Z} + Z_{jk}\frac{\pi_i - pY_i}{Z} + \frac{Z_{ijk}}{Z^2}\left[\frac{\pi_i \pi_j}{p} + Y_i Y_j p - (\pi_i Y_j + \pi_j Y_i)\right],
\end{aligned} \tag{A10}$$

which by inspection is convex in p, and strictly so if $Z_{ijk} > 0$. □

Proof of Proposition 1. Begin with the first claim, which identifies a necessary condition for stable prices. Suppose that the no-undercutting constraint of firm i is slack, so that either $i = 1$ and $p_1 < v$ or $i > 1$ and equation (4) holds as a strict inequality. Firm i can strictly raise p_i while maintaining undercut-proofness, and enter a price-cutting game with (by claim (iii) of Lemma 1) a unique Nash equilibrium which gives firm i a strictly higher expected profit.

For the sufficient condition, fix the candidate price profile. Given (6), this satisfies $p_1 = v$ and

$$p_i = p_{i-1} \frac{\sum_{B \subseteq \{1,\ldots,i-1\}} B_{i-1} \lambda(B)}{\sum_{B \subseteq \{1,\ldots,i\}} B_{i-1} \lambda(B)} \quad \text{for all } i \in \{2, \ldots, n\}. \tag{A11}$$

These prices are undercut-proof, and so $\tilde{p}_i = p_i$ for all i is the unique Nash outcome of a price-cutting game, by claim (iii) of Lemma 1. Firm i earns a profit $p_i \sum_{B \subseteq \{1,\ldots,i\}} B_i \lambda(B)$.

Consider a creep upward in price by $i > 1$ of $\Delta > 0$ sufficiently small such that $p_i + \Delta \leq p_{i-1}$, and that each firm $j < i - 1$ that strictly prefers p_j to undercutting p_i also strictly prefers p_j to undercutting $p_i + \Delta$. In the price-cutting game construct a strategy profile in which $j \notin \{i - 1, i\}$ choose $\tilde{p}_j = p_j$, while $j \in \{i - 1, i\}$ mix over $[p_i, p_i + \Delta)$ with distributions

$$F_j(p) = \frac{(p - p_i) \sum_{B \subseteq \{1,\ldots,i\}} B_k \lambda(B)}{p \sum_{B \subseteq \{1,\ldots,i\}} B_i B_{i-1} \lambda(B)} \quad \text{for } j, k \in \{i - 1, i\}, j \neq k, \tag{A12}$$

and then place remaining mass at p_{i-1} and $p_i + \Delta$ respectively. These are valid CDFs which continuously increase from $F_j(p_i) = 0$ and satisfy $F_j(p_i + \Delta) \leq 1$ for $j \in \{i-1, i\}$ if Δ is sufficiently small. Moreover, prices within this interval give the firms $j \in \{i-1, i\}$ their on-path expected profits. To see why, note that for $j \in \{i-1, i\}$ and $k \in \{i-1, i\}$ for $k \neq j$,

$$\underbrace{(p - p_i) \sum_{B \subseteq \{1,\ldots,i\}} B_j \, \lambda(B)}_{\text{gain from lifting price}} = \underbrace{pF_k(p) \sum_{B \subseteq \{1,\ldots,i\}} B_i B_{i-1} \, \lambda(B)}_{\text{lost sales to } k \neq j}. \tag{A13}$$

The left-hand side is the gain to j from charging a price higher than p_i. (The summation represents sales from being the cheapest of $\{1, \ldots, i\}$.) The right-hand side is then the value of sales lost to the competitor k, which incorporates the probability that k prices below p.

The condition (6) says that $i - 1$ is a firm that is indifferent to undercutting i. We chose Δ such that any firm that strictly prefers not to undercut p_i also strictly prefers not to undercut $p_i + \Delta$ and so prefers not to join the "tango" between $i - 1$ and i. It remains to check that any firm $j < i - 1$ that is indifferent to undercutting p_i is unwilling to join the dance (i.e., set some $p \in [p_i, p_i + \Delta)$). By Lemma A1 we only need to check that j does at least as well with p_j than both (i) p_i, and (ii) (undercutting) $p_i + \Delta$. As for (i), we know j is indifferent between p_j and p_i. For (ii), recall that Δ is sufficiently small such that $F_j(p_i + \Delta) \leq 1$ for $j \in \{i-1, i\}$. In fact, using (A12) we find $F_i^{-1}(1) = p_{i-1}$ (i.e., the function F_i reaches 1 at exactly p_{i-1}). Therefore, j gets a strictly lower profit from undercutting p_{i-1} than charging p_j (i has no mass at p_{i-1}, so no matter the mass $i - 1$ places there, j would not undercut p_{i-1} because the initial price profile is undercut-proof). It follows that undercutting $p_i + \Delta$ (which is $\leq p_{i-1}$) gets j an even lower expected profit than if it were to undercut p_{i-1}, and so j prefers p_j. □

Proof of Proposition 2. Fix a profile of maximal undercut-proof prices. We find the first $k \in \{1, \ldots, n-1\}$ such that $\lambda_k < \lambda_{k+1}$. We claim that for all $i \in \{2, \ldots, k+1\}$

$$p_i = p_{i-1} \frac{\lambda_{i-1} + X_{i-1}}{\lambda_{i-1} + X_i} \quad \text{and so} \quad p_i \equiv v \prod_{j=2}^{i} \frac{\lambda_{j-1} + X_{j-1}}{\lambda_{j-1} + X_j}. \tag{A14}$$

The first equality says that the local no-undercutting constraint binds at each step; setting $p_1 = v$ and repeated substitution gives the second equality. To prove this, we note that this must be true for $i = 2$ (forming an induction basis) because there is only one no-undercutting constraint that applies, and it must bind because prices are maximal. Now suppose that the claim holds (as an induction hypothesis) for all $j \in \{2, \ldots, i-1\}$. For firm i,

$$\begin{aligned}
p_i &= \min_{j<i} \left\{ p_j \frac{\lambda_j + X_j}{\lambda_j + X_i} \right\} = \min_{j<i} \left\{ p_j \frac{\lambda_j + X_j}{\lambda_j + X_i} \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \right) \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k} \right) \right\} \\
&= v \prod_{j=2}^{i} \frac{\lambda_{j-1} + X_{j-1}}{\lambda_{j-1} + X_j} \min_{j<i} \left\{ \frac{\lambda_j + X_j}{\lambda_j + X_i} \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \right) \right\} \\
&= v \prod_{j=2}^{i} \frac{\lambda_{j-1} + X_{j-1}}{\lambda_{j-1} + X_j} \left\{ 1, \min_{j<i-1} \left\{ \frac{\lambda_j + X_j}{\lambda_j + X_i} \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \right) \right\} \right\}
\end{aligned}$$

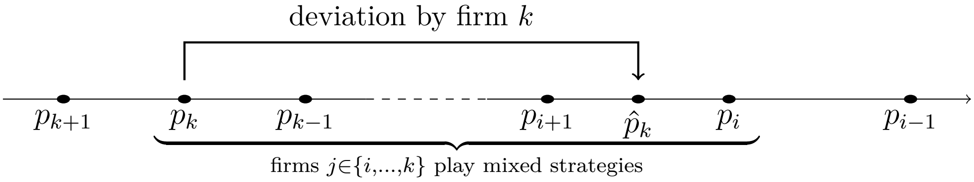

Figure 1. An Upward Deviation in Initial Price

$$= v \prod_{j=2}^{i} \frac{\lambda_{j-1} + X_{j-1}}{\lambda_{j-1} + X_j} = p_{i-1} \frac{\lambda_{i-1} + X_{i-1}}{\lambda_{i-1} + X_i}. \tag{A15}$$

The first four lines use algebraic re-arrangement. The final line holds because for each $j < i-1$,

$$\frac{\lambda_j + X_j}{\lambda_j + X_i} \prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \leq \frac{\lambda_j + X_j}{\lambda_j + X_i} \prod_{k=j+1}^{i} \frac{\lambda_j + X_k}{\lambda_j + X_{k-1}} = \frac{\lambda_j + X_j}{\lambda_j + X_i} \frac{\lambda_j + X_i}{\lambda_j + X_j} = 1. \tag{A16}$$

The inequality in the chain holds because $X_k \geq X_{k-1}$ in each of the ratio terms, which means that such terms are each decreasing in λ_{k-1}. An upper bound for each term is obtained by replacing λ_{k-1} with $\lambda_j \leq \lambda_{k-1}$, where this inequality holds because $j \leq k-1 \leq i-1$ and (by assumption) firms below i are in size order. The claim holds by the principle of induction.

Now consider firms k and $k+1$; the first out-of-order pair.

$$\frac{p_{k+1}}{p_k} = \frac{\lambda_k + X_k}{\lambda_k + X_{k+1}} < \frac{\lambda_{k+1} + X_k}{\lambda_{k+1} + X_{k+1}}. \tag{A17}$$

The equality is the (binding) no undercutting constraint. The inequality holds because $X_k < X_{k+1}$ and $\lambda_{k+1} > \lambda_k$. This means that if we switch the positions of k and $k+1$ in the profile of prices (while maintaining the actual prices) then the no-undercutting holds strictly. Firm k is now lower in the price order than before, but with the same profit, and so (as before) does not want to undercut any lower-priced firms. Firms $j < k$ face the same no-undercutting opportunities as before, and so their no-undercutting constraints still hold. Firm $k+1$ is now strictly better off. We have constructed a Pareto-superior undercut-proof profile. □

Proof of Lemma 3. Suppose that firm k (where necessarily $k > 1$) deviates upward to $\hat{p}_k > p_k$. There is some $i < k$ such that $p_{i+1} < \hat{p}_k \leq p_i$. For example, one case is where $i = k-1$, so that firm k deviates upward without crossing the initial price of another firm. Another case is when $i = 1$ and $\hat{p}_k = p_1 = v$, which means that k removes any restriction on its final price. We build a mixed-strategy equilibrium (illustrated in Figure 1) in which all firms earn their (common) on-path equilibrium expected profits, $v\lambda$. Firms $j \in \{i, \ldots, k\}$ mix (with atoms and gaps) over the interval $[p_k, p_i]$. Others set their initial prices: $p_j = p_j$ for $j \notin \{i, \ldots, k\}$.

Given that firms $\{1, \ldots, k\}$ price (by construction below) at or above p_k, any firm $l \in \{k+1, \ldots, n\}$ has no profitable deviation downward, and is constrained upward. Firms $j \in \{i, \ldots, k\}$ will (again by construction below) earn their on-path equilibrium profits, and this implies that firms $\{1, \ldots, i-1\}$ cannot profitably deviate to within $[p_k, p_i]$. This is because an upper bound to the expected profit a lower-indexed firm can achieve by doing so is that from “throwing some

$j \in \{i \ldots, k\}$ off the dance floor" and charging one of the prices j used to. Given the symmetry of firms, this gives the deviator the same expected profit j had before their ejection, $v\lambda$.

We now build the strategies used by the actively mixing firms $\{i, \ldots, k\}$. This group consists of the deviant firm k and all lower-indexed firms up to the firm i with the lowest initial price that weakly exceeds the deviant's new initial price. We consider three cases.

Case (i): a deviation that does not cross another first-stage price.

If $i = k-1$, so that $\hat{p}_k \in (p_k, p_{k-1}]$, then firms k and $k-1$ mix continuously over the single interval of prices $[p_k, \hat{p}_k)$, and then place atoms (these are strictly positive if and only if $\hat{p}_k < p_{k-1}$) at their respective initial prices $\hat{p}_k$ and p_{k-1}. They mix using the same distribution $F(p)$. Taking the indifference condition for firm k (the same condition holds for firm $k-1$), $F(p)$ satisfies

$$\begin{aligned} \lambda v &= p \sum_{B \subseteq \{1,\ldots,k-2\}} [\lambda(B \cup \{k\}) + (1 - F(p))\lambda(B \cup \{k, k-1\})] \\ &= p \sum_{x=0}^{k-2} \binom{k-2}{x} \left[\frac{I_{1+x}}{\binom{n}{1+x}} + \frac{[1 - F(p)] I_{2+x}}{\binom{n}{2+x}} \right], \end{aligned} \tag{A18}$$

where we define $I_1 \equiv n\lambda$.[35] The left-hand side is the expected profit of firm k. The right-hand side is the price p multiplied by the probability that firm k wins any comparisons. Firm k wins from comparisons which group it with any subset of $\{1, \ldots, k-2\}$ (these are the higher priced firms). Additionally, it wins comparisons that also include $k-1$ so long as $k-1$ prices above p, which happens with probability $1 - F(p)$. The second line computes the sizes of the relevant comparison sets. The summation over x ranges over the possible sizes of $B \subseteq \{1, \ldots, k-2\}$, noting that for each x there are $\binom{k-2}{x}$ relevant sets. Bringing in firm k, these comparison sets are of size $1+x$. The total mass of comparison sets of this size is I_{1+x}, and there are $\binom{n}{1+x}$ such sets. Hence $I_{1+x}/\binom{n}{1+x}$ is the size of each comparison set. Similar calculations apply when firm $k-1$ is added, where this time the combined mass of the relevant comparison sets is multiplied by $1 - F(p)$. The solution for $F(p)$ is strictly increasing in p, $F(p_k) = 0$, and $F(p_{k-1}) = 1$.

Case (ii): a deviation into the upper part of a higher price interval.

A second case is when the deviation of firm k crosses the initial price of at least one other firm, so that $i < k-1$ or equivalently $\hat{p}_k > p_{k-1}$, and when that deviant price is sufficiently high in $(p_{i+1}, p_i]$. We consider $\hat{p}_k \in [p_i^\diamond, p_i]$ where $p_i^\diamond \in (p_{i+1}, p_i)$ is a threshold to be determined below.

We build an equilibrium mixed-strategy profile in which there is a threshold $p_j^\diamond \in (p_{j+1}, p_j)$ for each $j \in \{i, \ldots, k-2\}$ such that the interval $(p_{j+1}, p_j^\diamond)$, which is the lower part of the interval between the initial prices of firms $j+1$ and j, is a gap in the mixing distributions of all firms. (This gap must exist because, for any price in that interval, a firm would prefer to undercut the price p_{j+1} in order to capture an atom which will be played by firm $j+1$.) For $j > i$, over the upper part of the interval $[p_j^\diamond, p_j)$ firms in $\{i, \ldots, j\} \cup \{k\}$ will mix continuously. Firm j will then place an atom at p_j. Turning to the top interval between the prices of $i+1$ and i, firms i

[35] Note that captive customers are included in the first line, with $\lambda(\emptyset \cup \{k\})$.

and k will mix continuously over $[p_i^\diamond, \hat{p}_k)$ and then will place remaining mass at their respective initial prices. Across the lowest interval $[p_k, p_{k-1})$ all firms mix continuously, with firm $k-1$ placing an atom at its initial price p_{k-1}. For $k=4$ and $i=1$ the basic plan of the equilibrium support of the firms' mixed strategies is illustrated in Figure 2.

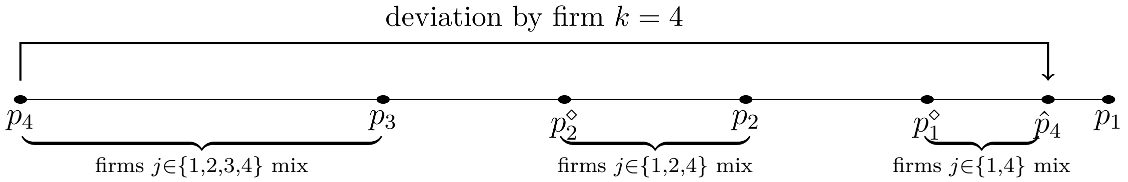

Figure 2. Mixing Supports for an Equilibrium of Type Case (ii)

For each $j \in \{i, \ldots, k-1\}$ (these are firms tempted to undercut following k's deviant first-stage choice), consider the interval of prices $[p_{j+1}, p_j)$. Firms $\{i, \ldots, j\} \cup \{k\}$ will actively used mixed strategies within this interval, where this is a strict subset for $j < k-1$. Note that there are $j-(i-1)+1$ such firms. Specify the cumulative distribution function, $F_j(p)$, to satisfy

$$\begin{aligned} \lambda v &= p \sum_{B \subseteq \{1,\ldots,i-1\}} \sum_{\widetilde{B} \subseteq \{i,\ldots,j\}} \lambda\left(B \cup \widetilde{B} \cup \{k\}\right) [1 - F_j(p)]^{|\widetilde{B}|} \\ &= p \sum_{x=0}^{i-1} \sum_{y=0}^{j-i+1} \binom{i-1}{x} \binom{j-i+1}{y} \frac{I_{1+x+y}}{\binom{n}{1+x+y}} [1 - F_j(p)]^y \end{aligned} \tag{A19}$$

The left-hand side is the (common) equilibrium expected profit of each firm. The right-hand side is k's expected profit when, at price p, all firms in $\{i, \ldots, j\}$ mix according to $F_j(p)$. The first summation collects together subsets of lower-indexed firms who always lose any comparisons with price p. The second summation deals with those who actively mix. For any set $\widetilde{B}$ there are $|\widetilde{B}|$ such firms, and so the price p wins comparisons against them all with probability $[1 - F_j(p)]^{|\widetilde{B}|}$. The second line follows from the various masses of consideration sets. This is an indifference condition for firm k. The same condition also holds for other firms in $\{i, \ldots, j\}$. The solution for $F_j(p)$ satisfies $F_j(p_{j+1}) = 0$ and is strictly increasing. Defining $F_j(p_j) = \lim_{p \uparrow p_j} F_j(p)$, the $k-i$ solutions satisfy $F_{k-1}(p_{k-1}) < F_{k-2}(p_{k-2}) < \cdots < F_i(p_i) = 1$.

Looking across the whole interval $[p_k, p_i)$, we might aim to join the $k-1$ functions to form a single distribution. However, such a function would jump downward at each initial price (to zero), and so would not be a valid distribution function. We "smooth out" these jumps as follows. For each $j \in \{i, \ldots, k-2\}$ we define $p_j^\diamond \in (p_{j+1}, p_j)$ to be the unique solution to $F_{j+1}(p_{j+1}) = F_j(p_j^\diamond)$. We now stitch together a full cumulative distribution as follows. First, we define $F(p) = F_{k-1}(p)$ for $p \in [p_k, p_{k-1}]$. For all other $j \in \{i, \ldots, k-2\}$ we define

$$F(p) = \begin{cases} F_{j+1}(p_{j+1}) & p \in (p_{j+1}, p_j^\diamond] \\ F_j(p) & p \in (p_j^\diamond, p_j] \end{cases} \tag{A20}$$

This distribution function continuously increases from $F(p_k) = 0$ to $F(p_i) = 1$. It is constant for each interval $[p_{j+1}, p_j^\diamond]$ for each $j \in \{i, \ldots, k-2\}$, but otherwise is strictly increasing.

We are finally ready to build our strategy profile. Firm k (the deviant) mixes according to $F(p)$ across $p \in [p_k, \hat{p}_k)$ and places any remaining mass (if $\hat{p}_k < p_i$) at its first-stage price, and so plays an atom of size $1 - F(\hat{p}_k)$ at $\hat{p}_k$. Firm i also mixes according to $F(p)$ for $p \in [p_k, \hat{p}_k)$ and then places its remaining mass $1 - F(\hat{p}_k)$ at p_i. (This means that firms i and k behave symmetrically save for the location of their atoms.) A firm $j \in \{i+1, \ldots, k-1\}$ mixes according to $F(p)$ across $p \in [p_k, p_j)$ and then places its remaining mass $1 - F(p_j)$ at p_j. This construction yields a mixed-strategy Nash equilibrium profile so long as the deviant initial price satisfies $\hat{p}_k \geq p_i^{\diamond}$.

We note that the constructed distribution function $F(p)$ is used by all firms below their respective initial prices. At any point in the support of a firm's strategy (so that $F(p)$ is strictly increasing) the function is constructed so that each mixing firm earns the on-path equilibrium expected profit, $v\lambda$. Any price within a gap (where $F(p)$ is constant) generates an expected profit strictly below $v\lambda$. (At such prices a firm performs strictly better by undercutting the next initial price below and so capturing the atom of another firm.)

The strategy profile constructed requires k to place an atom at its deviant price $\hat{p}_k$. If $\hat{p}_k \in (p_{i+1}, p_i^{\diamond})$, however, the deviant price lies strictly within an interval across which $F(p)$ is constant and so generates an expected profit strictly below $v\lambda$. We adapt to cover that case next.

Case (iii): a deviation into the lower part of a higher price interval.

We now consider $\hat{p}_k \in (p_{i+1}, p_i^{\diamond})$. What we do here is to construct an equilibrium in which firms follow the previous strategy profile up to some critical price $p^\star$, at which point firm i ceases to participate (in essence, this firm "leaves the dance floor") and places remaining mass at its initial price. Specifically, we define $p^\star$ to be the lowest price which satisfies $F(p^\star) = F_i(\hat{p}_k)$. Necessarily this critical price satisfies $p^\star < p_{i+1}$. We retain our definition of $F(p)$ for $p \leq p^\star$.

We now change firm i's strategy so that it mixes according to $F(p)$ for $p \in [p_k, p^\star]$ but then places remaining mass at its initial price, so that it has an atom at p_i of size $1 - F(p^\star) = 1 - F_i(\hat{p}_k)$.

This construction means that firm k earns its on-path equilibrium expected profit, $v\lambda$, from playing the price $\hat{p}_k$. For $p > p^\star$ firm i no longer actively mixes, and so we modify the behavior of other firms to maintain appropriate indifferences for each $j \in \{i+1, k-1\}$, and prices in the interval $[p_{j+1}, p_j)$ that are at or above $p^\star$ we specify $F_j^\star(p)$ to satisfy

$$\begin{aligned}\lambda v &= p \sum_{B \subseteq \{1,\ldots,i-1\}} \sum_{\widetilde{B} \subseteq \{i+1,\ldots,j\}} \left[1 - F_j^\star(p)\right]^{|\widetilde{B}|} \left[\lambda\left(B \cup \widetilde{B} \cup \{k\}\right) + (1 - F(p^\star))\lambda\left(B \cup \widetilde{B} \cup \{i,k\}\right)\right] \\ &= p \sum_{x=0}^{i-1} \sum_{y=0}^{j-i} \binom{i-1}{x} \binom{j-i}{y} [1 - F_j^\star(p)]^y \left[\frac{I_{1+x+y}}{\binom{n}{1+x+y}} + \frac{I_{2+x+y}[1 - F(p^\star)]}{\binom{n}{2+x+y}}\right]. \end{aligned} \tag{A21}$$

This is an indifference condition for k, but also applies to other relevant firms. It adjusts (A19) to treat i separately, as i prices above p with (constant) probability $1 - F(p^\star)$ for $p \in (p^\star, p_i)$. The solution satisfies $F_j^\star(p) > F_j(p)$ for $p > p^\star$ ($F_j^\star(p) = F_j(p)$ for $p = p^\star$). To proceed, we replace $F_j(p)$ with $F_j^\star(p)$ for $p > p^\star$. We then redefine $F(p)$ and the thresholds $p_j^{\diamond}$ appropriately. This modification ensures k is indifferent between $\hat{p}_k$ and slightly undercutting p_{i+1}. □

Proof of Proposition 4. Because of the symmetry of the full-exchangeability setting there is only one maximal undercut-proof profile, and so one candidate equilibrium prediction. That profile coincides with the profile given in Proposition 2. From that profile, there are no profitable downward first-stage (or any second-stage, by claim (iii) of Lemma 1) deviations. For subgames following any single-firm upward deviation, we apply Lemma 3. □

Proof of Proposition 5. This proposition is concerned with an exact (rather than approximate) captive-and-shopper model of sales: $I_m = 0$ for $m \in \{2, \ldots, n-1\}$. Pairwise consideration sets are empty and so the "twoness" property does not hold. This means that claim (i) of Lemma 1 does not apply: strictly positive undercut-proof prices are not necessarily distinct.[36]

Consider what must be true of any maximal undercut-proof prices. The lowest price cannot be zero: such a firm (even if there is more than one) could raise its price locally without violating no-undercutting constraints.[37] Given that the lowest price is strictly positive it must be charged by only a single firm (else a profitable undercut would be available). Firms who are not the cheapest, and so sell only to their captives, can raise their prices to the maximum, v, while maintaining undercut-proofness. We conclude that there must be only two price points in any maximal undercut-proof profile: a lowest (strictly positive) price and v.

Consider, then, price profiles in which one firm i sets $p_i < v$ while other firms set $p_j = v$ for $j \neq i$. All such firms j earn $v\lambda_j$, and so to dissuade undercutting we need $p_i \leq p_j^\dagger$, where $p_j^\dagger$ is from eq. (13). Thus, the maximal profile when i is cheapest must satisfy

$$p_i = \min_{j \neq i} p_j^\dagger = \begin{cases} p_{n-1}^\dagger & \text{if } i = n, \text{ or} \\ p_n^\dagger & \text{if } i \in \{1, \ldots, n-1\}. \end{cases} \tag{A22}$$

We have found n maximal price profiles, which vary according to the identity of the shopper-capturing cheapest firm. All firms $j \neq i$ earn $v\lambda_j$. If $i < n$, firm i earns strictly less than $v\lambda_i$; but if $i = n$ then firm i earns strictly more. From this we conclude that $i = n$ (the firm with fewest captives is cheapest) pins down the unique industry-optimal undercut-proof profile. Firm n commits to the lowest price such that it wins all the shoppers without a fight.

For the next statement (concerning pure-strategy play) consider the two-stage pricing game. Build a strategy profile in which firms choose the industry-optimal undercut-proof prices as initial prices, and maintain those prices on the equilibrium path in the second stage. By construction, no firm $i < n$ can profitably deviate in the first stage, and so the only candidate for a profitable deviation is for firm n to deviate upwards at the first stage.

A deviation by firm n to $\hat{p}_n \in (p_{n-1}^\dagger, v]$ leads to a subgame in which there is no pure-strategy Nash equilibrium. The following claim reports the profits in a mixed-strategy equilibrium.

[36] We noted earlier that if a "k-ness" property holds, so that all consideration sets comprising $k > 1$ firms have positive mass, then there can be at most $k-1$ tied prices. A model of sales has this property only for $k = n$, which leaves open the possibility of $n-1$ tied prices. Ultimately, this is what we predict. Claim (ii) of Lemma 1 does not hold as stated. However, this is simply because we need to adjust our notation to deal with cases of tied prices. Finally, claim (iii) of Lemma 1 continues to hold more generally even without the twoness property.

[37] Any higher-priced firm earns strictly positive profits, and would earn less by pricing close enough to zero.

Claim. *Consider the price-cutting game following $\hat{p}_n > p^\dagger_{n-1}$ and $p_i = v$ for $i < n$. There is a unique Nash equilibrium in which firm n earns $p^\dagger_{n-1}(\lambda_n + \lambda_S)$ and each firm i earns profit $v\lambda_i$.*

This result is covered by Proposition 7 of Myatt and Ronayne (2024b). In the current paper, marginal costs are symmetric and captive shares are strictly asymmetric, which implies the lowest dominated price of each firm, as defined in (23), is distinct, i.e., $p^\dagger_i \neq p^\dagger_j$ for $i \neq j$. As shown in Myatt and Ronayne (2024b, Appendix C), this removes the instances that can give multiple equilibria, and leaves us with a unique Nash equilibrium.

For example, if firm n deviates to a first-stage price $\hat{p}_n \in (p^\dagger_{n-1}, p^\dagger_{n-2})$, then in the subgame's Nash equilibrium each $i < n-1$ sets its first-stage price equal to v, while firms $n-1$ and n mix continuously (or "tango") over the interval $[p^\dagger_{n-1}, \hat{p}_n)$ with distribution functions

$$F_{n-1}(p) = \frac{(p - p^\dagger_{n-1})(\lambda_S + \lambda_n)}{p\lambda_S} \quad \text{and} \quad F_n(p) = \frac{(p - p^\dagger_{n-1})(\lambda_S + \lambda_{n-1})}{p\lambda_S}, \tag{A23}$$

with n and $n-1$ placing remaining mass at $\hat{p}_n$ and v, respectively. The profit earned by firm n is the same as reported in the statement of the proposition, making the upward deviation in its initial price non-profitable. Finally, within (off-path) subgames following any other choices of first-stage prices, any equilibrium may be played.[38] In summary, we have constructed a subgame perfect equilibrium that supports the on-path play of industry-optimal prices.

Turning to the final statement in the proposition, consider any other profile of maximal undercut-proof prices, i.e., one in which some firm $i \in \{1, \ldots, n-1\}$ chooses $p_i = p^\dagger_n$ while each $j \neq i$ chooses $p_j = v$. Suppose that firm i deviates upward at the first stage to an initial price $p^\dagger_n + \Delta < p^\dagger_{n-1}$. Over the interval $[p^\dagger_n, p^\dagger_n + \Delta)$, let firms i and n mix according to

$$F_i(p) = \frac{(p - p^\dagger_n)(\lambda_S + \lambda_n)}{p\lambda_S} \quad \text{and} \quad F_n(p) = \frac{(p - p^\dagger_n)(\lambda_S + \lambda_i)}{p\lambda_S}, \tag{A24}$$

and place remaining mass at their initial prices. The distributions are continuously increasing from $F_i(p^\dagger_n) = F_n(p^\dagger_n) = 0$, and satisfy $F_i(p) < F_n(p) \leq 1$ if p is not too large (guaranteed by Δ sufficiently small). By construction, firms i and n earn their original expected profits. We conclude that this price profile is creep resistant, and so forms a stable price profile. □

Proof of Lemma 4. We know that $p_1 = v$, from Lemma 2. The text following the lemma confirms $p_i \leq p_{i-1}(1 - \alpha_i)$ must hold for every $i \in \{2, \ldots, n\}$. Choosing maximal prices so that these constraints all bind generates the solutions stated in the lemma. More generally, there is a no-undercutting constraint $p_j \leq \prod_{i \geq k > j}(1 - \alpha_k)$ for every $i < j$, as derived in the text, and these constraints are all satisfied by the stated solutions for maximal undercut-proof prices. □

Proof of Lemma 5. The proof follows a similar structure to the proof of Lemma 3. Owing also to its length, it is relegated to our online supplemental Appendix C. □

[38]Again, an equilibrium exists there because Theorem 5 of Dasgupta and Maskin (1986, p.14) applies.

References

Ahrens, S., I. Pirschel, and D. J. Snower (2017): "A Theory of Price Adjustment under Loss Aversion," *Journal of Economic Behavior and Organization*, 134(C), 78–95.

Alford, B. L., and B. T. Engelland (2000): "Advertised Reference Price Effects on Consumer Price Estimates, Value Perception, and Search Intention," *Journal of Business Research*, 48(2), 93–100.

Anderson, S. P., A. Baik, and N. Larson (2023): "Price Discrimination in the Information Age: Prices, Poaching, and Privacy with Personalized Targeted Discounts," *Review of Economic Studies*, 90(5), 2085–2115.

Anderson, S. P., and A. De Palma (2005): "Price Dispersion and Consumer Reservation Prices," *Journal of Economics and Management Strategy*, 14(1), 61–91.

Anderson, S. P., N. Erkal, and D. Piccinin (2020): "Aggregative Games and Oligopoly Theory: Short-Run and Long-Run Analysis," *RAND Journal of Economics*, 51(2), 470–495.

Arbatskaya, M. (2007): "Ordered Search," *RAND Journal of Economics*, 38(1), 119–126.

Armstrong, M., and J. Vickers (2019): "Discriminating Against Captive Customers," *American Economic Review: Insights*, 1(3), 257–72.

——— (2022): "Patterns of Competitive Interaction," *Econometrica*, 90(1), 153–191.

Armstrong, M., J. Vickers, and J. Zhou (2009): "Prominence and Consumer Search," *RAND Journal of Economics*, 40(2), 209–233.

Armstrong, M., and J. Zhou (2011): "Paying for Prominence," *Economic Journal*, 121(556), 368–395.

Arnold, M. A. (2000): "Costly search, capacity constraints, and Bertrand equilibrium price dispersion," *International Economic Review*, 41(1), 117–132.

Baye, M. R., D. Kovenock, and C. G. de Vries (1992): "It Takes Two to Tango: Equilibria in a Model of Sales," *Games and Economic Behavior*, 4(4), 493–510.

Baye, M. R., and J. Morgan (2001): "Information Gatekeepers on the Internet and the Competitiveness of Homogeneous Product Markets," *American Economic Review*, 91(3), 454–474.

——— (2009): "Brand and Price Advertising in Online Markets," *Management Science*, 55(7), 1139–1151.

Bergemann, D., and A. Bonatti (2024): "Data, competition, and digital platforms," *American Economic Review*, 114(8), 2553–2595.

Bolton, L. E., L. Warlop, and J. W. Alba (2003): "Consumer Perceptions of Price (Un)Fairness," *Journal of Consumer Research*, 29(4), 474–491.

Burdett, K., and K. L. Judd (1983): "Equilibrium Price Dispersion," *Econometrica*, 51(4), 955–969.

Butters, G. R. (1977): "Equilibrium Distributions of Sales and Advertising Prices," *Review of Economic Studies*, 44(3), 465–491.

Campbell, M. C. (1999): "Perceptions of Price Unfairness: Antecedents and Consequences," *Journal of Marketing Research*, 36(2), 187–199.

——— (2007): ""Says Who?!" how the Source of Price Information and Affect Influence Perceived Price (Un)Fairness," *Journal of Marketing Research*, 44(2), 261–271.

Carlin, B. I. (2009): "Strategic Price Complexity in Retail Financial Markets," *Journal of Financial Economics*, 91(3), 278–287.

Chandra, A., and M. Tappata (2011): "Consumer Search and Dynamic Price Dispersion: An Application to Gasoline Markets," *RAND Journal of Economics*, 42(4), 681–704.

Chen, Y., and C. He (2011): "Paid Placement: Advertising and Search on the Internet," *The Economic Journal*, 121(556), 309–328.

Chioveanu, I. (2008): "Advertising, Brand Loyalty and Pricing," *Games and Economic Behavior*, 64(1), 68–80.

Chioveanu, I., and J. Zhou (2013): "Price Competition with Consumer Confusion," *Management Science*, 59(11), 2450–2469.

Dasgupta, P., and E. Maskin (1986): "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *Review of Economic Studies*, 53(1), 1–26.

Deneckere, R., D. Kovenock, and R. Lee (1992): "A Model of Price Leadership Based on Consumer Loyalty," *Journal of Industrial Economics*, 30(2), 147–156.

Diamond, P. A. (1971): "A model of price adjustment," *Journal of Economic Theory*, 3(2), 156–168.

Eaton, B. C., I. A. MacDonald, and L. Meriluoto (2010): "Existence Advertising, Price Competition and Asymmetric Market Structure," *B.E. Journal of Theoretical Economics*, 10(1), Article 38.

ECB (2005): "Digital Comparison Tools Market Study: Final Report," *Working Paper No. 535*, Report retrieved from `https://www.ecb.europa.eu/pub/pdf/scpwps/ecbwp535.pdf?3fe960921919feb5484df89c7475211f`, December 5, 2019.

——— (2019): "Price-Setting Behaviour: Insights from a Survey of Large Firms," Report retrieved from `https://www.ecb.europa.eu/pub/economic-bulletin/focus/2019/html/ecb.ebbox201907_05~99afe2b4fe.en.html`, December 5, 2019.

Eliaz, K., and R. Spiegler (2011): "Consideration Sets and Competitive Marketing," *Review of Economic Studies*, 78(1), 235–262.

Fabra, N., and M. Reguant (2020): "A Model of Search with Price Discrimination," *European Economic Review*, 129, 103571.

Fershtman, C., and A. Fishman (1992): "Price Cycles and Booms: Dynamic Search Equilibrium," *American Economic Review*, 82(5), 1221–1233.

Galenianos, M., R. L. Pacula, and N. Persico (2012): "A Search-Theoretic Model of the Retail Market for Illicit Drugs," *Review of Economic Studies*, 79(3), 1239–1269.

Gill, D., and J. Thanassoulis (2009): "The Impact of Bargaining on Markets with Price Takers: Too Many Bargainers Spoil the Broth," *European Economic Review*, 53(6), 658–674.

——— (2016): "Competition in Posted Prices with Stochastic Discounts," *Economic Journal*, 126(594), 1528–1570.

Golding, E. L., and S. Slutsky (2000): "Equilibrium Price Distributions in an Asymmetric Duopoly," in *Advances in Applied Microeconomics*, vol. 8, pp. 139–159.

Gorodnichenko, Y., V. Sheremirov, and O. Talavera (2018): "Price Setting in Online Markets: Does IT click?," *Journal of the European Economic Association*, 16(6), 1764–1811.

Grewal, D., K. B. Monroe, and R. Krishnan (1998): "The Effects of Price-Comparison Advertising on Buyers' Perceptions of Acquisition Value, Transaction Value, and Behavioral Intentions," *Journal of Marketing*, 62(2), 46–59.

Grossman, G. M., and C. Shapiro (1984): "Informative Advertising with Differentiated Products," *Review of Economic Studies*, 51(1), 63–81.

Hagiu, A., and J. Wright (2024): "Optimal Discoverability on Platforms," *Management Science*.

Heidhues, P., J. Johnen, and B. Kőszegi (2021): "Browsing Versus Studying: A Pro-Market Case for Regulation," *Review of Economic Studies*, 88(2), 708–729.

Hong, H., and M. Shum (2010): "Using Price Distributions to Estimate Search Costs," *RAND Journal of Economics*, 37(2), 257–275.

Inderst, R. (2002): "Why Competition May Drive Up Prices," *Journal of Economic Behavior and Organization*, 47(4), 451–462.

Inderst, R., and M. Obradovits (2020): "Loss Leading with Salient Thinkers," *The RAND Journal of Economics*, 51(1), 260–278.

Ireland, N. J. (1993): "The Provision of Information in a Bertrand Oligopoly," *Journal of Industrial Economics*, 41(1), 61–76.

Janssen, M. C. W., and J. L. Moraga-González (2004): "Strategic Pricing, Consumer Search and the Number of Firms," *Review of Economic Studies*, 71(4), 1089–1118.

Johnen, J., and D. Ronayne (2021): "The Only Dance in Town: Unique Equilibrium in a Generalized Model of Sales," *Journal of Industrial Economics*, 69(3), 595–614.

Kahneman, D., J. L. Knetsch, and R. H. Thaler (1986): "Fairness as a Constraint on Profit Seeking: Entitlements in the Market," *American Economic Review*, 76(4), 728–741.

Kahneman, D., and A. Tversky (1979): "Prospect Theory: An Analysis of Decisions under Risk," *Econometrica*, 47(2), 263–291.

Kan, C., D. R. Lichtenstein, S. J. Grant, and C. Janiszewski (2013): "Strengthening the Influence of Advertised Reference Prices through Information Priming," *Journal of Consumer Research*, 40(6), 1078–1096.

Kaplan, G., and G. Menzio (2015): "The Morphology of Price Dispersion," *International Economic Review*, 56(4), 1165–1206.

Kaplan, G., G. Menzio, L. Rudanko, and N. Trachter (2019): "Relative Price Dispersion: Evidence and Theory," *American Economic Journal: Microeconomics*, 11(3), 68–124.

Lach, S. (2002): "Existence and Persistence of Price Dispersion: An Empirical Analysis," *Review of Economics and Statistics*, 84(3), 433–444.

Lach, S., and J. L. Moraga-González (2017): "Asymmetric Price Effects of Competition," *Journal of Industrial Economics*, 65(4), 767–803.

Lichtenstein, D. R., S. Burton, and E. J. Karson (1991): "The Effect of Semantic Cues on Consumer Perceptions of Reference Price Ads," *Journal of Consumer Research*, 18(3), 380–391.

Manzini, P., and M. Mariotti (2014): "Stochastic Choice and Consideration Sets," *Econometrica*, 82(3), 1153–1176.

Maskin, E., and J. Tirole (1988a): "A Theory of Dynamic Oligopoly I: Overview and Quantity Competition with Large Fixed Costs," *Econometrica*, 56(3), 549–569.

——— (1988b): "A Theory of Dynamic Oligopoly II: Price Competition, Kinked Demand Curves, and Edgeworth Cycles," *Econometrica*, 56(3), 571–599.

McAfee, R. P. (1994): "Endogenous Availability, Cartels, and Merger in an Equilibrium Price Dispersion," *Journal of Economic Theory*, 62(1), 24–47.

Moen, E. R., F. Wulfsberg, and Ø. Aas (2020): "Price Dispersion and the Role of Stores," *Scandinavian Journal of Economics*, 122(3), 1181–1206.

Moraga-González, J. L., Z. Sándor, and M. R. Wildenbeest (2021): "Simultaneous Search for Differentiated Products: the Impact of Search Costs and Firm Prominence," *Economic Journal*, 131(635), 1308–1330.

Moraga-González, J. L., and M. R. Wildenbeest (2008): "Maximum Likelihood Estimation of Search Costs," *European Economic Review*, 52(5), 820–848.

——— (2012): "Comparison Sites," in *The Oxford Handbook of the Digital Economy*, ed. by M. Peitz, and J. Waldfogel. Oxford University Press, Oxford, England.

Myatt, D. P., and D. Ronayne (2024a): "Advertising Stable Prices," *Working Paper in Progress.*

——— (2024b): "Asymmetric Models of Sales with Innovative Firms," *Working Paper.*

——— (2024c): "Finding a Good Deal: Stable Price Dispersion with Robust Search," *Working Paper.*

——— (2024d): "Product Prominence with Stable Price Dispersion," *Working Paper in Progress.*

Nakamura, E., and J. Steinsson (2008): "Five Facts about Prices: A Reevaluation of Menu Cost Models," *Quarterly Journal of Economics*, 123(4), 1415–1464.

Narasimhan, C. (1988): "Competitive Promotional Strategies," *Journal of Business*, 61(4), 427–449.

Nermuth, M., G. Pasini, P. Pin, and S. Weidenholzer (2013): "The Informational Divide," *Games and Economic Behavior*, 78(1), 21–30.

Nocke, V., and N. Schutz (2018): "Multiproduct-Firm Oligopoly: An Aggregative Games Approach," *Econometrica*, 86(2), 523–557.

Obradovits, M. (2014): "Austrian-Style Gasoline Price Regulation: How It May Backfire," *International Journal of Industrial Organization*, 32(1), 33–45.

Okun, A. (1981): *Prices and Quantities: A Macroeconomic Analysis.* The Brookings Institution.

Pennerstorfer, D., P. Schmidt-Dengler, N. Schutz, C. Weiss, and B. Yontcheva (2020): "Information and Price Dispersion: Theory and Evidence," *International Economic Review*, 61(2), 871–899.

Piccione, M., and R. Spiegler (2012): "Price Competition Under Limited Comparability," *Quarterly Journal of Economics*, 127(1), 97–135.

Reinganum, J. F. (1979): "A Simple Model of Equilibrium Price Dispersion," *Journal of Political Economy*, 87(4), 851–858.

Robert, J., and D. O. Stahl (1993): "Informative Price Advertising in a Sequential Search Model," *Econometrica*, 61(3), 657–686.

Ronayne, D. (2021): "Price Comparison Websites," *International Economic Review*, 62(3), 1081–1110.

Ronayne, D., and G. Taylor (2020): "Competing Sales Channels," *Working Paper, University of Oxford.*

——— (2022): "Competing Sales Channels with Captive Consumers," *Economic Journal*, 132(642), 741–766.

Rosenthal, R. W. (1980): "A Model in which an Increase in the Number of Sellers Leads to a Higher Price," *Econometrica*, 48(6), 1575–1579.

Shelegia, S., and C. M. Wilson (2021): "A Generalized Model of Advertised Sales," *American Economic Journal: Microeconomics*, 13(1), 195–223.

Simon, L. K., and W. R. Zame (1990): "Discontinuous games and endogenous sharing rules," *Econometrica*, pp. 861–872.

Sorensen, A. T. (2000): "Equilibrium Price Dispersion in Retail Markets for Prescription Drugs," *Journal of Political Economy*, 108(4), 833–850.

Stahl, D. O. (1989): "Oligopolistic Pricing with Sequential Consumer Search," *American Economic Review*, 79(4), 700–712.

Urbany, J. E., W. O. Bearden, and D. C. Weilbaker (1988): "The Effect of Plausible and Exaggerated Reference Prices on Consumer Perceptions and Price Search," *Journal of Consumer Research*, 15(1), 95–110.

Varian, H. R. (1980): "A Model of Sales," *American Economic Review*, 70(4), 651–659.

Wulfsberg, F. (2016): "Inflation and Price Adjustments: Micro Evidence from Norwegian Consumer Prices 1975–2004," *American Economic Journal: Macroeconomics*, 8(3), 175–194.

Xia, L., K. B. Monroe, and J. L. Cox (2004): "The Price is Unfair! A Conceptual Framework of Price Fairness Perceptions," *Journal of Marketing*, 68(4), 1–15.

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Supplemental On-Line Appendices B and C for "Stable Price Dispersion"

by David P. Myatt and David Ronayne

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Appendix B. Extensions

In this appendix we supplement and extend various results or points in the main text.[39] We:

1. detail the model of prominence outlined in Section 6;
2. detail the n-firm model of advertising outlined for duopoly in Section 7;
3. discuss the model of costly search sketched in Section 7;
4. consider the impact of risk-aversion making equilibrium strict; and
5. describe a two-stage Stackelberg-style game in the captive-shopper setting.

1. Prominence. In Section 6 we described a triopoly in which one firm is prominently considered. One industry-optimal undercut-proof profile is supported by the equilibrium play of pure strategies (Proposition 10). Profits for the firms (which we order so that $\phi_2 \geq \phi_3$) are

$$\pi_1 = v\phi_1, \quad \pi_2 = p_2\phi_2 = \frac{v\phi_1\phi_2}{\phi_1+\phi_2+\phi_3}, \quad \text{and} \quad \pi_3 = p_3\phi_3 = \frac{v\phi_1\phi_3}{\phi_1+\phi_3}. \tag{B1}$$

Here we describe a Nash equilibrium from the play of the standard single-stage pricing game. (This is an equilibrium in a subgame of our two-stage game following $p_1 = p_2 = p_3 = v$ at $t = 1$.) In this equilibrium, firm 2 mixes over the interval $[p_2, p_3]$ according to the distribution

$$F_2(p) = \frac{\phi_1+\phi_2+\phi_3}{\phi_2} - \frac{v\phi_1}{\phi_2 p}, \tag{B2}$$

where $p_2 = v\phi_1/(\phi_1+\phi_2+\phi_3)$ and $p_3 = v\phi_1/(\phi_1+\phi_3)$ are the equilibrium-supported initial prices from Proposition 10. Firm 3 then mixes over the interval $[p_3, v]$ according to

$$F_3(p) = \frac{\phi_1+\phi_3}{\phi_3} - \frac{v\phi_1}{\phi_3 p}. \tag{B3}$$

Finally, the prominent firm 1 mixes over the entire interval $[p_2, v)$ with the distribution

$$F_1(p) = 1 - \frac{v\phi_1}{p(\phi_1+\phi_2+\phi_3)}, \tag{B4}$$

with remaining mass as an atom of size $\phi_1/(\phi_1+\phi_2+\phi_3)$ at v. It is straightforward to confirm that all firms are indifferent across all $p \in [p_2, v)$. In this equilibrium firms 1 and 2 earn the expected profits reported above in (B1). However, the expected profit of firm 3 is

$$\tilde{\pi}_3 = \frac{v\phi_1\phi_3}{\phi_1+\phi_2+\phi_3} < \frac{v\phi_1\phi_3}{\phi_1+\phi_3} = \pi_3 \tag{B5}$$

[39] Some of these connect to our related work: product prominence (Myatt and Ronayne, 2024d), endogenous advertising (Myatt and Ronayne, 2024a), two-stage pricing with risk aversion, and a sequential-move captive-and-shopper game (Myatt and Ronayne, 2024b, Appendix C).

and so firm 3 is strictly worse off than it would be on the equilibrium path of our two-stage game. As noted in our concluding remarks, this is a setting in which our profit predictions do not coincide with those from a Nash equilibrium of the corresponding single-stage game.

We note that the single-stage game studied here is one studied by Inderst (2002, Section 3). We obtain an equivalence by setting $\phi_2 = \phi_3$ (so that the non-prominent firms are symmetric) and $\delta = 0$ in his paper (which eliminates any captives for non-prominent firms). Lemma 3 of Inderst (2002) suggests that the non-prominent firms must mix over the same support, whereas we have an equilibrium in which their supports are non-overlapping.[40]

This setting can further illustrate how our pricing framework can be a component of a deeper model. Inspired by papers in which suppliers pay for prominence (Armstrong and Zhou, 2011; Chen and He, 2011), we introduce a prominence provider that sells that position to firms.

Suppose that all three firms begin with exclusive local customer bases, so that firm $i \in \{1, 2, 3\}$ would charge v to ϕ_i customers within its locality. A monopolist prominence provider, M, offers, in a preliminary (pre-pricing) stage, to bring one firm to national prominence. For example, a provider may be a department store that chooses a product to display in the window, or a website that shows a product on its home page or highlights it at the top of search results.

Specifically, M makes a take it or leave it offer to one firm, and commits to make a specified competitor prominent if the offer is refused. We assume firms have differently sized bases and label them so that $\phi_1 > \phi_2 > \phi_3$. Following the allocation of prominence, we assume that firms set prices that are supported by the equilibrium play of pure strategies (in which the larger non-prominent firm is cheapest, as per Proposition 10).

Because firms' profits are increasing in the size of the prominent firm's base, and the largest firm is the cheapest when it is not prominent, M maximizes its fee (which is accepted in equilibrium) by offering prominence to the firm with the largest base, firm 1, while threatening to make their rival with the smallest base, firm 3, prominent if it refuses.

In essence, a small non-prominent firm has a threatening lean and hungry look, which strengthens the ability of M to extract a fee from a large firm. As such, in equilibrium, M bestows prominence upon firm 1. The prominence provider profits by exploiting the asymmetries between the largest and smallest firm. It then compounds this asymmetry by making firm 1 prominent. This is to the detriment of customers, for whom firm 1 is the worst choice.

2. Advertising. In Section 7 we sketched an extension to the independent awareness specification in which adverting decisions influence consideration sets. Here we flesh out that extension with n firms, noting that some details are reported elsewhere (Myatt and Ronayne, 2024a).

Specifically, we now think of firms that play the following three-stage perfect-information game:

[40] It is possible that the source of the difference in predictions might lie within derivation of the second displayed equation of the proof of Lemma 3 in the appendix of Inderst (2002).

($t = 1$) firms simultaneously choose their awareness parameters $\alpha_i \in [0, 1]$; and then
($t = 2$) firms simultaneously choose their initial price positions $p_i \in [0, v]$; and last
($t = 3$) firms simultaneously choose price cuts to $\tilde{p}_i \in [0, p_i]$.

A firm's payoff is its operating profit minus the cost of advertising, where that advertising determines the awareness of the firm. Firm i's advertising cost $C_i(\alpha_i)$ is smoothly increasing, convex, $C_i(0) = 0$, and $C_i'(0) < v$. When firms are asymmetric we index them so that $C_1'(\alpha) < \cdots < C_n'(\alpha)$ for all $\alpha \in (0, 1]$. This differs from McAfee (1994) by allowing for asymmetric firms, while in Ireland (1993) firms face no costs of advertising.[41]

We seek subgame-perfect equilibria with the play of pure strategies (for advertising choices, initial prices, and final retail prices) along the equilibrium path, and we also look for the play of pure strategies following any first-stage deviations in advertising choices.

Following any first-stage advertising choices, we know that any subgame-perfect equilibrium involves the on-path play of pure strategies (in prices) only if the associated prices are undercut-proof and industry optimal (Proposition 7). The profits of firms (before the deduction of advertising costs) are uniquely defined in such a case. This means that we can simply refer to an equilibrium of the advertising game (with pure on-path strategies) with payoffs $\pi_i - C_i(\alpha_i)$.

Definition (Equilibrium with Endogenous Advertising). *A profile of advertising strategies is* supported by the equilibrium play of pure strategies *if there is a subgame-perfect equilibrium in which pure strategies are played, both on the equilibrium path and on any path beginning within any second-stage subgame. Such a profile is a pure strategy Nash equilibrium of a simultaneous-move advertising game in which firms' payoffs are $\pi_i - C_i(\alpha_i)$ where π_i is the profit of firm i from any industry-optimal profile of undercut-proof prices.*

Given that firms are not yet ordered by their (now endogenous) choice of advertising exposure, we can write the expected sales revenues as

$$\pi_i = \begin{cases} v\alpha_i \prod_{j \neq i}(1 - \alpha_j) & \alpha_i > \max_{j \neq i}\{\alpha_j\} \text{ and} \\ v\alpha_i(1 - \alpha_i) \prod_{j \notin \{i,k\}}(1 - \alpha_j) & \alpha_i < \alpha_k \text{ where } \alpha_k = \max_{j \neq i}\{\alpha_j\}, \end{cases} \tag{B6}$$

and where both expressions apply when firm i ties to be the largest firm.

A firm's sales revenue reacts differently to its advertising reach depending on whether that firm is the largest. The largest firm sets the highest (monopoly) price and so does not worry about another firm undercutting them. Therefore for the largest firm, an increase in α_i increases its expected revenue linearly. In contrast, smaller firms' prices must be set to deter undercutting by larger firms. For them, there are two competing effects: fixing second-period prices, an increase in α_i scales up sales; however, it also forces its second-period price down (and that of

[41] McAfee (1994) also related his paper to that of Robert and Stahl (1993), who specified the simultaneous (rather than sequential) choice of advertising exposure and price.

any smaller firms because of the recursive nature of prices). In fact,

$$\frac{\partial \pi_i}{\partial \alpha_i} = \begin{cases} v \prod_{j \neq i}(1-\alpha_j) & \alpha_i > \max_{j\neq i}\{\alpha_j\} \text{ and} \\ v(1-2\alpha_i)\prod_{j\notin\{i,k\}}(1-\alpha_j) & \alpha_i < \alpha_k \text{ where } \alpha_k = \max_{j\neq i}\{\alpha_j\}. \end{cases} \tag{B7}$$

For a smaller firm, revenue is decreasing in advertising exposure when a firm reaches a majority of customers, that is, when $\alpha_i > 1/2$. If not, then this revenue kinks upward as α_i passes through the maximum advertising exposure of competing firms. Specifically,

$$\frac{\lim_{\alpha_i \downarrow \max_{j\neq i}\alpha_j} \partial \pi_i/\partial \alpha_i}{\lim_{\alpha_i \uparrow \max_{j\neq i}\alpha_j} \partial \pi_i/\partial \alpha_i} = \frac{1-\max_{j\neq i}\alpha_j}{1-2\max_{j\neq i}\alpha_j} > 1, \tag{B8}$$

where the inequality is strict because (once dominated strategies have been eliminated) every firm chooses positive exposure. This implies that no firm chooses its advertising reach to be exactly equal to the maximum of others, and so there is a unique largest firm.

For smaller firms, advertising increases sales revenue only if $\alpha_i \leq 1/2$. This implies firms other than the largest restrict awareness to a minority of potential customers (no matter the cost).

The proofs of Lemma B1, and Propositions B1 and B2 can be found in Appendix C.

Lemma B1 (Properties of Advertising Choices). *In any profile of advertising choices supported by the equilibrium play of pure strategies there is a unique largest firm, and all other firms advertise to a minority of customers.*

On the revenue side, the largest firm always faces an incentive to increase its exposure. Labeling this firm as k, it is straightforward to confirm that, in equilibrium, $\partial \pi_k/\partial \alpha_k \geq 1/2^{n-1}$. Hence, if $C'(1) < 1/2^{n-1}$ then firm k chooses $\alpha_k = 1$ and advertises to everyone.

An advertising equilibrium is characterized by the specification of a leading (and largest) firm k, and n advertising choices which satisfy the n first-order conditions

$$\frac{C'_k(\alpha_k)}{v} = \prod_{j\neq k}(1-\alpha_j) \quad \text{and} \quad \frac{C'_i(\alpha_i)}{v} = (1-2\alpha_i)\prod_{j\notin\{i,k\}}(1-\alpha_j)\ \forall i \neq k. \tag{B9}$$

Because payoffs can be written to rely on a product of all firms' advertising choices, we can (and do, in the proof of Proposition B1) treat this as an aggregative game and solve accordingly (see, e.g., Anderson, Erkal, and Piccinin, 2020; Nocke and Schutz, 2018).

To fully characterize an equilibrium we also need to check for any non-local deviations. For example, one of the smaller firms $i \neq k$ has the option to deviate and choose $\alpha_i > \alpha_k$, and become the largest firm. The proof of Proposition B1 checks such remaining details.

Proposition B1 (Pure Strategies on Path: Endogenous Advertising). *There is at least one profile of advertising choices supported by the equilibrium play of pure strategies.*

In any such equilibrium, one firm chooses a strictly higher advertising level than all the others, sets a price equal to the monopoly price, and only sells to customers who are uniquely aware of its product. Other firms advertise to at most half of potential customers and set lower prices.

In equilibrium, one leading firm advertises distinctly more than others. Proposition B1 does not identify this firm. If the advertising cost functions are not too different then any firm can play this role.[42] If they are different then the leading firm is one with relatively low advertising costs.[43] The other minority-audience firms can, however, be ordered given the structure of the advertising cost functions. For example, if $k = 1$ then advertising choices satisfy $\alpha_1 > \cdots > \alpha_n$.

If firms are symmetric ($C_i(\alpha_i) = C(\alpha_i)$ for all i) then the first-order conditions simplify appreciably. Writing α for the common advertising choice of the smaller firms,[44]

$$\frac{C'(\alpha_k)}{v} = (1-\alpha)^{n-1} \quad \text{and} \quad \frac{C'(\alpha)}{v} = (1-2\alpha)(1-\alpha)^{n-2}. \tag{B10}$$

A special case is when advertising is free (Ireland, 1993), where there is a pathological equilibrium in which multiple firms choose $\alpha_i = 1$ and profits are subsequently driven to zero. Putting this aside (or by allowing costs to be close to free) the "free advertising" case yields $\alpha = 1/2$ for $n-1$ firms, and complete coverage, $\alpha_k = 1$, for one firm.

Another case of interest is the cost specification derived from the random mailbox postings technology suggested by Butters (1977).[45] Equivalently, this is what McAfee (1994) called constant returns to scale in the availability of a firm's price.[46] This is obtained by setting $C(\alpha) = \gamma \log[1/(1-\alpha)]$, so that the marginal cost of increased advertising satisfies $C'(\alpha) = \gamma/(1-\alpha)$. Setting $\gamma = 1$ without loss of generality (this cost coefficient only matters relative to the valuation v of customers for the product) and requiring $v > 1$ (otherwise all firms choose zero advertising) the relevant first-order conditions become

$$\frac{1}{v(1-\alpha_k)} = (1-\alpha)^{n-1} \quad \text{and} \quad \frac{1}{v} = (1-2\alpha)(1-\alpha)^{n-1}. \tag{B11}$$

These equations solve recursively. Substituting the second into the first, we find that $\alpha_k = 2\alpha$: no matter what the level of cost, the large firm reaches twice as many customers as each smaller firm. The solution for α satisfies the natural comparative-static property that α is increasing in the product valuation v, and so is decreasing in the advertising cost parameter γ.

Proposition B2 (Equilibrium with Symmetric Advertising Costs). *If advertising is free, as it is under the specification of Ireland (1993), then, in an equilibrium in which firms earn positive profits, the largest firm chooses maximum advertising exposure, while others advertise to half of potential customers. The largest firm earns twice the profit of each smaller firm.*

If the cost of advertising reach is determined by a random mailbox postings technology, as it is under the constant returns case of McAfee (1994), so that $C(\alpha) = -\gamma \log(1-\alpha)$, then the

[42] This is true for the specifications of Ireland (1993) and McAfee (1994), under which costs are symmetric.
[43] Formally: there is some $k^\star$ such that there is an equilibrium in which any $k \in \{1, \dots, k^\star\}$ leads the industry.
[44] The expressions in (B10) are precisely the equilibrium conditions stated by McAfee (1994).
[45] Suppose that customers are divided into $1/\Delta$ segments each of size Δ. Each segment corresponds to a mailbox. An advertisement costs $\gamma_i\Delta$ for firm i, and randomly hits one of the segments. Hence, with a total spend of $C_i(\alpha_i)$, a firm is able to distribute $C_i(\alpha_i)/(\gamma_i\Delta)$ advertisements. It follows that $\alpha_i = 1-(1-\Delta)^{C_i(\alpha_i)/(\gamma_i\Delta)}$. Taking the limit as $\Delta \to 0$, we observe that $(1-\Delta)^{C_i(\alpha_i)/(\gamma_i\Delta)} \to \exp(-C(\alpha_i)/\gamma_i)$. Solving suggests a cost specification $C_i(\alpha_i) = \gamma_i \log[1/(1-\alpha_i)]$ where (for asymmetric firms) we assume that $\gamma_n > \cdots > \gamma_1 > 0$.
[46] Under "constant returns" two merging firms do not save advertising costs. The probability that a customer considers firm i or j is $1-(1-\alpha_i)(1-\alpha_j)$. There are constant returns if $C(\alpha_i)+C(\alpha_j) = C(1-(1-\alpha_i)(1-\alpha_j))$.

largest firm chooses advertising awareness equal to double that of the competing small firms. Advertising is increasing in customers' willingness to pay.

In both cases, with firms' labels chosen appropriately, prices satisfy $p_i = v/2^{i-1}$.

The "independent awareness" advertising technology and its endogenous selection are not new to this paper: Ireland (1993) and McAfee (1994) both report that the leading firm is twice the size (in terms of advertising reach) and earns twice the profit of other firms. Other authors have, more recently, studied versions of the single-stage model but with a pre-pricing stage in which firms determine their captive shares and have also found asymmetric equilibrium advertising outlays (Chioveanu, 2008).[47] In contrast to those papers, our result maintains the prediction of asymmetric advertising intensities while allowing for the on-path play of pure strategies. We identify (as the final claim of Proposition B2) an interesting pricing sequence: the margin of each firm in the pricing ladder is half that of the firm above.

3. Costly Search. In Section 7 we also sketched a model of customer search. In a related paper (Myatt and Ronayne, 2024c) we consider more fully that search model. We build upon the fixed-sample search technology of Burdett and Judd (1983) and Janssen and Moraga-González (2004): customers choose how many (costly) quotations to request and then select the best available price. Firms set prices using the approach of this paper. Again we predict that firms choose entirely distinct prices. Search behavior and the comparative-static properties differ from those of Janssen and Moraga-González (2004) including the number of quotations customers obtain and the relationship between expected price and entry to the industry.

4. Risk Aversion. In our concluding remarks we observe that in our two-stage pricing game firms are typically indifferent to raising their initial prices. If a firm deviates by doing so, then (for each setting—see the proofs of Lemmas 3 and 5 and Propositions 8 and 10) we constructed a mixed-strategy equilibrium for the ensuing subgame which generates the on-path profit for the deviator. This means that there is only a weak incentive to maintain an initial price.

This is all underpinned by the assumption (otherwise maintained throughout) that firms are risk neutral. Suppose instead that we split each firm into two players: a manager, and an operational pricing agent. We define a game (of perfect information) with $2n$ players in which

($t = 1$) the firms' managers simultaneously choose initial price positions $p_i \in [0, v]$; and then
($t = 2$) the firms' agents simultaneously choose where to cut prices to $\tilde{p}_i \in [0, p_i]$.

Agents' payoffs are simply profits, and so they are assumed to be risk neutral and maximize expected profit. The manager of firm i, however, has payoff $u_i(\pi_i)$, which is a smoothly increasing and concave function of the firm's profit. (The more general and key assumption here is that the manager is more risk averse than the pricing agent.) Equilibrium play in any subgame is

[47] A similar result, but in a setting with a comparison site that advertises alongside sellers for its captive base can be found in an earlier version of Ronayne and Taylor (2022): Ronayne and Taylor (2020, Appendix W.3).

unaffected by the move to this "$2n$ player" environment. If firms' managers choose the initial prices p_i described in our results, then they obtain payoffs $u_i(\pi_i)$ where π_i is a the corresponding profit of firm i under the relevant price profile. Any upward deviation leads to a subgame with the same expected profit, but a lower expected utility. This means that manager i's choice of p_i is the unique best reply to the initial prices, p_j, of managers $j \neq i$.

Moreover, in the captive-shopper setting we find (see Myatt and Ronayne, 2024b, Proposition C2) conditions under which the prices reported in Proposition 5 are the unique subgame-perfect equilibrium of this two-stage manager-agent game.

5. A Stackelberg Version of the Captive-and-Shopper Game. Also in our concluding remarks, we mentioned a Stackelberg interpretation. Keeping (for simplicity of discussion) to the captive-shopper setting with firms ordered by their masses of captive customers, $\lambda_1 > \cdots > \lambda_n > 0$, suppose that a choice of initial price in the first stage is a commitment to a final retail price (a firm that does so becomes, endogenously, a Stackelberg leader) that can be neither raised nor lowered, but that every firm has the option to remain unconstrained, i.e., such a commitment is optional. In the second stage, all unconstrained firms proceed (as endogenous Stackelberg followers) to select their final retail prices. Just as before, we look for an equilibrium in which pure strategies are chosen along the equilibrium path.[48]

There is a subgame-perfect equilibrium in which firm n commits (as the unique Stackelberg leader) to $p_n = p^\dagger_{n-1}$ in the first stage, while other (follower) firms remain unconstrained. In the second stage, firms $i < n$ charge $p_i = v$ and sell to captives, while firm n serves the shoppers.

It is easy to see that no firm $i < n$ has a profitable first-stage or second-stage deviation (to capture shoppers requires a dominated price) and that firm n loses strictly with a lower first-stage price choice (this firm already serves all shoppers with $p^\dagger_{n-1}$). If firm n deviates to a higher price in the first stage, then it loses the shoppers (some of the time) to $n-1$ in the second stage. If firm n deviates to remain unconstrained in the first stage then we revert to a subgame in firm n (once again) does not gain from this deviation.

This example motivates a richer exercise in which firms can choose to commit to (e.g., advertise) a price at any point of a T-stage game, where firms face a flow of customers, who arrive each period. In that setting we show (see Myatt and Ronayne, 2024b, Proposition 8) that so long as T is not too small, then in any subgame-perfect equilibrium firm n commits to the distinctly low price $p^\dagger_{n-1}$ in the first period and sells to the shoppers, while all other firms $i < n$ charge the monopoly price and sell only to their captives.

Appendix C. Other Omitted Proofs and Results

Proof of Lemma 1. We first prove claim (i). If strictly positive prices tie, then a firm in that tie recognizes there is a positive mass of customers who compare them to another firm in that tie

[48] For technical reasons, we retain a free choice of how to break ties when two firms charge the same price.

(and no other). Such a firm strictly improves by undercutting. Thus, any strictly positive prices within a profile are distinct. A special case is claim (i), when all prices are strictly positive.

Turning to claim (ii), if prices are undercut-proof, then no higher priced firm $j < i$ wishes to undercut a cheaper competitor i. If $p_i = 0$ then this is trivially true. If $p_i > 0$, then the strictly positive prices p_i and p_j are distinct and so firm j earns p_j from any comparisons that exclude any higher indexed (strictly cheaper) firms. These are all the comparison sets $B \subseteq \{1, \ldots, j\}$, each of which has mass $\lambda(B)$, that include firm j, which is incorporated by the use of the indicator $B_j \in \{0, 1\}$. Hence $p_j \sum_{B\subseteq\{1,\ldots,j\}} B_j \, \lambda(B)$ is the profit of j. The same logic says that j can achieve (arbitrarily close to) a profit $p_i \sum_{B\subseteq\{1,\ldots,i\}} B_j \, \lambda(B)$ by undercutting p_i and so winning any comparisons amongst the first i firms. Thus, the no-undercutting constraint is

$$p_j \sum_{B\subseteq\{1,\ldots,j\}} B_j \, \lambda(B) \geq p_i \sum_{B\subseteq\{1,\ldots,i\}} B_j \, \lambda(B) \quad \Leftrightarrow \quad p_i \leq \frac{p_j \sum_{B\subseteq\{1,\ldots,j\}} B_j \, \lambda(B)}{\sum_{B\subseteq\{1,\ldots,i\}} B_j \, \lambda(B)}. \tag{C1}$$

This must hold for all $j < i$, giving condition (4) in the lemma. Now suppose that we have a price profile that satisfies (4). The inequality (C1) holds for any pair $j < i$. This inequality is the correct no-undercutting constraint so long as the positive prices involved are distinct. However, the inequalities holding imply that the prices are distinct. To see this, note that

$$p_i \leq \frac{p_j \sum_{B\subseteq\{1,\ldots,j\}} B_j \, \lambda(B)}{\sum_{B\subseteq\{1,\ldots,i\}} B_j \, \lambda(B)} \leq \frac{p_j \sum_{B\subseteq\{1,\ldots,j\}} B_j \, \lambda(B)}{\lambda(\{i,j\}) + \sum_{B\subseteq\{1,\ldots,j\}} B_j \, \lambda(B)} < p_j, \tag{C2}$$

where the final strict inequality follows from our maintained "twoness" assumption. □

Proof of Lemma 5. The proof follows a similar structure to the proof of Lemma 3.

If firm k (where necessarily $k > 1$) deviates upward to $\hat{p}_k > p_k$, then there is some $i < k$ such that $\hat{p}_k \in (p_{i+1}, p_i]$. Just as in the proof of Lemma 3, we build a mixed-strategy equilibrium (illustrated in Figure 1) in which all firms $j \in \{i, \ldots, k\}$ mix (with atoms and gaps) over the interval $[p_k, p_i]$. Other firms maintain their initial prices: $p_j = p_j$ for $j \notin \{i, \ldots, k\}$.

Just as before, $l \in \{k + 1, \ldots, n\}$ has no profitable deviation, for the usual reasons. A firm $l \in \{1, \ldots, i-1\}$ cannot profitably deviate to within $[p_k, p_i]$. An upper bound on its profit from doing so is what it would get by "stealing" the price position of some $j \in \{i \ldots, k\}$. Specifically, suppose l sets a price in $[p_k, p_i]$ and could arrange for j to price above it. Under independent awareness, l's expected profit from a price position in competition with other firms is the same as it was for j, save for the fact that their expected profits are scaled by α_l and α_j, respectively. However, those scalings also apply to the on-path equilibrium expected profits. This means that l does not profitably gain by "stepping on to the dancefloor" with higher-indexed firms.

We now build the strategies used by the actively mixing firms $\{i, \ldots, k\}$.

Case (i): a local deviation upward to $\hat{p}_k \in (p_k, p_{k-1}]$.

Consider a strategy profile in which any firm $j \notin \{k-1, k\}$ maintains its initial price, while firms $j \in \{k-1, k\}$ continuously mix over $[p_k, \hat{p}_k)$ according to distribution functions

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{p_k}{p}\right), \tag{C3}$$

and place remaining mass at their initial prices. These CDFs satisfy $F_j(p_k) = 0$. Because $\alpha_k \leq \alpha_{k-1}$ implies $F_{k-1}(p) \leq F_k(p)$, we need only check that $F_k(p)$ is a valid CDF:

$$F_k(p) \leq 1 \quad \Leftrightarrow \quad p \leq \frac{p_k}{1-\alpha_k} = p_{k-1}, \tag{C4}$$

which holds because $\hat{p}_k \leq p_{k-1}$. Prices within this interval generate the expected profit

$$\pi_k(p) = p\alpha_k\left(1 - \alpha_{k-1}F_{k-1}(p)\right)\prod_{j>k}(1-\alpha_j) = p_k\alpha_k\prod_{j>k}(1-\alpha_j), \tag{C5}$$

which is the on-path expected profit of firm k, π_k. A similar calculation holds for $k-1$.

For the remaining cases, firm k deviates to $\hat{p}_k \in (p_{i+1}, p_i]$ where $i < k-1$.

Case (ii): a deviation to the upper part of a higher price interval, so that $\hat{p}_k \in ((1-\alpha_k)^{1/2}p_i, p_i]$.

We write $F_j(p)$ for the mixed strategy of j. In the lowest interval of prices $[p_k, p_{k-1})$ we set

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \left(\frac{p_k}{p}\right)^{1/(k-i)}\right), \tag{C6}$$

for each firm $j \in \{i, \ldots, k\}$. These are well-defined continuously increasing CDFs. Note that

$$\lim_{p\uparrow p_{k-1}} F_j(p) = \frac{1}{\alpha_j}\left(1 - (1-\alpha_k)^{1/(k-i)}\right) \leq \frac{1}{\alpha_k}\left(1 - (1-\alpha_k)^{1/(k-i)}\right) < 1, \tag{C7}$$

and so these solutions require $k-1$ (this firm faces the constraint $p_{k-1} \leq p_{k-1}$) to place an atom at its initial price p_{k-1}. The expected profit for j from any price within this interval is,

$$p\alpha_j\prod_{l\neq j}(1-\alpha_l F_l(p)) = p_k\alpha_j\prod_{l>k}(1-\alpha_l) = v\alpha_j\prod_{l>1}(1-\alpha_l), \tag{C8}$$

which is the on-path equilibrium expected profit for firm j, π_j.

Next, for each $j \in \{i+1, \ldots, k-2\}$ consider the price interval $[p_{j+1}, p_j)$. This interval lies above the initial price of any firm $l \in \{j+1, \ldots, k-1, k+1, \ldots, n\}$, and so $F_l(p) = 1$ for all such firms. The firms $l \in \{i, \ldots, j\} \cup \{k\}$ (there are $j-i+2$ such firms) all actively mix via

$$F_l(p) = \begin{cases} \frac{1}{\alpha_l}\left(1 - (1-\alpha_k)^{1/(j-i+2)}\right) & p \in [p_{j+1}, (1-\alpha_k)^{1/(j-i+2)}p_j) \\ \frac{1}{\alpha_l}\left(1 - \left(\frac{p_j(1-\alpha_k)}{p}\right)^{1/(j-i+1)}\right) & p \in [(1-\alpha_k)^{1/(j-i+2)}p_j, p_j) \end{cases} \tag{C9}$$

$$= \max\left\{\lim_{p^\diamond\uparrow p_{j+1}} F_l(p^\diamond), \frac{1}{\alpha_l}\left(1 - \left(\frac{p_j(1-\alpha_k)}{p}\right)^{1/(j-i+1)}\right)\right\}. \tag{C10}$$

This means that the CDF remains flat (there is a gap in the support) across the lower part of the interval $[p_{j+1}, p_j)$. For any price in such a gap, a firm would prefer to deviate and undercut the initial price p_{j+1} given that firm $j+1$ places an atom there. Indeed,

$$\lim_{p \uparrow p_j} F_l(p) = \frac{1}{\alpha_l}\left(1 - (1-\alpha_k)^{1/(j-i+1)}\right) \leq \frac{1}{\alpha_k}\left(1 - (1-\alpha_k)^{1/(j-i+1)}\right) < 1, \tag{C11}$$

and so firm j places an atom at its initial price position. Any price $p \in [(1-\alpha_k)^{1/(j-i+2)} p_j, p_j)$ generates the on-path expected profits for any mixing firm. For example, firm k gets

$$p\alpha_k \left[\prod_{h \in \{j+1,\ldots,k-1,k+1,\ldots,n\}} (1-\alpha_h)\right]\left[\prod_{l \in \{i,\ldots,j\}} (1-\alpha_l F_l(p))\right]$$
$$= p_j \alpha_k \left[\prod_{h \in \{j+1,\ldots,k-1,k+1,\ldots,n\}} (1-\alpha_h)\right](1-\alpha_k) = p_n \alpha_k = \pi_k. \tag{C12}$$

For the top price interval (this is for $j = i$), the same formulae apply up to $\hat{p}_k$. That is,

$$F_l(p) = \begin{cases} \frac{1}{\alpha_l}\left(1 - (1-\alpha_k)^{1/2}\right) & p \in [p_{i+1}, p_i (1-\alpha_k)^{1/2}) \\ \frac{1}{\alpha_l}\left(1 - \left(\frac{p_i(1-\alpha_k)}{p}\right)\right) & p \in [p_i (1-\alpha_k)^{1/2}, p_i). \end{cases} \tag{C13}$$

The two firms $l \in \{i, k\}$ then place their remaining mass on their initial prices. (If $\hat{p}_k = p_i$ then the CDFs described above specify $F_k(p_i) = 1$ and so firm k has no atom.)

Case (iii): a deviation to the lower part of a higher price interval, so that $\hat{p}_k \in (p_{i+1}, p_i(1-\alpha_k)]$.

For this case we build the same strategy profile that we would use if $\hat{p}_k = p_{i+1}$. There, firm i does not participate, and always chooses its initial price so that $p_i = p_i$. If $i = k-2$ then we build the strategy profile described in case (i), and if $i < k-2$ then we use the strategy profile from case (ii). In both cases, for prices just below p_{i+1}, the two firms k and $i+1$ mix. Specifically, for $p \in [p_{i+1} (1-\alpha_k)^{1/2}, p_{i+1})$ and $l \in \{k, i+1\}$,

$$F_l(p) = \frac{1}{\alpha_l}\left(1 - \left(\frac{p_{i+1}(1-\alpha_k)}{p}\right)\right). \tag{C14}$$

We know $\alpha_k \leq \alpha_l$ and so $F_l(p) \leq F_k(p)$. Moreover, $\lim_{p \uparrow p_{i+1}} F_k(p) = 1$. This means that k places all mass continuously below p_{i+1}, and so does not use any prices within $(p_{i+1}, \hat{p}_k]$. However, for $\alpha_k < \alpha_l$, l places an atom at p_{i+1}. Firms earn their equilibrium expected profits.

Notice that firm k places all mass below p_{i+1}, which captures the atom of firm $i+1$. We need to check that k does not get more than its equilibrium expected profit, π_k, by charging $\hat{p}_k$:

$$\hat{p}_k \alpha_k \prod_{j \in \{i+1,\ldots,k-1,k+1,\ldots,n\}} (1-\alpha_j) = \frac{\hat{p}_k \alpha_k}{1-\alpha_k} \prod_{j=i+1}^{n} (1-\alpha_j) \leq \pi_k \quad \Leftrightarrow \quad \hat{p}_k \leq p_i(1-\alpha_k), \tag{C15}$$

where this inequality holds by assumption in this case.

Case (iv): a deviation to an intermediate range, so that $\hat{p}_k \in (p_i(1-\alpha_k), p_i(1-\alpha_k)^{1/2}]$.

Case (iii) of Lemma 3's proof is similar in nature. For $p \in [p_k, p_{k-1})$, the lowest interval, define:

$$F_l^+(p) = \frac{1}{\alpha_l}\left(1 - \left(\frac{p_k}{p}\right)^{1/(k-i)}\right) \qquad l \in \{i, \dots, k\} \tag{C16}$$

$$F_l^-(p) = \frac{1}{\alpha_l}\left(1 - \left(\frac{p_k}{p}\frac{\hat{p}_k}{p_i(1-\alpha_k)}\right)^{1/(k-i-1)}\right) \qquad l \in \{i+1, \dots, k\}. \tag{C17}$$

Next, for each $j \in \{i+1, \dots, k-2\}$ and the corresponding price interval $[p_{j+1}, p_j)$, define

$$F_l^+(p) = \frac{1}{\alpha_l}\left(1 - \min\left\{(1-\alpha_k)^{\frac{1}{j-i+2}}, \left(\frac{p_j(1-\alpha_k)}{p}\right)^{\frac{1}{j-i+1}}\right\}\right) \qquad l \in \{i, \dots, j\} \cup \{k\} \tag{C18}$$

$$F_l^-(p) = \frac{1}{\alpha_l}\left(1 - \min\left\{\left(\frac{\hat{p}_k}{p_i}\right)^{\frac{1}{j-i+1}}, \left(\frac{p_j}{p}\frac{\hat{p}_k}{p_i}\right)^{\frac{1}{j-i}}\right\}\right) \qquad l \in \{i+1, \dots, j\} \cup \{k\}. \tag{C19}$$

For the largest-awareness firm i and $p \in [p_k, p_i)$ define

$$F_i(p) = \min\left\{F_i^+(p), \frac{1}{\alpha_i}\left(1 - \frac{p_i(1-\alpha_k)}{\hat{p}_k}\right)\right\}, \tag{C20}$$

and let i place its remaining mass at the initial price p_i.

For other firms $l \in \{i+1, \dots, k\}$ and prices $p < p_l$ define

$$F_l(p) = \begin{cases} F_l^+(p) & F_i(p) = F_i^+(p) \\ F_l^-(p) & \text{otherwise,} \end{cases} \tag{C21}$$

with remaining mass at the firm's initial price (so that $F_l(p) = 1$ for $p \geq p_l$). □

Proof of Proposition 8. Consider the profile from (20). As usual we construct a strategy profile for our two-stage game in which firms charge those prices in the first stage, and maintain those prices in the second stage. The prices are undercut-proof and so there is no profitable second-stage deviation, nor any profitable downward first-stage deviation. It remains to consider upward deviations by either firm 2 or 3 in the first stage. (As usual, we can specify the play of any equilibrium in games that are further from the equilibrium path.)

If firm 2 deviates upward to $\hat{p}_2 > p_2$ then we construct an equilibrium in which firm 3 charges p_3 (earning its on-path expected profit) while 1 and 2 mix using the distributions

$$F_2(p) = \frac{(\lambda_1 + X_2)(p - p_2)}{pX_2} \quad \text{and} \quad F_1(p) = \frac{(\lambda_2 + X_2)(p - p_2)}{pX_2} \tag{C22}$$

over the interval $[p_2, \hat{p}_2)$ with (if $\hat{p}_2 < p_1$) both firms placing remaining mass at their initial prices. If $\hat{p}_2 = p_1 = v$ then the solutions above yield $F_2(p_1) = F_2(v) = 1$ and so only firm 1 plays an atom at its initial price $p_1 = v$. These strategies generate the on-path equilibrium expected payoffs for both firms across the support of their mixed strategies, and it is straightforward to confirm that they have no incentive to deviate elsewhere.

As noted in the text, the more difficult case involves firm 3 deviating upward to $\hat{p}_3 > p_3$. We construct an equilibrium in which firm 1 sets $\tilde{p}_1 = p_1$. We then (as explained in the main text) build mixed strategies for firms 2 and 3 over $[p_3, \min\{\hat{p}_3, p_2\})$ with distributions

$$F_2(p) = \frac{(\lambda_3 + X_3)(p - p_3)}{p(X_3 - X_2)} \quad \text{and} \quad F_3(p) = \frac{(\lambda_2 + X_3)(p - p_3)}{p(X_3 - X_2)}, \tag{C23}$$

where both firms place remaining mass at their initial prices. These distributions give both firms their on-path expected profits across the support. As noted in the text, $F_3(p_2) = 1$. This means that if $\hat{p}_3 > p_2$ then firm 3 cannot play any price $\tilde{p}_3 \in (p_2, \hat{p}_3]$. We need to check that firm 3 does not wish to play such a price. By the argument in the text, that is true if and only if $\hat{p}_3(\lambda_3 + X_2) \leq p_3(\lambda_3 + X_3)$, which is satisfied for all $\hat{p}_3 \leq v$ if and only if $v(\lambda_3 + X_2) \leq p_3(\lambda_3 + X_3)$. Rearranging this gives the inequality (24) stated in the proposition.

So far we have shown that, if (24) holds, there is a strategy profile in which firms 2 and 3 mix and obtain their on-path expected profits, and where they have no incentive to deviate anywhere else. However, we need to check that firm 1 does not wish to deviate from charging $\tilde{p}_1 = p_1 = v$. If $\hat{p}_3 < p_2$ then any deviation $p_1 \in (\hat{p}_3, p_2)$ should be to just below p_2 to capture the atom of firm 2. However, this is not profitable owing to the no-undercutting constraint. This means that we need to check firm 1's expected profit from deviating to some price $\tilde{p}_1 \in [p_3, \min\{\hat{p}_3, p_2\})$ which is (in essence) the "dance floor" across which firms 2 and 3 tango. By Lemma A1, that expected profit, $\pi_1(p_1)$, is quasi-convex in p_1 over the interval $[p_3, p_2)$, which means that

$$\begin{aligned}\pi_1(p_1) \leq \max\left\{\pi_1(p_3), \lim_{p_1 \uparrow p_2} \pi_1(p_1)\right\} &= \max\left\{p_3(\lambda_1 + X_3), p_2(\lambda_1 + X_2(1 - \lim_{p_1 \uparrow p_2} F_2(p_1))\right\} \\ &< \max\left\{v\lambda_1, p_2(\lambda_1 + X_2)\right\} = v\lambda_1. \end{aligned} \tag{C24}$$

The strict inequality holds for both of the components over which the maximum is taken. Specifically, $p_2(\lambda_1 + X_2(1 - \lim_{p_1 \uparrow p_2} F_2(p_1))) < p_2(\lambda_1 + X_2)$ because firm 2 places an atom at p_2. Also $p_3(\lambda_1 + X_3) < v\lambda_1$ because firm 1 strictly prefers not to undercut firm 3. Explicitly:

$$\begin{aligned} p_3(\lambda_1 + X_3) =& v\lambda_1 \frac{\lambda_1 + X_3}{\lambda_1 + X_2} \frac{\lambda_2 + X_2}{\lambda_2 + X_3} \\ =& v\lambda_1 \frac{\lambda_1\lambda_2 + X_2X_3 + (\lambda_1 + \lambda_2)X_2 + \lambda_2(X_3 - X_2)}{\lambda_1\lambda_2 + X_2X_3 + (\lambda_1 + \lambda_2)X_2 + \lambda_1(X_3 - X_2)} < v\lambda_1. \end{aligned} \tag{C25}$$

From this we conclude that firm 1 does not wish to step onto the dance floor.

In summary, we have constructed an equilibrium in deviant subgames with expected profits equal to those on path so long as $\hat{p}_3(\lambda_3 + X_2) \leq p_3(\lambda_3 + X_3)$, which is necessarily true if the inequality (24) holds. Now suppose that this inequality fails, which means that a deviation $\hat{p}_3(\lambda_3 + X_2) > p_3(\lambda_3 + X_3)$ is possible. Our construction (such that 3 earns its on-path expected profit also in the deviant second-stage subgame) no longer works, as we now explain.

We know that firm 1 can achieve at least $v\lambda_1$ by charging $p_1 = v$. This means that the price p_3, and prices just above it, are strictly dominated for 1. It follows that the support of any mixed strategy for firm 1 lies strictly above p_3. If the support for the mixed strategy of firm 2 were to lie strictly above p_3, then firm 3 could achieve strictly more than its on path expected profit. (There would be a price $\tilde{p}_3 > p_3$ below the support of the competitors which would allow firm 3 to win all comparisons and so earn $\tilde{p}_3(\lambda_3 + X_3) > p_3(\lambda_3 + X_3)$.)

We conclude that firm 2 must mix down to p_3 or below. Suppose that p_3 is indeed the lower bound. (We can make the same argument for a strictly lower lower bound.) Firms 2 and 3 must mix continuously as we move up from that lower bound. Given that their expected profits are determined by capturing all comparisons at the lower bound, we can solve for their mixed strategies with the solutions for F_2 and F_3 as before. As we move up the price range, we can evaluate $\pi_1(p)$ from firm 1 joining in at any price p. We have already showed that this is strictly less than $v\lambda_1$. We conclude that firm 1 never joins the dance floor as we move up through the prices. Eventually we reach the same conclusion that we did before: firm 3 has a strict incentive to set $p_3 = \hat{p}_3$, and our intended construction fails. □

Proof of Proposition 9. Fix the profile of maximal undercut-proof prices stated in the text. As usual, we construct a strategy profile in which $\tilde{p}_i = p_i$ is on the path of subgame-perfect equilibrium. There are no profitable downward deviations at the first stage, for the usual reason: the same deviation at the second stage does weakly better. We can, of course, specify any equilibrium in subgames that are not reached with a unilateral deviation in the first stage.

We now focus on upward deviations by either firm 2 or firm 3 in the first stage. Suppose that firm 2 raises its initial price to $\hat{p}_2$. In the subgame, firms $j \in \{1, 2\}$ mix according to

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{p_2}{p}\right) \quad \text{for} \quad p \in [p_2, \hat{p}_2), \tag{C26}$$

and place remaining mass at their initial prices. Firm 3 plays $p_3 = p_3$. This profile yields equilibrium expected profits. For example, for prices in this interval, $j, k \in \{1, 2\}$ and $j \neq k$,

$$\pi_k(p) = \alpha_k p(1-\alpha_3)\left(1 - \alpha_j F_j(p)\right) = p_2\alpha_k(1-\alpha_3) = v\alpha_k(1-\alpha_3)(1-\alpha_2) = \pi_k. \tag{C27}$$

Moreover, $F_1(p) \leq F_2(p) \leq F_2(p_1) = \frac{1}{\alpha_2}\left(1 - \frac{p_2}{p_1}\right) = 1$, and so these are valid CDFs.

Next consider deviations by firm 3 to $\hat{p}_3 > p_3$. One possibility is $p_3 < \hat{p}_3 \leq v(1-\alpha_3) < p_2$ (the last inequality holds because $p_2 = v(1-\alpha_2)$ and $\alpha_3 > \alpha_2$.) Suppose firms $j \in \{2, 3\}$ mix via

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{p_3}{p}\right) \quad \text{for} \quad p \in [p_3, \hat{p}_3), \tag{C28}$$

and place remaining mass at their first-stage prices. Firm 1 plays p_1. Straightforwardly, prices by $j \in \{2, 3\}$ in $[p_3, \hat{p}_3)$ yield equilibrium expected profits, and prices by firm 1 there earn it strictly less than in equilibrium. We need to check $F_j(p)$ are valid CDFs:

$$F_3(p) \leq F_2(p) \leq F_2(\hat{p}_3) = \frac{1}{\alpha_2}\left(1 - \frac{p_3}{\hat{p}_3}\right) \leq 1 \quad \Leftrightarrow \quad \hat{p}_3 \leq \frac{p_3}{1-\alpha_2} = v(1-\alpha_3), \tag{C29}$$

which holds by assumption in this case.

The remaining deviations by firm 3 are to $\hat{p}_3 > v(1-\alpha_3)$. The strategy profiles that we construct specify mixing by each $j \in \{1, 2, 3\}$ via

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \left(\frac{p_3}{p}\right)^{1/2}\right) \quad \text{for} \quad p \in [p_3, p^\diamond), \tag{C30}$$

for some $p^\diamond$. It is simple to check that all firms earn equilibrium expected profits in this interval.

Different cases involve different choices for $p^\diamond$. First suppose $(1-\alpha_3)^{1/2} \leq 1-\alpha_2$. This is true if and only if $\frac{v(1-\alpha_3)}{1-\alpha_2} \leq p_2$. For this parameter case, suppose $\hat{p}_3 < \frac{v(1-\alpha_3)}{1-\alpha_2}$ and set:

$$p^\diamond = \frac{(\hat{p}_3)^2(1-\alpha_2)}{v(1-\alpha_3)}. \tag{C31}$$

Firm 1 places remaining mass at its first-stage price. Firms $j \in \{2,3\}$ mix according to:

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{p_3}{p(1-\alpha_1 F_1(p^\diamond))}\right) = \frac{1}{\alpha_j}\left(1 - \frac{p_3}{p}\left(\frac{p^\diamond}{p_3}\right)^{1/2}\right) \quad \text{for} \quad p \in [p^\diamond, \hat{p}_3). \tag{C32}$$

Both earn equilibrium expected profits across this interval, the CDFs are continuous at $p^\diamond$ and $F_2(\hat{p}_3) = 1$. We complete the specification with 3 placing all remaining mass as an atom at $\hat{p}_3$.

If $\hat{p}_3 = \frac{v(1-\alpha_3)}{1-\alpha_2}$, there is no interval with exactly two firms mixing. Expressions (C30) and (C31) give the equilibrium strategies, and firms place any remaining mass at their first-stage prices.

Now suppose instead that $\hat{p}_3 > \frac{v(1-\alpha_3)}{1-\alpha_2}$. For this case we set $p^\diamond = \frac{v(1-\alpha_3)}{1-\alpha_2}$, and we note that the solution for the CDFs below $p^\diamond$ satisfies $F_2(p^\diamond) = 1$. Hence firm 2 prices only below $p^\diamond$, and does not use the ability to price in $(p^\diamond, p_2]$. Firms $j \in \{1,3\}$ then mix according to

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{p_3}{p(1-\alpha_2)}\right) \quad \text{for} \quad p \in [p^\diamond, \hat{p}_3)\,. \tag{C33}$$

Both firms then place remaining atoms at their first-stage prices. (If $\hat{p}_3 = v$ then this formula specifies $F_3(v) = 1$, and so only firm 1 has an atom.)

It remains to consider parameters satisfying $(1-\alpha_3)^{1/2} > (1-\alpha_2)$, so that $p_2 < \frac{v(1-\alpha_3)}{1-\alpha_2}$.

If $\hat{p}_3 \in (v(1-\alpha_3), p_2]$, then we use the same approach as before by setting $p^\diamond$ as per (C31), and building an equilibrium in which firms 2 and 3 mix over $[p^\diamond, \hat{p}_3)$ which exhausts the CDF for firm 2 as $\hat{p}_3$ is reached, at which point firm 3 places an atom.

If $\hat{p}_3 \in \left(p_2, v\,(1-\alpha_3)^{1/2}\right]$, we also set $p^\diamond$ as per (C31). All firms mix up to $p^\diamond$, firm 1 puts remaining mass on its first-stage price, and $j \in \{2,3\}$ mix via

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{p_3}{p(1-\alpha_1 F_1(p^\diamond))}\right) = \frac{1}{\alpha_j}\left(1 - \frac{p_3}{p}\left(\frac{p^\diamond}{p_3}\right)^{1/2}\right) \quad \text{for} \quad p \in [p^\diamond, p_2), \tag{C34}$$

with 2 playing an atom at p_2 and 3 at $\hat{p}_3$. The value of $p^\diamond$ ensures that 3 earns its equilibrium expected profit from $\hat{p}_3$, making it just indifferent to undercutting firm 2's atom at p_2.

The final case is $\hat{p}_3 \in \left(v\,(1-\alpha_3)^{1/2}, v\right]$. We set $p^\diamond = p_2$. This means that all three firms mix up p_2, with firm 2 playing an atom at p_2. The remaining firms $j \in \{1,3\}$ play

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{p_3}{p(1-\alpha_2)}\right) \quad \text{for} \quad p \in \left[v\,(1-\alpha_3)^{1/2}, \hat{p}_3\right), \tag{C35}$$

with their CDFs remaining constant for $p \in \left(p_2, v\,(1-\alpha_3)^{1/2}\right)$. □

Proof of Proposition 10. We now prove the claims without complete proofs in the main text, which we divide into three parts: Parts 1 and 2 address Nash equilibria in subgames following local and non-local deviations, respectively; Part 3 covers the n-firm symmetric-size case.

Part 1. Here, we provide a Nash equilibrium strategy for each firm in the subgame following local deviations from the profile of prices p_1, p_i, p_j, where $p_1 > p_i > p_j$, from Proposition 10.

The first class of local deviations has firm i setting some $\hat{p}_i \in (p_i, p_1]$. The following strategies constitute a Nash equilibrium of the ensuing subgame. Firms 1 and i mix over $[p_i, \hat{p}_i)$ via

$$F_1 = 1 - \frac{p_i}{p}, \quad F_i = 1 - \frac{(v-p)\phi_1}{p\phi_i}, \tag{C36}$$

with residual mass placed at p_1 and $\hat{p}_i$ respectively. Firm j sets $p_j = p_j$.

The second class of local deviations has firm j setting some $\hat{p}_j \in (p_j, p_i]$. The following strategies constitute a Nash equilibrium of the ensuing subgame. Firms 1 and j mix over $[p_j, \hat{p}_j)$ via

$$F_1 = 1 - \frac{p_j}{p}, \quad F_j = 1 - \frac{(v-p)\phi_1 - p\phi_i}{p\phi_j}, \tag{C37}$$

with residual mass placed at p_1 and $\hat{p}_j$, respectively; firm i sets $p_i = p_i$, earning $p_i\phi_i(1-F_1(\hat{p}_j)) = p_i\phi_i(p_j/\hat{p}_j)$. We confirm i does not have an incentive to deviate to some $p \in [p_j, \hat{p}_j)$:

$$p\phi_i(1 - F_1(p)) = p_j\phi_i \leq p_j\phi_i(p_i/\hat{p}_j). \tag{C38}$$

Part 2. Here we address "non-local" deviations. The first case is that when the smaller non-prominent firm is cheaper, i.e., $i = 2$ and $j = 3$, and $\phi_2 > \phi_3$, with $p_1 > p_2 > p_3$ as stated in Proposition 10. We now prove that in any Nash equilibrium of the subgame following first-stage prices p_1, p_2, and $\hat{p}_3 \in (p_2, v\phi_1/(\phi_1 + \phi_3))$, firm 3 gets a strictly greater profit than $p_3\phi_3$.

In any Nash equilibrium of such a subgame:

(i) No firm places an atom strictly below its first-stage price: if a firm did, then no competitor would ever price at or just above this atom, and so the firm could safely move the atom upward.

(ii) The prominent firm uses a mixed strategy: if pure, each firm's price equals their first-stage price, and the prominent firm would find it profitable to undercut p_2 and capture all customers.

(iii) For the prominent firm, prices $p < p_3$ are strictly dominated, as are $p \in (p_2, v\phi_1/(\phi_1 + \phi_3))$. A firm $k \in \{2, 3\}$ can secure all the relevant customers by charging p_3 and so can guarantee an expected profit of $p_3\phi_k > 0$. Take the highest price charged by any non-prominent firm. This wins customers with strictly positive probability (as it must to generate a positive expected profit) only if the prominent firm prices above it with strictly positive probability. Thus the prominent firm places an atom at $p_1 = v$, which implies its expected profit is $v\phi_1$.

(iv) Excluding the atom at $p_1 = v$, consider the support of the prominent firm's (continuous) mixed strategy. This lies within the union of the competitors' supports: any other price can be safely raised (that is, without losing sales) which strictly raises profit. The support of any competitor lies within the support of the prominent firm, and for the same reason. It follows that the two supports (the prominent firm's, and the union of the competitors') coincide. At the lower bound of that support, the prominent firm sells to everyone, a mass $\phi_1 + \phi_2 + \phi_3$. This firm's profit is $v\phi_1$, and so that lower-bound price must equal $p_3 = v\phi_1/(\phi_1 + \phi_2 + \phi_3)$.

(v) Consider the interval $[p_3, p_2)$. Price p_3 is the lower bound of firm 1's support and therefore also for some $h \in \{2, 3\}$. There cannot be any gaps in the union of all firms' supports in $[p_3, p_2)$. For all other prices in that interval that h plays, h must be indifferent: $p_3\phi_h = p\phi_h(1 - F_1(p)) \Leftrightarrow F_1(p) = 1 - p_3/p$. For any $p \in [p_3, p_2)$ charged by $k \neq h$, k must be indifferent to p and the infimum of those, x, implying k's expected profit is $x\phi_k(1 - F_1(x)) = p_3\phi_k$ and so 1 must again price by the same CDF for p charged by k: $F_1(p) = 1 - p_3/p$ over all $p \in [p_3, p_2)$.

(vi) No first-stage price is in $[p_3, p_2)$, and so there are no atoms. Within this interval there is no gap within the support of the prominent firm: if so, then there would be a gap in the support of the competitors' strategies, and so the prominent firm could safely (i.e., without losing sales) move a price from the bottom of the gap upward, and so strictly gain. Similarly, there is no gap with the union of opponents' supports. Given that, at least one $h \in \{2, 3\}$ is willing to set p_2, earning an expected profit of least $p_3\phi_h$. Because $F_1(p)$ does not depend on which firm has p in their support, the two non-prominent firms face the same expected profit when pricing against the prominent firm, and so 3 can guarantee at least $p_3\phi_3$ by setting p_2.

(vii) Firm 3 earns $p_3\phi_3$ on the equilibrium path, and at least that much by playing $p_3 = p_2$ in the deviant subgame. Recall that the prominent firm 1 never prices just above p_2. Hence, prices p slightly above p_2 earn $p\phi_3(1 - F_1(p_2)) = p_3\phi_3(p/p_2) > p_3\phi_3$. We conclude that any equilibrium in this subgame yields a profitable deviation, and that the profile of prices with firm 3 as the cheapest is not supported by the equilibrium play of pure strategies.

The second case is that when the larger non-prominent firm is cheaper, i.e., $i = 3$ and $j = 2$, with $p_1 > p_3 > p_2$ as stated in Proposition 10. Consider the subgame following a deviation of firm 2 to some $\hat{p}_2 \in (p_3, p_1]$, then the following strategy profile constitutes a Nash equilibrium.

All firms mix: firm 1 over $[p_2, \hat{p}_2)$, 2 over $[v\phi_1/(\phi_1 + \phi_2), \hat{p}_2)$ and 3 over $[p_2, v\phi_1/(\phi_1 + \phi_2))$ via

$$F_1(p) = 1 - \frac{p_2}{p}, \quad F_2 = 1 - \frac{(v-p)\phi_1}{p\phi_2}, \quad \text{and} \quad F_3 = 1 - \frac{(v-p)\phi_1 - p\phi_2}{p\phi_3}, \tag{C39}$$

with any residual mass for firms 1 and 2 placed at p_1 and $\hat{p}_2$, respectively. Firm 2 earns $\phi_2 p_2$, the same as without the deviation. We conclude that the prices in the proposition with $i = 3$ and $j = 2$ are supported as the on-path strategies of a subgame-perfect equilibrium.

Part 3. The remaining claim concerns n firms and $\phi_1 = \cdots = \phi_n \equiv \phi$. Without loss of generality, label the firms inversely to price so that $p_1 > \cdots > p_n > 0$ where firm 1 is the prominent firm. As usual, $p_1 = v$ in any industry-optimal undercut-proof profile. Now consider p_i for $i > 1$. The prominent firm's no-undercutting constraints (one for each local firm) are

$$v\phi_1 \geq p_i \left(\textstyle\sum_{j=1}^{i} \phi_j \right) \quad \Leftrightarrow \quad p_i \leq \frac{v\phi_1}{\sum_{j=1}^{i} \phi_j} = \frac{v}{i}. \tag{C40}$$

For efficiency these bind, and so $p_i = v/i$. As usual, no firm has an incentive to lower its first-stage price or undercut in the second stage. It remains to check upward first-stage deviations.

Suppose that firm $i > 1$ raises its first-stage price to $\hat{p}_i > p_i$. Consider this strategy profile in the subgame. Firm 1 mixes over $[p_i, \hat{p}_i)$ using the distribution function

$$F_1(p) = 1 - \frac{p_i}{p}, \tag{C41}$$

and places all remaining mass at $p_1 = v$. Any cheaper firm, $j > i$ sets $p_j = p_j$ as a pure strategy. Any firm $j < i$, which satisfies $p_j \geq \hat{p}_i$ also plays a pure strategy, $p_j = p_j$. Any other firm $j \neq i$ satisfies $p_i < p_j < \hat{p}_i$. Such a firm mixes over $[p_{j+1}, p_j)$ using the distribution function

$$F_j(p) = 1 - \frac{v - jp}{p}. \tag{C42}$$

Finally, consider the deviant firm i. Take the lowest index k (and so highest first-stage price p_k) which satisfies $p_k < \hat{p}_i$. Firm i mixes over $[p_k, \hat{p}_i)$ using the distribution function

$$F_i(p) = 1 - \frac{v - (k-1)p}{p}, \tag{C43}$$

and places all remaining mass at the deviant price $\hat{p}_i$.

This is a Nash equilibrium of the subgame. Firm 1 earns $v\phi_1$, the same as without the deviation; by undercut-proofness of the initial prices, firm 1 does not do better with a price outside $[p_i, \hat{p}_i)$. For any local firm $j < i$, a price satisfying $p_i \leq p \leq \hat{p}_i$ is in firm 1's support and yields an expected profit of $p_i\phi$. If $p_j \leq \hat{p}_i$ then a firm can do no better than this, and optimally plays the prescribed strategy. If $p_j > \hat{p}_i$ then j is strictly better off with $\tilde{p}_j = p_j$, and so does so. □

This next proposition is a more general version of Proposition 2.

Proposition C3. *For firms placed in size order, $\lambda_1 \geq \cdots \geq \lambda_n$, define the following prices:*

$$p_1^\ddagger = v \quad \text{and} \quad p_i^\ddagger \equiv v \prod_{j=2}^{i} \frac{\lambda_{j-1} + X_{j-1}}{\lambda_{j-1} + X_j}. \tag{C44}$$

Next, for any *given order of firms, consider the set of maximal undercut-proof prices:*

(1) These prices satisfy $p_i \leq p_i^\ddagger$ for all i.

(2) If $\lambda_1 \geq \cdots \geq \lambda_{i-1}$ (so that firms indexed below i are in size order) then $p_i = p_i^\ddagger$.

(3) The i^{th} highest price is highest when firms are in size order, for all i.

(4) Placing firms in size order maximizes the industry profit.

(5) All firms would unanimously prefer to be placed in size order.

Proof. Claim (1) is straightforward. It holds trivially for $i = 1$. If it holds for all $j < i$ then

$$p_i = \min_{j<i} \left\{ p_j \frac{\lambda_j + X_j}{\lambda_j + X_i} \right\} \leq p_{i-1} \frac{\lambda_{i-1} + X_{i-1}}{\lambda_{i-1} + X_i} \leq p_{i-1}^\ddagger \frac{\lambda_{i-1} + X_{i-1}}{\lambda_{i-1} + X_i} = p_i^\ddagger, \tag{C45}$$

and so it holds also for i, and, by the principle of induction, for all i.

Claim (2) can also be proved inductively. It holds for $i = 2$. If it holds for all $j < i$ then

$$\begin{aligned} p_i = \min_{j<i} \left\{ p_j \frac{\lambda_j + X_j}{\lambda_j + X_i} \right\} &= \min_{j<i} \left\{ p_j^\ddagger \frac{\lambda_j + X_j}{\lambda_j + X_i} \right\} \\ &= \min_{j<i} \left\{ p_j^\ddagger \frac{\lambda_j + X_j}{\lambda_j + X_i} \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \right) \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k} \right) \right\} \\ &= p_i^\ddagger \min_{j<i} \left\{ \frac{\lambda_j + X_j}{\lambda_j + X_i} \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \right) \right\} \\ &= p_i^\ddagger \left\{ 1, \min_{j<i-1} \left\{ \frac{\lambda_j + X_j}{\lambda_j + X_i} \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \right) \right\} \right\} = p_i^\ddagger. \end{aligned} \tag{C46}$$

The first line holds by the inductive hypothesis. The second line introduces product terms which cancel each other. The third line recognizes that the second product term multiplied by $p_j^\ddagger$ is $p_i^\ddagger$. The fourth line is obtained by separating out the first term for $j = i - 1$ and the remaining terms for $j < i - 1$. The final line is obtained by noting that for each $j < i - 1$,

$$\frac{\lambda_j + X_j}{\lambda_j + X_i} \prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \leq \frac{\lambda_j + X_j}{\lambda_j + X_i} \prod_{k=j+1}^{i} \frac{\lambda_j + X_k}{\lambda_j + X_{k-1}} = \frac{\lambda_j + X_j}{\lambda_j + X_i} \frac{\lambda_j + X_i}{\lambda_j + X_j} = 1. \tag{C47}$$

The inequality in the chain holds because $X_k \geq X_{k-1}$ in each of the ratio terms, which means that such terms are each decreasing in λ_{k-1}. An upper bound for each term is obtained by replacing λ_{k-1} with $\lambda_j \leq \lambda_{k-1}$, where this inequality holds because $j \leq k - 1 \leq i - 1$ and (by assumption) firms below i are in size order. The claim holds by the principle of induction.

For Claim (3), suppose that firms are not in size order. Consider the first firm k that is out of order: $\lambda_1 \geq \cdots \geq \lambda_{k-1}$ but $\lambda_k > \lambda_{k-1}$. We know that $p_i = p_i^\ddagger$ for all $i \leq k$. This means that

$$p_k = p_{k-1} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k}, \tag{C48}$$

which (given that $X_{k-1} < X_k$) is strictly increasing in λ_{k-1}. The next price is

$$\begin{aligned} p_{k+1} &= \min \left\{ p_{k-1} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_{k+1}}, p_k \frac{\lambda_k + X_k}{\lambda_k + X_{k+1}} \right\} \\ &= p_{k-1} \min \left\{ \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_{k+1}}, \frac{\lambda_k + X_k}{\lambda_k + X_{k+1}} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k} \right\} \\ &= p_{k-1} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k} \min \left\{ \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}}, \frac{\lambda_k + X_k}{\lambda_k + X_{k+1}} \right\} = p_{k-1} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_{k+1}}. \end{aligned} \tag{C49}$$

Suppose that we interchange the two firms; we swap λ_k and λ_{k-1}. Prices p_i for $i < k$ remain unchanged. We write $p_k^\diamond$ for the remaining maximal undercut-proof prices. Clearly,

$$p_k^\diamond = p_{k-1} \frac{\lambda_k + X_{k-1}}{\lambda_k + X_k} > p_{k-1} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k} = p_k, \tag{C50}$$

where the inequality holds because $\lambda_{k-1} < \lambda_k$. Next,

$$p_{k+1}^\diamond = \min \left\{ p_{k-1} \frac{\lambda_k + X_{k-1}}{\lambda_k + X_{k+1}}, p_k^\diamond \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}} \right\}$$

$$
\begin{aligned}
&= p_{k-1} \min\left\{\frac{\lambda_k + X_{k-1}}{\lambda_k + X_{k+1}}, \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}}, \frac{\lambda_k + X_{k-1}}{\lambda_k + X_k}\right\} \\
&= p_{k-1} \frac{\lambda_k + X_{k-1}}{\lambda_k + X_k} \min\left\{\frac{\lambda_k + X_k}{\lambda_k + X_{k+1}}, \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}}\right\} \\
&= p_{k-1} \frac{\lambda_k + X_{k-1}}{\lambda_k + X_k} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}} \\
&> p_{k-1} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}} = p_{k-1} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_{k+1}} = p_{k+1}. \quad \text{(C51)}
\end{aligned}
$$

Straightforwardly all other prices for $i > k+1$ must also (at least weakly) rise.

For Claim (4), from Claim (3) prices are highest by placing firms in order. This maximizes the industry profit $\sum_{i=1}^{n} p_i X_i$ from comparator customers. The profit from captives is maximized when the largest firms charge the highest prices, and this is so when firms are in size order.

This claim also holds when we correct a misstep in the first group of firms: if $\lambda_1 \geq \cdots \geq \lambda_{k-1}$ but $\lambda_k < \lambda_{k-1}$, then switching $k-1$ and k raises the profits earned by the first k firms.

For Claim (5), consider again the procedure above of switching $k-1$ and k into the correct order. Firm $k-1$ benefits: this firm was previously indifferent to charging p_{k-1} and charging p_k, but now gains strictly because $p_k^{\diamond} > p_k$. Firm k benefits from this switch if

$$
p_k(\lambda_k + X_k) \geq p_{k-1}(\lambda_k + X_{k-1}) \quad \Leftrightarrow \quad \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k} > \frac{\lambda_k + X_{k-1}}{\lambda_k + X_k}, \quad \text{(C52)}
$$

where this last inequality holds because $\lambda_k < \lambda_{k-1}$. This means that the first pair of misordered firms both gain by "correcting" their order, as well as raising the profits of all firms $i > k$. □

Proof of Lemma B1. The first claim follows from the argument in the text. The second claim holds because if a firm is not the largest then the derivative of its profit π_i (ignoring awareness costs) with respect to α_i has the same sign as $1 - 2\alpha_i$, which is strictly negative if $\alpha_i > \frac{1}{2}$. □

Proof of Proposition B1. We seek an equilibrium of the advertising game, where firm i's payoff is $\pi_i - C_i(\alpha_i)$. We write $\alpha_i^\star$ for the (pure strategy) equilibrium choice of firm i. Recalling that we ordered firms according to $C_i'(\cdot)$, we will show there is an equilibrium in which firm 1 (the firm with the lowest (marginal) cost of advertising) chooses $\alpha_1^\star > \max_{i \neq 1}\{\alpha_i^\star\}$. Advertising choices (for $\alpha_i \in (0,1)$) satisfy the first-order conditions (B9). With $k = 1$, those become

$$
\frac{C_1'(\alpha_1)}{v} = \prod_{j>1}(1 - \alpha_j) \quad \text{and} \quad \frac{C_i'(\alpha_i)}{v} = \frac{1 - 2\alpha_i}{1 - \alpha_i} \prod_{j>1}(1 - \alpha_j) \ \forall i > 1, \quad \text{(C53)}
$$

where for the set of $n-1$ first-order conditions for $i > 1$ we have divided by $1 - \alpha_i$ knowing that the equilibrium must satisfy $\alpha_i^\star \leq 1/2$. Define $R = \prod_{j>1}(1 - \alpha_j)$. If $Rv > C_i'(0)$, then we define $A_i(R)$ to be the $\alpha_i \in (0, 1/2)$ that satisfies firm i's first-order condition. That is,

$$
\frac{C_i'(A_i(R))}{v} = \frac{1 - 2A_i(R)}{1 - A_i(R)} R. \quad \text{(C54)}
$$

This is uniquely defined because the left-hand side is continuously increasing in $A_i(R)$ and the right-hand side is continuously decreasing (beginning from R and decreasing to zero at $A_i(R) = 1/2$). Furthermore, this solution is strictly increasing in R. If $Rv \leq C_i'(0)$ (so that the right-hand side lies everywhere below the left-hand side) then we set $A_i(R) = 0$. To find R we seek a solution to $R = \prod_{j>1}(1 - A_j(R))$. The right-hand side lies within $[0, 1]$, begins above zero, and is decreasing in R, and so we can find a unique solution $R^\star$. We then set $\alpha_i^\star = A_i(R^\star)$ for $i > 1$. Finally, we can find $\alpha_1^\star$, where $\alpha_1^\star = 1$ if $C_1'(1) < vR^\star$, but otherwise $\alpha_1^\star$ is the unique positive solution (and one which satisfies $\alpha_1^\star > \alpha_i^\star$ for $i > 1$) to the condition $C_1'(\alpha_1) = vR^\star$.

The remaining deviation checks are non-local:

(i) 1 deviates to $\hat{\alpha}_1 \leq \alpha_j^\star$ where $j \colon \alpha_j^\star = \max_{i>1}\{\alpha_i^\star\}$. Firm j satisfies a first-order condition at $\alpha_j^\star$. Therefore, the best such deviation for 1 is to $\hat{\alpha}_1 = \alpha_j^\star$ (1's revenue (cost) curve is the same (flatter) for $\hat{\alpha}_1 \in [0, \alpha_j^\star]$ than j's over the same interval when $\alpha_i = \alpha_i^\star$ for $i \neq j$). By continuity and 1's first-order condition, 1's profit at any $\hat{\alpha}_1 \geq \alpha_j^\star$ is less than at $\alpha_1^\star$.

(ii) $i > 1$ deviates to $\hat{\alpha}_i \geq \alpha_1^\star$. Firm 1 satisfies their first-order condition at $\alpha_1^\star$. Therefore, the best such deviation for $i > 1$ is to $\hat{\alpha}_i = \alpha_1^\star$ (i's revenue (cost) curve is flatter (steeper) for $\hat{\alpha}_i \in [\alpha_1^\star, 1]$ than 1's over the same interval when $\alpha_i = \alpha_i^\star$ for $i > 1$). But by continuity and i's first-order condition, i's profit at any $\hat{\alpha}_i \leq \alpha_1^\star$ is less than at $\alpha_i^\star$.

The other claims of the proposition follow from Lemma B1. □

Proof of Proposition B2. When all firms have zero costs, we put aside trivial equilibria where more than one firm chooses $\alpha_i = 1$ which lead to zero profit outcomes. From symmetry it follows that although the profile of equilibrium advertising choices we report is unique, the assignment of firms is not. Subject to this disclaimer, the main text explains that one firm will advertise with the outright highest intensity, and we label this firm 1.

By (B6), the profit of firm 1 is strictly (and linearly) increasing in α_1 for any $\alpha_i < 1$ for $i > 1$, hence $\alpha_1^\star = 1$. Given $\alpha_1^\star = 1$, (B6) shows that the profit of the non-largest firms is maximized at $\alpha_i = 1/2$ for any $\alpha_j < 1$ where $j \neq 1, i$, hence $\alpha_i^\star = 1/2$ for $i > 1$.

For positive costs, firms $i, j > 1$ must satisfy their first-order conditions given in (B9) but with $C_i = C$. Taking the ratio of i's and j's condition yields

$$\frac{C'(\alpha_i)}{C'(\alpha_j)} = \frac{(1 - 2\alpha_i)(1 - \alpha_j)}{(1 - 2\alpha_j)(1 - \alpha_i)}. \tag{C55}$$

If $\alpha_i > (<)\alpha_j$ the left-hand side $> 1(< 1)$ but the right-hand side $< 1(> 1)$. However, if $\alpha_i = \alpha_j$, (C55) is satisfied. Hence $\alpha_i^\star = \alpha_j^\star$. Letting $C(\alpha) = -\log(1 - \alpha)$ gives (B11), the solution to which gives the values of $\alpha_1^\star$ and $\alpha_i^\star$ for $i > 1$, and that $\alpha_1^\star = 2\alpha_i^\star$. Similar reasoning to that in the proof of Proposition B1 rules out profitable non-local deviations. □