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The Term Structure of Sharpe Ratios and Arbitrage–Free Asset Pricing in Continuous Time*

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Abstract

Recent empirical studies suggest a downward sloping term structure of Sharpe ratios. We present a theoretical framework in continuous time that can cope with such a non–flat forward curve of risk prices. The approach departs from an arbitrage-free and incomplete market setting when different pricing measures are possible. Involved pricing measures now depend on the time of evaluation or the maturity of payoffs. This results in a time inconsistent pricing scheme. The dynamics can be captured by a *time–delayed* backward stochastic Volterra integral equation, which to the best of our knowledge, has not yet been studied.

Key words and phrases: Term Structures, Sharpe Ratio, Incomplete Markets, Asset Pricing, Time Inconsistency, Arbitrage, (Time–Delayed) Volterra Equations

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1 Introduction

We present a dynamic asset–pricing methodology in continuous time. Our considerations depart from the basic risk–return trade–off that is captured by the Sharpe ratio (SR). In particular, we allow for non–flat term–structures of SRs, which reveal an additional dependency on the payoff's maturity. New types of dynamics for the stochastic–discount factor (SDF) emerge. Such a new primitive structure for arbitrage–free asset pricing can no longer be captured by a single equivalent martingale measure (EMM). Instead, these risk–neutral transformations depend on the time to maturity.

Our theoretical results respond to the recent empirical findings that, at least since the beginning of the financial crisis, the SR (market price of risk in the sense of Section 6 G in Duffie (1996) or Section 14 in Björk (2009), rather than the ratio of the first two moments of the stochastic discount factor) has begun to depend significantly on the considered time to maturity of the asset prices from which the respective SRs are estimated. The stochastic dynamics of this entity are well understood when the phenomenon of maturity dependency is absent; see Duffie (1996) for classic risk–neutral asset pricing and a microeconomic foundation based on risk preferences. In this regard most theoretical and empirical considerations implicitly assume a flat term structure of SRs. Moreover, such a property is an implication of the respective consumption–based CAPM and the resulting SDF. The new perspective on SRs allows us to incorporate the dependency on maturity as a new source for modeling dynamic aspects of asset pricing.

On the empirical side, the recent evidence for a maturity dependency in SRs is manifold.[1] This basic observation is supported by a systematic empirical investigation by van Binsbergen, Hueskes, Koijen, and Vrugt (2013), into which the pricing of dividend strips[2] with different maturities is incorporated. Their study finds a slope in the term structure of excess returns that moves pro-cyclically. Van Binsbergen, Brandt, and Koijen (2012) offer additional findings on the term structure of Sharpe ratios and present a comparison with theoretical benchmark models. One main observation is a decrease in the SR when the maturity increases. Consequently, the SR–forward curve contains information about SDFs at different horizons. From the general perspective of discount factors this suggests a comparable approach to the one applied to zero–coupon bonds that have information about the (locally risk–free) forward curve of interest rates.

On the theoretical side, it is important to realize that the nature of a

[1] See Lettau and Wachter (2007) and Hansen, Heaton, and Li (2008) for early research on maturity–dependent risk pricing.

[2] Dividend strips are discounted sums of dividends in a small time interval with a constant length and a varying position on the time axis.

non–flat SR–forward curve reveals the presence of an incomplete market. At time zero, the market reveals a different EMM for each maturity of payoffs. However, this can only be possible if there is not only *one* EMM, or equivalently, the market is incomplete; see Harrison and Kreps (1979) for this basic insight.[3] In the incomplete market setting, we introduce a maturity–dependent path of EMMs. We call this object an *EMM–string* which exactly captures a (market consistent) term structure of instantaneous forward Sharpe ratios. The interdependencies of time and maturity result in a second time parameter for the maturity.

In the case of incomplete markets, an important question refers to a meaningful choice of an EMM. This is a classical task in mathematical finance. For instance, the minimal–variance EMM of Föllmer and Schweizer (1991) and the minimal–entropy EMM of Frittelli (2000) build two classic methods to select a pricing measure.[4] However, in both cases the EMM is chosen at time 0 and applies for any time of pricing and any maturity of the claim. No term structure of Sharpe ratios can emerge. In contrast to that, the present approach to asset pricing via an EMM–string allows us to make a selection within the set of EMMs in a dynamic manner.[5]

Motivated by these empirical and theoretical observations, we introduce a formal model to describe the dynamics of pricing that is based on the term structure of SRs. The basic principle rests on the idea that the EMM-string should be encoded in today's asset prices with different maturities. At this stage our approach is in a similar vein as Heath, Jarrow, and Morton (1992) (HJM), in which the dynamics of the term structure of interest rates are analyzed. Recently, the HJM–methodology has found variations and applications in different fields.[6] However, a straight forward dimension–check indicates that the instantaneous forward rate $f(t,\tau)$ and the instantaneous Sharpe ratio $\theta(t,\tau)$, at time t and maturity τ, have essentially the same dimensionality. To clarify this point, recall for a given forward rate $u \mapsto f(t,u)$ the resulting locally risk–free discount factor $\Lambda(t,\tau)$ in the HJM–approach:

$$\Lambda(t,\tau) = \exp\left(-\int_t^\tau f(t,u)\mathrm{d}u\right) \tag{1}$$

The natural counterpart of (1) with respect to the stochastic discount factor

[3]From the reverse perspective, the range of the SRs' forward curve already contains a lower bound for the degree of incompleteness of the financial market.

[4]See also Elliott and Madan (1998) for an alternative choice of an EMM by means of an extended Girsanov principle.

[5]From a technical stance, our notion of an EMM-string is comparable with a shifting of martingale measures in Biagini, Föllmer, and Nedelcu (2014), see Example 1.

[6] For an application of the HJM approach to volatility surface modeling see Schweizer and Wissel (2008) and Carmona and Nadtochiy (2009).

$\Psi(t,\tau)$ and a given term structure of SRs $u \mapsto \theta(t,u)$ can be written as follows:

$$\Psi(t,\tau) = \exp\left(-\frac{1}{2}\int_t^\tau \theta(t,u)^2 \mathrm{d}u - \int_t^\tau \theta(t,u)\mathrm{d}B_u\right), \tag{2}$$

where B denotes a standard Brownian motion.

Pricing via an EMM–string and the resulting implications for dynamic asset pricing are facilitated by the use of a new *time–delayed* backward stochastic Volterra integral equation (BSVIE); see Yong (2006) for a standard BSVIE and Detemple and Rindisbacher (2010) for the first application in dynamic asset pricing. The necessity for this generalization stems from the maturity dependent continuum of EMMs in the pricing by means of the SDF in (2). The induced pricing principle turns out to satisfy several standard properties, such as positive homogeneity, monotonicity and a form of linearity. However, the time consistency[7] of pricing no longer persists. Intuitively, the updating of pricing is now perturbed by the time–dependency of the EMM–string.

It is already standard practice to formulate continuous–time recursive utility under risk (Duffie and Epstein (1992)) or under ambiguity (Chen and Epstein (2002)), value processes of optimal portfolios (Rouge and El Karoui (2000)), dynamic indifference pricing (Mania and Schweizer (2005)), dynamic risk measures (Barrieu and El Karoui (2005) and Rosazza Gianin (2006)) or even nonlinear conditional expectation (Coquet, Hu, Mémin, and Peng (2002)) through a backward stochastic differential equation (BSDE), see El Karoui, Peng, and Quenez (1997) for a general overview. A dynamic pricing scheme under *one* EMM refers to a linear BSDE. One way of generalizing this continuous–time analog of backward induction makes use of a Volterra–type formulation that perfectly describes our EMM–string and the resulting price process. The idea behind a BSVIE is a second time parameter that allows, in essence, a change at any instant in time of the BSDE that drives the pricing for a fixed EMM. From this perspective, when it comes to pricing via an EMM–string, a class of time–delayed BSVIEs captures the maturity dependence.

To deliver a microeconomic foundation for a term structure of Sharpe ratios, we present a Lucas (1978)–type economy. As in Duffie and Skiadas (1994), we develop a utility gradient approach. Instead of focusing on hyperbolic discounting, as a typical source of time inconsistency, see Krusell and Smith (2003) for such an approach, we consider the agent's belief formation across time yielding an equilibrium expectation that is consistent with a maturity dependent SR. We also show that the resulting

[7] For the pricing under a given EMM, time consistency directly follows from the law of iterated expectations, see also Pelsser and Stadje (2014) for more general results in this direction.

pricing scheme can be the result of an asset pricing model under multiple prior uncertainty. As a byproduct, this framework allows to cope with time inconsistencies that arise in the robust control approach of Hansen and Sargent (2001).

To summarize, the main contributions of the paper are the following. First, we introduce the notion of EMM–strings in order to give more flexibility to new dynamic pricing methods here proposed. Second, by focusing on two different pricing methods via EMM–strings and based on the evaluation-time or on the time to maturity we show that the corresponding pricing dynamics follow Volterra-type equations. To this aim, we introduce a new family of equations that are both of Volterra-type and time-delayed, namely time–delayed BSVIEs, and prove existence and uniqueness results. Finally, we present an equilibrium asset pricing model that is consistent with the EMM–string approach.

The rest of the paper is organized as follows. Section 2 recasts the classical complete market case in continuous time. Section 3 introduces the EMM-string and the relation to the term structure of SR. Section 4 presents a consumption–based asset pricing model that generates nontrivial EMM-strings. Section 5 concludes. The appendix presents a primer on BSVIE and proofs of the results.

2 Preliminaries and Asset Pricing in Complete Markets

Fix a continuous trading economy in the time interval $[0, T]$. Let $(\Omega, \mathcal{F}_T, \mathbb{P})$ be a probability space and let $(\mathcal{F}_t)_{t\in[0,T]}$ be an augmented filtration, generated by the m–dimensional Brownian motion $(B_t)_{t\in[0,T]}$. Suppose that there is an underlying financial market consisting of one risky and one riskless asset. We assume absence of arbitrage, such that the classical fundamental theorem of asset pricing (FTAP) holds: the set of equivalent martingale measures (EMM) is nonempty, that is $\mathcal{Q} \neq \emptyset$. For a definitive version see Delbaen and Schachermayer (1994). For simplicity, we set the interest rate to $r = 0$ and we denote $L_t = L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ for $t \in [0, T]$ and $\mathbb{L} := \{x : x_s \in L_s,\ \forall s \in [0, T]\}$.

For perspective we assume in this section $m = 1$ and a complete financial market, that is $\mathcal{Q} = \{\mathbb{Q}\}$. Following El Karoui, Peng, and Quenez (1997) and Duffie (1996), we recap standard results from asset pricing by formulating the dynamics through linear BSDEs. We divide the analysis into the SDF–approach and the recursive–approach. Each approach considers the pricing of a contingent claim and a payoff stream separately. This allows us to compare the same steps for the pricing approaches via EMM-strings in Section 3.

(i.) SDF–Approach – the case of contingent claims

The price $p_t(X)$ at time $t \in [0, T]$ of a contingent claim X with maturity

T and finite variance, i.e. $X \in L_T$, is given by the $\mathcal{F}_t$–conditional expectation

$$p_t(X) = E_t^{\mathbb{Q}}[X] = E_t^{\mathbb{P}}\left[\frac{\psi_T}{\psi_t}X\right], \tag{3}$$

where $\psi_T = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is the Radon-Nykodym derivative and $\psi_t = E_t^{\mathbb{P}}[\psi_T]$ is the state price density. As formulated in (4), $\frac{\psi_T}{\psi_t}$ is known as the *stochastic discount factor* (SDF) and appears in the Riesz representation $p_0(\cdot) = \langle \psi_T, \cdot \rangle$ of a viable equilibrium price system $p_0 : L_T \to \mathbb{R}$.

Furthermore, it is well known (see Revuz and Yor (2013) for details) that in a Brownian setting ψ_t can be formulated as an exponential martingale, that is

$$\psi_t = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}(-\theta_\bullet B)_t := \exp\left(-\frac{1}{2}\int_0^t \theta_u^2 \mathrm{d}u - \int_0^t \theta_u \mathrm{d}B_u\right) \tag{4}$$

for some square integrable stochastic process $(\theta_t)_{t\in[0,T]}$ that is known as the instantaneous Sharpe ratio. Equivalently, $\psi_t = \mathcal{E}(-\theta_\bullet B)_t$ uniquely solves $\mathrm{d}\mathcal{E}(-\theta_\bullet B)_t = -\theta_t \mathcal{E}(-\theta_\bullet B)_t \mathrm{d}B_t$, $\mathcal{E}(-\theta_\bullet B)_0 = 1$. In a more concise way, (4) will sometimes be written as $\psi_t = \exp\left(-\theta_\bullet B_t - \frac{1}{2}\langle \theta_\bullet B\rangle_t\right)$, where $\theta_\bullet B_t = \int_0^t \theta_s \mathrm{d}B_s$.

(ii.) Recursive Approach – the case of contingent claims

An application of the martingale representation theorem under $\mathbb{Q}$ gives us a stochastic process $(\sigma_t^{\mathbb{Q}})_{t\in[0,T]}$ which allows for a formulation of $\{p_t(X)\}_{t\in[0,T]}$ as the solution to a stochastic integral equation

$$p_t(X) = E_0^{\mathbb{Q}}[X] + \int_0^t \sigma_s^{\mathbb{Q}} \mathrm{d}B_s^{\mathbb{Q}} \quad a.s., \tag{5}$$

where the variance process is endogenous and depends on the evaluated claim X and $B_t^{\mathbb{Q}} = B_t + \int_0^t \theta_s \mathrm{d}s$ is a $\mathbb{Q}$–Brownian motion. This gives an equivalent formulation in terms of a linear BSDE:

$$\mathrm{d}p_t(X) = \sigma_t^{\mathbb{Q}} \mathrm{d}B_t^{\mathbb{Q}} = \theta_t \sigma_t^{\mathbb{Q}} \mathrm{d}t + \sigma_t^{\mathbb{Q}} \mathrm{d}B_t, \tag{6}$$

with $p_T(X) = X$, or equivalently $p_t(X) = \int_t^T -\theta_\tau \sigma_\tau^{\mathbb{Q}} \mathrm{d}\tau + \int_t^T \sigma_\tau^{\mathbb{Q}} \mathrm{d}B_\tau + X$. Taking conditional expectation $E_t^{\mathbb{P}}$ on both sides yields a recursive formulation $p_t(X) = E_t^{\mathbb{P}}[\int_t^T -\theta_\tau \sigma_\tau^{\mathbb{Q}} \mathrm{d}\tau + X]$. Using the formulation of $p_t(X)$ in (6), yield the time consistency of pricing: $p_t(X) = p_t(p_{t+s}(X))$ for any $X \in L_T$, t, s.

(iii.) SDF–Approach – the case of payoff streams

When we move to a valuation of payoff streams $\{x_s\} \in \mathbb{L}$ the pricing payoff streams is defined on $\mathbb{L}$, that is, $p_t : \mathbb{L} \to L_t$. To distinguish the two cases, we now write $p_t(x_{[t,T]})$, where $x_{[t,T]} := \{x_\tau\}_{\tau\in[t,T]}$. Applying Fubini's theorem and Bayes rule for conditional expectation, (4) yields:

$$\begin{aligned} p_t\big(x_{[t,T]}\big) &= E_t^{\mathbb{Q}}\left[\int_t^T x_\tau \mathrm{d}\tau\right] = \int_t^T E_t^{\mathbb{P}}\left[\frac{\psi_\tau}{\psi_t}x_\tau\right]\mathrm{d}\tau \\ &= E_t^{\mathbb{P}}\left[\int_t^T \exp\left(-\frac{1}{2}\int_t^\tau \theta_u^2 \mathrm{d}u - \int_t^\tau \theta_u \mathrm{d}B_u\right)x_\tau \mathrm{d}\tau\right]. \end{aligned} \tag{7}$$

The SR $\theta : [0,T]\times\Omega\to\mathbb{R}$ and the SDF ψ are in one–to–one correspondence. The relation is formulated by means of the exponential martingales; see (4).

(iv.) Recursive Approach – the case of payoff streams

The price of the remaining stream at time t also allows for a recursive and equivalent formulation. Similarly to (6), we have for the dynamics of the pricing scheme $\mathrm{d}p_t(x_{[t,T]}) = \big(\theta_t\sigma_t^{\mathbb{Q}} - x_t\big)\mathrm{d}t + \sigma_t^{\mathbb{Q}}\mathrm{d}B_t$, $p_T(x_{[t,T]}) = 0$, where $\sigma_t^{\mathbb{Q}}$ depends on (x_t, θ_t). Hence

$$p_t\big(x_{[t,T]}\big) = E_t^{\mathbb{Q}}\left[\int_t^T x_\tau \mathrm{d}\tau\right] = E_t^{\mathbb{P}}\left[\int_t^T \big(-\theta_\tau\sigma_\tau^{\mathbb{Q}} + x_\tau\big)\mathrm{d}\tau\right]. \tag{8}$$

We wish to emphasize that in (4), (6), (7) and (8) the SR only depends on the maturity. We will see in the following that it will be no more the case in the pricing approach via EMM–strings introduced here below.

3 EMM–Strings and Term Structure of SRs

We move to an arbitrage–free incomplete financial market model with a nonempty set $\mathcal{Q}$ of EMMs and without specifying the dynamics of the price processes. The Brownian uncertainty model is that of Section 2. Again, in order to focus solely on the dynamics of pricing measures we consider a flat term structure of interest rates and set the risk-free rate $r = 0$. From a dynamic perspective, the nonempty set of EMMs is well behaved. Example 1 below gives some intuition with regard to the time consistency of the set of EMM's $\mathcal{Q}$. More precisely, $\mathcal{Q}$ is m-stable[8] (also known as rectangular or fork convex). This is a crucial concept both from a mathematical point of a view and for applications to economics and finance, see Delbaen (2006) for details.

Definition 1 *A sequence $\mathfrak{Q} = \{\mathbb{Q}^\tau\}_{\tau\in[0,T]}$ in the set of equivalent martingale measures is called an* EMM–string *if the process $\mathfrak{Q} : [0,T]\times\Omega\to\mathcal{Q}$ taking EMM measures as values is adapted.*

[8]Roughly speaking, a set of probability measures is m-stable whether any pasting of different probability measures within $\mathcal{Q}$ corresponds to another probability belonging again to that set.

From the definition we see directly that complete markets, that is $\mathcal{Q} = \{\mathbb{Q}\}$, only allow for exactly one EMM–string that reduces to $\mathbb{Q}^\tau = \mathbb{Q}$ for any τ. To understand the generality of Definition 1, note that, for any fixed $\mathbb{Q}_0 \in \mathcal{Q}$, the EMM–string defined by $\mathbb{Q}^\tau = \mathbb{Q}_0$ for all $\tau \in [0, T]$ corresponds to a trivial choice.

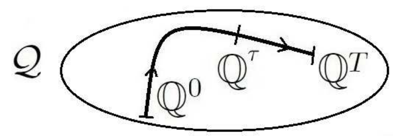

Figure 1: Illustration of an EMM-string

As in (4), we introduce for any $\mathfrak{Q} = \{\mathbb{Q}^\tau\}_{\tau\in[0,T]}$ the induced Sharpe ratio structure $\{\theta(t,\tau)\}_{t\leq\tau}$. More precisely, $\theta(t,\tau) = \theta_\tau^{\mathbb{Q}^t}$ is the instantaneous Sharpe ratio associated to $\mathbb{Q}^t$, that is $\frac{\mathrm{d}\mathbb{Q}^t}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_\tau} = \mathcal{E}(-\theta(t,\cdot)_\bullet B)_\tau$. For technical reasons we make the following standing assumption on the term structure of SR:

Assumption 1 *Let the EMM-string $\mathfrak{Q}$ be deterministic. Let θ be bounded and adapted: for any fixed $\tau \in [0, T]$, $\theta(t, \tau)$ is $\mathcal{F}_t$-measurable for any $t \leq \tau$.*

We analyze two methods for asset pricing via a given EMM–string $\mathfrak{Q}$:

> **Method 1:** at time t we pick an EMM from $\mathcal{Q}$ whatever the maturity $\tau > t$ of the claim is.
>
> **Method 2:** to evaluate a claim with maturity τ, we take an EMM from $\mathcal{Q}$ whatever the evaluation time $t < \tau$ is.

For both pricing methods we repeat the four steps in Section 2.

3.1 Method 1: Current time–based pricing

Let us begin with the method in which the time-t EMM $\mathbb{Q}^t$ is invariant with respect to the maturity of the priced claim. This method is denoted by p^*.

(i.) SDF–Approach – the case of contingent claims

Fix a claim $X \in L_T$ with maturity T. The analog of (3) is now given by

$$p_t^*(X) = E_t^{\mathbb{Q}^t}[X] = E_t^{\mathbb{P}}\left[\frac{\psi(t,T)}{\psi(t,t)}X\right] \tag{9}$$

where $\psi(t,\tau)=E_\tau^{\mathbb{P}}[\frac{\mathrm{d}\mathbb{Q}^t}{\mathrm{d}\mathbb{P}}]=:\frac{\mathrm{d}\mathbb{Q}^t}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_\tau}$, with $t\le\tau$. The SDF now depends on the time of evaluation and the maturity $\tau=T$:

$$\frac{\psi(t,\tau)}{\psi(t,t)}=\frac{\mathrm{d}\mathbb{Q}^t}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_\tau}\left(\frac{\mathrm{d}\mathbb{Q}^t}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t}\right)^{-1} = \frac{\mathcal{E}(-\theta(t,\cdot)_\bullet B)_\tau}{\mathcal{E}(-\theta(t,\cdot)_\bullet B)_t} = \exp\left(-\frac{1}{2}\int_t^\tau\theta(t,u)^2\mathrm{d}u-\int_t^\tau\theta(t,u)\mathrm{d}B_u\right) \tag{10}$$

This calculation follows similar lines as in (4) and motivates the following definition. In the notation of (2), we have $\Psi(t,\tau)=\frac{\psi(t,\tau)}{\psi(t,t)}$.

Definition 2 *Fix an EMM–string* $\mathfrak{Q}$. *We call the induced two–parameter process* $\theta(t,\tau)$ *in* (10) *(or equivalently in* (17)*) the* term structure of Sharpe ratios*.*

Specifically, $\theta(t,\tau)$ is the Sharpe ratio (SR) at time t for pricing a certain claim with maturity τ. If $t>\tau$, set $\theta(t,\tau)=0$ for time–maturity pairs below the diagonal in the left part of Figure 2.

(ii.) Recursive Approach – the case of contingent claims

Let us repeat the same arguments as in (6) of Section 2 for any fixed evaluation time $t\in[0,T]$. This yields with the notation $\sigma_\tau^{\mathbb{Q}^t}=\sigma(t,\tau)$ that the price $p_t^*(X)$ solves

$$p_t^*(X)=\int_t^T-\theta(t,\tau)\sigma(t,\tau)\mathrm{d}\tau+\int_t^T\sigma(t,\tau)\mathrm{d}B_\tau+X \tag{11}$$

or, equivalently,

$$p_t^*(X)=E_t^{\mathbb{Q}^t}[X]=E_t^{\mathbb{P}}\left[\int_t^T-\theta(t,\tau)\sigma(t,\tau)\mathrm{d}\tau+X\right]. \tag{12}$$

The second equality in (12) follows immediately from (11). At any time $t\in[0,T]$ the pricing selects an EMM $\mathbb{Q}^t$ corresponding to a Sharpe ratio $(\theta(t,\tau))_{\tau\in[t,T]}$.[9]

An alternative way to determine the validity of (12) is a comparison with the formulation of $p_t^*(X)$ in (9) and the applied SDF in (10). It follows that only the SR on the path $\tau\mapsto\theta(t,\tau)$ matters. The left part of Figure 2 illustrates this issue. Equation (11) is a Backward Volterra Integral Equation that will be discussed in detail later on and in Appendix A.

[9] To continue this point, for any fixed t we can think of $\mathbb{Q}^t=\mathbb{Q}'$ as an EMM from t onwards and set $p_{t,u}(X)=E_u^{\mathbb{Q}'}[X]=E_u^{\mathbb{P}}[\frac{\psi(t,T)}{\psi(t,t)}X]$ for any $u\in[t,T]$. Hence, $(p_{t,u}(X))_{u\in[t,T]}$ is a $\mathbb{Q}'$-martingale. According to the martingale representation theorem, $p_{t,T}(X)=p_{t,t}(X)-\int_t^T\sigma(t,s)\mathrm{d}B_s^{\mathbb{Q}'}$ holds for some $\sigma(t,s)$. By $p_{t,T}(X)=X$, $p_{t,t}(X)=p_t^*(X)$ and $\mathrm{d}B_s^{\mathbb{Q}'}=\mathrm{d}B_s+\theta(t,s)\mathrm{d}s$ it follows equation (11). Taking expectation, we obtain equation (12).

(iii.) SDF–Approach – the case of payoff streams

Pricing a payoff stream $x_{[t,T]}$ at time t via SDFs is based on *(i.)*. Again, a term structure of SRs $\theta(t,\tau)$, with $t \le \tau$, arises when we aim to describe the dynamics of an EMM–string through the kernel of the explicit Radon–Nykodym densities of $\mathbb{Q}^t$. This can be seen from the following computation

$$\begin{aligned} p_t^*\big(x_{[t,T]}\big) &= E_t^{\mathbb{Q}^t}\left[\int_t^T x_\tau \mathrm{d}\tau\right] = E_t^{\mathbb{P}}\left[\int_t^T \frac{\psi(t,\tau)}{\psi(t,t)} x_\tau \mathrm{d}\tau\right] \\ &= E_t^{\mathbb{P}}\left[\int_t^T \exp\left(-\frac{1}{2}\int_t^\tau \theta(t,u)^2 \mathrm{d}u - \int_t^\tau \theta(t,u)\mathrm{d}B_u\right) x_\tau \mathrm{d}\tau\right], \end{aligned} \tag{13}$$

where we apply (10) to every $\tau \in [t,T]$ for fixed $t \in [0,T]$, Fubini's theorem and Bayes rule for conditional expectation.

(iv.) Recursive Approach – the case of payoff streams

As in Section 2, we again move to the recursive pricing of a payoff stream. By proceeding as in (ii.), this leads to an alternative formulation of p^*:

$$p_t^*\big(x_{[t,T]}\big) = E_t^{\mathbb{Q}^t}\left[\int_t^T x_\tau \mathrm{d}\tau\right] = E_t^{\mathbb{P}}\left[\int_t^T -\theta(t,\tau)\sigma(t,\tau) + x_\tau \mathrm{d}\tau\right]. \tag{14}$$

As in (13), for the pricing at time t, only the SRs $(\theta(t,\tau))_{\tau\in[t,T]}$ matter. The left part of Figure 2 provides an illustration of the present method 1.

We continue with the properties of the pricing under p^*. When the choice of $\mathbb{Q}$ will take place at any time t independently of the claim's maturity, the recursive pricing of a claim can be formulated in terms of Backward Stochastic Volterra Integral Equation (BSVIE) (see Proposition 1 for a precise statement). The recursive pricing of a payoff stream $x_{[t,T]}$ in (14) can be formulated as

$$p_t^*(x_{[t,T]}) = \int_t^T x_\tau - \theta(t,\tau)\sigma(t,\tau)\mathrm{d}\tau + \int_t^T \sigma(t,\tau)\mathrm{d}B_\tau. \tag{15}$$

Apart from the linearity of pricing with respect to different payoff streams, several standard properties remain valid.

Proposition 1 *Under Assumption 1, let $(\theta(t,\tau))_{\tau\in[t,T]}$ be progressively measurable in τ and such that $\theta(t,\tau)$ is Lipschitz continuous in t for any given τ. Then the pricing schemes $p_t^*(X)$ in* (12) *and $p_t^*(x_{[t,T]})$ in* (14) *uniquely solve (if coupled with suitable processes $\sigma(t,s)$) the linear BSVIEs* (11) *and* (15)*, respectively.*

Moreover, for any $t \in [0,T]$ we have:

1. *monotonicity:* $x_s \le y_s$ *on* $[t,T]$ *implies* $p_t^*(x_{[t,T]}) \le p_t^*(y_{[t,T]})$
2. *homogeneity:* $p_t^*(\lambda x_{[t,T]}) = \lambda p_t^*(x_{[t,T]})$ *for any* $\lambda \in \mathbb{R}$

3. *conditional homogeneity:* $p_t^*(\Lambda x_{[t,T]}) = \Lambda p_t^*(x_{[t,T]})$ *for any* $\Lambda \in L_t$
4. *static linearity:* $p_t^*(x_{[t,T]} + y_{[t,T]}) = p_t^*(x_{[t,T]}) + p_t^*(y_{[t,T]})$

These properties also hold for the pricing of contingent claims $X, Y \in L_T$ *with maturity* T.

Another property of dynamic pricing via an EMM–string is the absence of time-consistency. In other words, for all contingent claims $X \in L_T$ and $t, s \in [0,T]$, $p_t^*(X) = p_t^*\big(p_{t+s}^*(X)\big)$ no longer holds.

Proposition 2 *If the pricing scheme is induced by a non–constant EMM–string, that is,* $\mathbb{Q}^u \neq \mathbb{Q}^v$ *for some* $u, v \in [0,T]$*, then time-consistency fails.*

A simple but important consequence distinguishes EMM-strings from the usual pasting operations, see Example 1. For more details on the pasting of probability measures and on time–consistency, please refer to Delbaen (2006).

Corollary 1 *There is an EMM–string that cannot be identified by pasting.*

The proofs of the previous three results are postponed to Appendix B. Three examples, in which time–consistency fails, are provided below.

Example 1 Fix two different EMM's $\mathbb{Q}^0, \mathbb{Q}^1$.

1. Let us consider a typical non m-stable set. Set $\psi^k = \frac{\mathrm{d}\mathbb{Q}^k}{\mathrm{d}\mathbb{P}}$, $k = 0, 1$. Consider the convex set $\mathcal{Q}^{0,1} := \{Q_t : Q_t = t\mathbb{Q}^0 + (1-t)\mathbb{Q}^1 \textit{ with } t \in [0,T]\}$ and the given filtration (in continuous time). The set $\mathcal{Q}^{0,1}$ is not m-stable. To see this, define $\mathbb{Q}^* \notin \mathcal{Q}^{0,1}$ as the pasting of $\mathbb{Q}^0$ (up to τ) and $\mathbb{Q}^1$ (on $[\tau, T]$) or in terms of $\psi^* = \frac{d\mathbb{Q}^*}{d\mathbb{P}}$ as

$$E_t^{\mathbb{P}}[\psi^*] = 1_{[0,\tau]}(t) E_t^{\mathbb{P}}[\psi^0] + 1_{[\tau,T]}(t) \frac{E_\tau^{\mathbb{P}}[\psi^1] \cdot E_t^{\mathbb{P}}[\psi^0]}{E_\tau^{\mathbb{P}}[\psi^0]}.$$

2. Fix $t \in [0,T]$ and consider the simple EMM-string $s \mapsto \mathbb{Q}^s = 1_{[0,t]}(s)\mathbb{Q}^0 + 1_{]t,T]}(s)\mathbb{Q}^1$. At time t the EMM jumps from $\mathbb{Q}^0$ to $\mathbb{Q}^1$. As in 1., the pasting is accomplished in a time–inconsistent manner.

3. Let there be two gurus whose views on the market are given by pessimistic and optimistic pricing measures $\mathbb{Q}^0$ and $\mathbb{Q}^1$. At any time t, each guru has a fraction of followers $\mu_t \in [0,1]$. The resulting EMM–string is then $t \mapsto \mathbb{Q}^t = \mu_t\mathbb{Q}^0 + (1-\mu_t)\mathbb{Q}^1$. See also Biagini, Föllmer, and Nedelcu (2014) for a more detailed account.

Finally, we mention that we have assumed a zero interest rate to simplify our analysis. Nevertheless, taking into account a non-zero interest rate (r_t) poses no problem in our approach. In that case, indeed, the price process $p_t^*(X)$ would be again the solution to

$$p_t^*(X) = X + \int_t^T -\theta(t,s)\sigma(t,s) - r_t p_s^*(X)\mathrm{d}s + \int_t^T \sigma(t,s)\mathrm{d}B_s \tag{16}$$

and therefore would correspond to a linear BSVIE with driver $g(t,s,y,z) = -r_t y - \theta(t,s)z$ also depending on y. A non zero interest reappears in Section 4.2.

3.2 Method 2: Maturity–based pricing

In the case of maturity-based pricing of payoff streams the whole term structure of Sharpe ratios enters in the pricing scheme. Moreover, the pricing of a claim is rather simple. Maturity–based pricing is denoted by $\hat{p}$ while the pricing measure used to evaluate at time t a claim with maturity τ is denoted by $\mathbb{Q}^\tau$ (for any $t \in [0,\tau]$). The choice of the EMM depends then only on the maturity of the claim to be priced.

(i.)/(ii.) SDF/Recursive Approach – the case of contingent claims
Since only the maturity T of the claim matters for choosing the pricing measure, this case takes the same form as in steps *(i.)* and *(ii.)* of Section 2, whenever we suppose $\mathbb{Q} = \mathbb{Q}^T$.

Preliminaries for (iii.)/(iv.) To prepare the maturity-based pricing for payoff streams, it is crucial to realize that the time and maturity parameters t and τ may change separately. In contrast to method 1, the changing maturity selects the pricing measure. In other words, each maturity $\tau \in [0,T]$ generates a linear BSDE. The τ-dependent SR induces a τ-dependent stochastic discount factor $\{\psi(t,\tau)\}_{t\in[0,\tau]}$.

The explicit form of this SDF clarifies the dependence on the particular EMM $\mathbb{Q}^\tau$. For the present method and in view of (4), we have

$$\begin{aligned} \frac{\psi(\tau,\tau)}{\psi(t,\tau)} = \frac{\mathrm{d}\mathbb{Q}^\tau}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_\tau} \left(\frac{\mathrm{d}\mathbb{Q}^\tau}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t}\right)^{-1} &= \frac{\mathcal{E}(-\theta(\cdot,\tau)_\bullet B)_\tau}{\mathcal{E}(-\theta(\cdot,\tau)_\bullet B)_t} \\ &= \exp\left(-\frac{1}{2}\int_t^\tau \theta(u,\tau)^2\mathrm{d}u - \int_t^\tau \theta(u,\tau)\mathrm{d}B_u\right) \end{aligned} \tag{17}$$

for all t,τ with $0 \le t \le \tau \le T$. With these derivations, we may equivalently formulate Definition 2 via (17) instead of (10). Moreover, in contrast to method 1, the SRs on $\{(u,r) \in [t,T]^2 : u < r\}$ (the gray triangle in the right part of Figure 2) are completely exhausted to derive the SDF $\frac{\psi(\tau,\tau)}{\psi(t,\tau)}$ in the present method 2.

(iii.) SDF–Approach – the case of payoff streams

So far the derivation in (17) relates to a fixed $\mathbb{Q}^\tau$. On the conceptual level, the crucial point is that the martingale representation theorem is applied under $\mathbb{Q}^\tau$; for a comparison see (5). When it comes to the pricing of a payoff stream $\{x_\tau\}$ under an EMM-string via method 2, the whole term structure of SRs determines the pricing $\hat{p}$. We write by virtue of (17) the analog of (7):

$$\begin{aligned} \hat{p}_t\big(x_{[t,T]}\big) &= \int_t^T E_t^{\mathbb{Q}^\tau}[x_\tau]\mathrm{d}\tau = \int_t^T E_t^{\mathbb{P}}\left[\frac{\psi(\tau,\tau)}{\psi(t,\tau)}x_\tau\right]\mathrm{d}\tau \\ &= E_t^{\mathbb{P}}\left[\int_t^T \exp\left(-\frac{1}{2}\int_t^\tau \theta(u,\tau)^2\mathrm{d}u - \int_t^\tau \theta(u,\tau)\mathrm{d}B_u\right)x_\tau\mathrm{d}\tau\right]. \end{aligned} \tag{18}$$

At this point, the similarity to the HJM-methodology becomes apparent. Specifically, the HJM approach is mainly focused on a locally riskless formulation of discounting, which is not the case in the present situation.[10]

For perspective we compute a concrete example, in which the term structure is downward sloping.

Example 2 Let $t = 0$, $x_\tau = e^{B_\tau}$ and $\theta(u,\tau) = \bar{\theta} + \exp(\theta_0 + g(\tau - u))$, with $g < 0$ and $\bar{\theta}, \theta_0 \in (0,1)$, the long run and short run Sharpe ratio. We obtain a downward sloping term structure of SRs. As investigated by Van Binsbergen, Brandt, and Koijen (2012), this type of SR is consistent with the Gabaix (2012) rare disaster model and the long run risk model studied in Lettau and Wachter (2007). We derive

$$\hat{p}_0\left(\left\{e^{B_\tau}\right\}_{\tau\in[0,T]}\right) = \int_0^T \exp\left\{\left(\frac{1}{2}+\bar{\theta}\right)\tau - \frac{1-\exp(\theta_0+g\tau)}{\theta_0+g}\right\}\mathrm{d}\tau.$$

For comparison, we have with the standard pricing approach of (8) and $\theta(u,\tau) = \bar{\theta}$ (constant in time and maturity):

$$p_0\left(\left\{e^{B_\tau}\right\}_{\tau\in[0,T]}\right) = \int_0^T \exp\left\{\left(\frac{1}{2}+\bar{\theta}\right)\tau\right\}\mathrm{d}\tau = \frac{\exp\left\{\left(\frac{1}{2}+\bar{\theta}\right)T\right\}-1}{\frac{1}{2}+\bar{\theta}}.$$

(iv.) Recursive Approach – the case of payoff streams

In contrast to method 1, the time and maturity parameters in the term structure are now moving simultaneously when pricing a payoff stream. To see

[10] The concept of forward measures, introduced by Jarrow (1987), is conceptually different from the idea of changing equivalent martingale measures; see also Musiela and Rutkowski (1997). Such measures rest on a numéraire change via different maturities of zero-coupon bond valuation. In contrast, for an EMM-string it is essential to have an incomplete market setting. Conversely, the forward measure has the same structure under complete markets as under incomplete markets, as the SDF remains fixed.

this, we repeat the same steps as in (8) but with maturity–based usage of the EMM–string:

$$\begin{aligned}\hat{p}_t\big(x_{[t,T]}\big) &= \int_t^T E_t^{\mathbb{Q}^\tau}[x_\tau]\mathrm{d}\tau = \int_t^T E_t^{\mathbb{P}}\left[\int_t^\tau -\theta(u,\tau)\sigma(u,\tau)\mathrm{d}u + x_\tau\right]\mathrm{d}\tau \\ &= E_t^{\mathbb{P}}\left[\int_t^T\left(\int_t^\tau -\theta(u,\tau)\sigma(u,\tau)\mathrm{d}u + x_\tau\right)\mathrm{d}\tau\right] \qquad (19)\end{aligned}$$

Here, we apply to each $\tau \in [t,T]$ the Girsanov theorem, via a linear BSDE, and the Fubini theorem for conditional expectations. The right part of Figure 2 provides a graphical illustration of the approach. Moreover, we see that in this case a new time–delayed aspect enters the recursive formulation of (19), which we call a time–delayed BSVIE (TD-BSVIE for short).

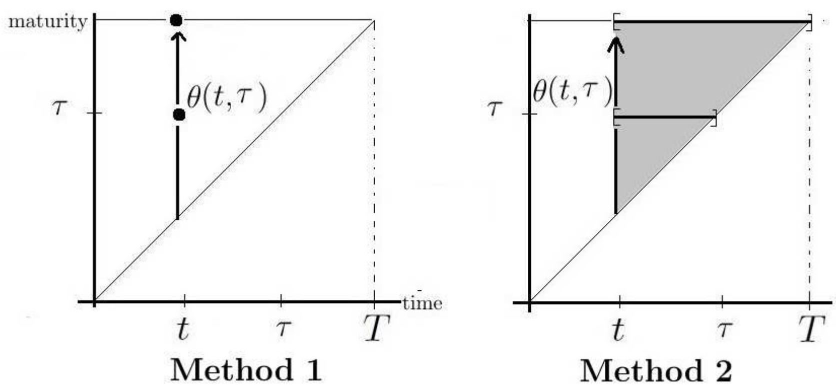

Figure 2: Two ways to employ the random field of SR $\theta(t,\tau)_{t\le\tau}$. *Method 1* uses the EMM string at the time of evaluation; see (13). *Method 2* uses the EMM-string at the maturity of the claim and employs the whole gray triangle; see (18).

The next result justifies that this pricing method leads to a new type of recursive equation.

Theorem 1 *Under Assumption 1, if T is sufficiently close to* 0 *then the maturity–based pricing uniquely solves the following time–delayed BSVIE:*

$$\hat{p}_t(x_{[t,T]}) = \int_t^T\left(-\int_t^\tau \theta(u,\tau)\sigma(u,\tau)\mathrm{d}u + x_\tau\right)\mathrm{d}\tau + \int_t^T \sigma(t,\tau)\mathrm{d}B_\tau. \qquad (20)$$

The proof can be found in Appendix C. Appendix B presents a more detailed account of backward stochastic equations of the form in (20).

The two methods of pricing via an EMM–string or a fixed EMM (classical approach) can be summarized as in Table 1.

	Claim X		Payoff Stream $\{x_\tau\}_{\tau\in[t,T]}$	
Approach	(i) **SDF**	(ii) **Recursive**	(iii) **SDF**	(iv) **Recursive**
$p-$ classical	$\frac{\psi(T,T)}{\psi(t,t)}$	*BSDE*	$\left\{\frac{\psi(\tau,\tau)}{\psi(t,t)}\right\}_{\tau\in[t,T]}$	*BSDE*
p^*- time	$\frac{\psi(t,T)}{\psi(t,t)}$	*BSVIE*	$\left\{\frac{\psi(t,\tau)}{\psi(t,t)}\right\}_{\tau\in[t,T]}$	*BSVIE*
$\hat{p}-$ maturity	$\frac{\psi(T,T)}{\psi(t,T)}$	*BSDE*	$\left\{\frac{\psi(\tau,\tau)}{\psi(t,\tau)}\right\}_{(t,\tau):\,t\leq\tau}$	*TD–BSVIE*

Table 1: Summary of the methods for pricing at time t. The first row is discussed in Section 2. The recursive columns state the type of the related backward stochastic equation (see Appendices A and B for details). The SDF columns present the involved stochastic discount factors. ψ is indexed by time–maturity pairs.

Remark 1 *Under Assumption 1, the pricing at time* 0 *of a contingent claim via p^* or $\hat{p}$ belongs to the no arbitrage interval*

$$[\underline{p}(X), \overline{p}(X)] = \left[\inf_{\mathbb{Q}\in\mathcal{Q}} E^{\mathbb{Q}}[X], \sup_{\mathbb{Q}\in\mathcal{Q}} E^{\mathbb{Q}}[X]\right].$$

See Delbaen (1992), El Karoui and Quenez (1995) or Karatzas and Kou (1996) for further details on the no arbitrage interval in a static and dynamic setting.

4 Lucas Asset Pricing and EMM-strings

We present an equilibrium asset pricing model that is compatible with the two pricing methods in Section 3. The underlying economy contains an agent with changing beliefs over time.[11]

For perspective, recall the complete market case in Section 2, which yields a unique SDF. An underlying equilibrium price system $p_0(\cdot) = \langle\psi, \cdot\rangle_{\mathbb{L}}$ can be related to a representative agent economy, having additive separable utility and a rational belief with $U_t^{\mathbb{P}}(c) = E_t^{\mathbb{P}}[\int_t^T e^{-\beta(s-t)}u(c_s)\mathrm{d}s]$. The corresponding SDF $\psi_t = e^{-\beta t}u'(c_t)$ is then based on the marginal utility at time t. Note that one can define the utility process $U_t^{\mathbb{P}}(c)$ as a linear BSDE:

$$\mathrm{d}U_t^{\mathbb{P}}(c) = \beta U_t^{\mathbb{P}}(c) - u(c_t)\mathrm{d}t + \sigma_t \mathrm{d}B_t, \quad U_T^{\mathbb{P}}(c) = 0.$$

Applying Itô's lemma to ψ_t yields the Sharpe ratio θ_t of the consumption–based CAPM; see Duffie (1996). The resulting equilibrium pricing scheme $p_t(\cdot)$ can then be written as a linear BSDE in the form of (6).

[11] Such a type of fragility can be caused by learning. We follow the perspective of Kurz (1994) and consider the belief as time dependent. An alternative viewpoint refers to the concept of optimal beliefs considered in the study by Brunnermeier and Parker (2005), in which a forward–looking agent maximizes average felicity over beliefs.

4.1 The Utility Process of a Time Inconsistent Agent

Let the subjective belief of an agent change over time, specifically, $\mathbb{P}_t$ is the belief system at time t and mutually absolute continuous with respect to $\mathbb{P}$ with $\frac{\mathrm{d}\mathbb{P}_t}{\mathrm{d}\mathbb{P}} = \mathcal{E}(-\vartheta^t_\bullet B)_T$ for some bounded $\vartheta^t \in \mathbb{L}$. The underlying idea is based on an adaptive learning principle, in which the belief at the beginning may be far away from the true law. Testing the validity with the use of data may lead to an update. By observing a signal or the price process, the increasing amount of available data about the true law yields

$$\lim_{t\to\infty} \mathbb{P}_t = \mathbb{P}.$$

Since we consider only finite horizons, the case $\mathbb{P}_T = \mathbb{P}$ is rather unlikely. However, $\|\mathbb{P} - \mathbb{P}_s\| \geq \|\mathbb{P} - \mathbb{P}_t\| \to 0$,[12] with $s > t$, imposes a monotone path to $\mathbb{P}$. In comparison with Example 1, this description of belief formation yields a further case, in which asset pricing, based on the utility gradient, becomes time–inconsistent.

The present type of belief formation leads to a specific functional form of utility:

$$U_t(c) = \mathbb{E}_t^{\mathbb{P}_t}\left[\int_t^T u(c_s)\mathrm{d}s\right]. \tag{21}$$

For simplicity we set $\beta = 0$ and fix a positive consumption rate process (c_t) in the consumption set $\mathbb{L}_+ = \{c \in \mathbb{L} : c \geq 0 \ \mathbb{P} \otimes \mathrm{d}t\}$. Proceeding as in the recursive approach (*case (iii)*) for method 1 with $u(c_s) = x_s$, the utility in (21) allows for an equivalent formulation as the unique solution of the following linear BSVIE:

$$U_t(c) = \int_t^T \Big(-\vartheta(t,s)\sigma(t,s) + u(c_s)\Big)\mathrm{d}s + \int_t^T \sigma(t,s)\mathrm{d}B_s, \tag{22}$$

where $\vartheta^t = \vartheta(t,\cdot)$ is induced by $\frac{\mathrm{d}\mathbb{P}_t}{\mathrm{d}\mathbb{P}}$; see (9) and (10).

4.2 Asset Pricing under Time Dependent Subjective Expected Utility

Similarly to the Lucas (1978) model, we derive a stochastic Euler equation by means of the first–order conditions of equilibrium. Let there be a long–lived risky asset in the security market that pays dividend stream $\{D_\tau\}_{\tau\in[0,T]}$ and a riskless asset paying that pays the risk free rate as a dividend, that is $D_t^{\texttt{free}} = 0$. The price for the risk–free security is denoted by $S_t^{\texttt{free}}$ and the equilibrium price

[12] An alternative is to consider $d(\mathbb{P}_s, \mathbb{P}) \geq d(\mathbb{P}_t, \mathbb{P}) \to 0$, a generalized distance d such as in Csiszar (1975) and Frittelli (2000).

for the risky asset, S_t, is the solution of a linear BSVIE and is therefore no longer a semi-martingale so the continuous–time formulation of self–financing trading strategies fails. For this particular reason, our argument relies on discrete–time trading strategies and a variational argument in continuous time.

The gain process is denoted by $G_t = (S_t^{\texttt{free}}, S_t) + (0, D_t)$. We assume that assets are available in zero net supply. Moreover, the instantaneous dividend $\{D_\tau\}_{\tau\in[0,T]}$ can be described as an Itô process. A plan (c, η) consists of a positive consumption stream $c = (c_t)_{t\in[0,T]}$ and an S-integrable portfolio composition $\eta = (\eta_t^0, \eta_t^1)$. Both processes are adapted. We assume that η_t^0 and η_t^1 are simple processes.[13] An agent with time–dependent subjective expected utility is confronted with the following problem at time t:

$$\max_{c_t,\eta_t} \qquad U_t(c) = E_t^{\mathbb{P}_t}\left[\int_t^T u(c_s)\mathrm{d}s\right] \tag{23}$$

$$s.t. \qquad \eta_r(1, S_r) = \int_t^r \eta_s \mathrm{d}G_s + \int_t^r (e_s - c_s)\,\mathrm{d}s, \text{ for all } r \in [t, T]. \tag{24}$$

where the endowment (e_t) is assumed to be an Ito process. Condition (24) expresses the usual self-financing condition.

A *no–trade equilibrium* is given by a pair of price processes $(S_t^{\texttt{free}}, S_t)$ such that $(e, 0) = (c, \eta)$ is optimal.

Theorem 2 *In equilibrium it is necessary and sufficient that the (risky) asset price process satisfies the following linear BSVIE:*

$$S_t = \int_t^T (-\vartheta(t,s)\sigma(t,s) + u'(e_s)D_s)\,\mathrm{d}s + \int_t^T \sigma(t,s)\mathrm{d}B_s \tag{25}$$

or, equivalently, in conditional terms $S_t = \frac{1}{u'(c_t)}E_t^{\mathbb{P}_t}[\int_t^T u'(c_\tau)D_\tau \mathrm{d}\tau]$.

To recover now the EMM-string $(\mathbb{Q}^t)$ from the stochastic Euler equation of Theorem 2, we have to specify the fundamental value of the (risky) asset at any time t by the conditional expectation under some EMM $\mathbb{Q}^t \in \mathcal{Q}$, that is, $S_t = E_t^{\mathbb{Q}^t}[\int_t^T e^{\int_t^\tau r_s^* \mathrm{d}s} D_\tau \mathrm{d}\tau]$, where, with an abuse of notation, the equilibrium risk–free rate is given by $r_t^* = -\frac{1}{\mathrm{d}t}E_t^{\mathbb{P}_t}[\frac{\mathrm{d}u'(e_t)}{u'(e_t)}]$ and defined by $\mathrm{d}S_t^{\texttt{free}} = r_t^* S_t^{\texttt{free}}\mathrm{d}t$. Note that, at this stage, we can no longer assume a zero interest rate, since r^* is now an equilibrium outcome, see (16) for an extension of asset pricing via EMM-strings when the interest rate is not zero.

Assume $\frac{\mathrm{d}e_t}{e_t} = \mu_t^e \mathrm{d}t + \sigma_t^e \mathrm{d}B_t$ for some bounded $\mu^e, \sigma^e \in \mathbb{L}$ and denote by $R(e_t) = -\frac{u''(e_t)}{u'(e_t)}e_t$ the relative risk aversion at time t. We then infer via Ito's formula

$$\frac{\mathrm{d}\mathbb{Q}^t}{\mathrm{d}\mathbb{P}_t} = \mathcal{E}\big(-[R(e)\cdot\sigma^e]_\bullet B\big)_T. \tag{26}$$

[13] For the variational argument in the proof of Theorem 2, it suffices to consider simple processes.

With the relation in (26), we are now in the position to connect the equilibrium–based EMM string with the induced term structure of SR $\theta(t,\tau)$, and finally get

$$\frac{\mathrm{d}\mathbb{Q}^t}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(-\theta(t,\cdot)_\bullet B\right)_T, \quad \text{where} \quad \theta(t,\tau) = \vartheta(t,\tau) + R(e_t)\sigma^e_t.$$

4.3 Examples

The following example specifies $(\mathbb{P}_t)_{t\in[0,T]}$ on behavioral grounds.

Example 3 *(Dynamics of Optimism and Pessimism)* Consider an agent with a pessimistic belief $\underline{P} = \mathbb{P}_0$ at time 0, with $\frac{\mathrm{d}\underline{P}}{\mathrm{d}\mathbb{P}} = \mathcal{E}(-\underline{\vartheta}_\bullet B)_T$, and a rather optimistic a belief $\overline{P}$, with $\frac{\mathrm{d}\overline{P}}{\mathrm{d}\mathbb{P}} = \mathcal{E}(-\overline{\vartheta}_\bullet B)_T$, with $\underline{\vartheta} < \overline{\vartheta}$ in $\mathbb{L}$. The degree of pessimism $\alpha_t : [0,T] \to [0,1]$ starts in 0, is of bounded variation and continuous. This yields the following flow of beliefs

$$\mathbb{P}_t = (1-\alpha_t)\underline{P} + \alpha_t\overline{P}.$$

In terms of conditional expectations, we then have by Lemma 3.7 of Biagini, Föllmer, and Nedelcu (2014) and $E^{\overline{P}}_t[\frac{\mathrm{d}\underline{P}}{\mathrm{d}\overline{P}}] = E^{\mathbb{P}}_t[\frac{\mathcal{E}(-\underline{\vartheta}_\bullet B)_T}{\mathcal{E}(-\overline{\vartheta}_\bullet B)_t}]$

$$E^{\mathbb{P}_t}_t[\cdot] = \rho_t E^{\overline{P}}_t[\cdot] + (1-\rho_t)E^{\underline{P}}_t[\cdot], \quad \textit{where } \rho_t = \frac{\alpha_t}{\alpha_t + (1-\alpha_t)\frac{\mathcal{E}(-\underline{\vartheta}_\bullet B)_t}{\mathcal{E}(-\overline{\vartheta}_\bullet B)_t}}.$$

This yields a specific case to formulate the utility process in (21).

Thus far the sequence of beliefs has been considered as given. The following example discusses the robust control approach of Hansen and Sargent (2001). As discussed in Hansen, Sargent, Turmuhambetova, and Williams (2006), a restriction by means of the relative entropy determines a set of priors around the reference model $\mathbb{P}$ and results into time consistencies.

Example 4 *(Robust Control via an Entropy–Based Penalty Term)* In a dynamic setting under Knightian uncertainty, it is common to suppose an m-stable set of possible priors; see Example 1 for a discussion. In discrete time, the latter concept was introduced and axiomatized in Epstein and Schneider (2003). The continuous–time analogue with respect to drift ambiguity is

$$\mathcal{P}^\kappa = \left\{ P_\vartheta : \frac{\mathrm{d}P_\vartheta}{\mathrm{d}\mathbb{P}} = \mathcal{E}(-\vartheta_\bullet B)_T \textit{ for some } \vartheta \in \mathbb{L} \textit{ and } \vartheta \in [-\kappa,\kappa]\, \mathbb{P}\otimes \mathrm{d}t - \textit{a.e.} \right\}$$

for some $\kappa > 0$, see Chen and Epstein (2002). Rectangularity yields, for every time t with information $\mathcal{F}_t$, a certain subset of priors $\mathcal{P}^\kappa_t := \{P \in \mathcal{P} : P =$

$\mathbb{P}$ *on* $\mathcal{F}_t\}$.[14] The functional form of the penalized worst case utility at $c = \{c_t\}$ then reads as follows[15]:

$$U_t(c) = \underset{P \in \mathcal{P}_t^\kappa}{\operatorname{essinf}} \left\{ E_t^P \left[\int_t^T u(c_s) \mathrm{d}s \right] + \mathscr{R}_t(P \| \mathbb{P}) \right\} \tag{27}$$

where the relative entropy at time t is $\mathscr{R}_t(P\|\mathbb{P}) = E_t^P[\int_t^T \log(\frac{\mathrm{d}P}{\mathrm{d}\mathbb{P}})\mathrm{d}\tau]$.

Let c be a fixed optimal consumption. At each time t, pick a conditional minimizer $\mathbb{P}^t \in \arg\min_{P \in \mathcal{P}^\kappa} \left\{ E_t^P \left[\int_t^T u(c_s)\mathrm{d}s \right] + \mathscr{R}_t(P\|\mathbb{P}) \right\}$, see Appendix A.1 in Pelsser and Stadje (2014) for the existence $\mathbb{P}^t$. Each $\mathbb{P}^t$ is an entropy minimal choice at time t and is directly related to the super gradient. In the special case of rectangularity of $\mathcal{P}$, the pasting of all conditional minimizers $\{\mathbb{P}^t\}_{t\in[0,T]}$ guarantees a universal minimizer $\mathbb{P}^{\min}$ that satisfies $\mathbb{P}^{\min}_{|\mathcal{F}_t} = \mathbb{P}^t_{|\mathcal{F}_t}$ for every t. This allows us to remain in the time–consistent world of BSDEs.

In contrast to $\mathcal{P}^\kappa$, for any entropy level $\eta > 0$ we define now a set of priors $\mathcal{P}^\eta$ that fails to satisfy rectangularity. This alternative specification to discipline ambiguity is widely employed in robust control:

$$\mathcal{P}^\eta = \left\{ P_\vartheta \sim \mathbb{P} : \frac{\mathrm{d}P_\vartheta}{\mathrm{d}\mathbb{P}} = \mathcal{E}(-\vartheta_\bullet B)_T \textit{ for some } \vartheta \in \mathbb{L} \textit{ and } \mathscr{R}_0(P_\vartheta\|\mathbb{P}) \leq \eta \right\}$$

where $\mathscr{R}_0(P_\vartheta\|\mathbb{P}) = \frac{1}{2}E^P[\int_0^T \vartheta_t^2 \mathrm{d}t]$. When employing this set in the utility specification from (27), no universal minimizer $\mathbb{P}^{\min}$ exists. In view of Theorem 2, at each time $t \in [0,T]$ a different risk–neutral world arises. With $u(x) = \ln(x)$ and as in (26), this results into an EMM–string via

$$\mathrm{d}\mathbb{Q}^t = \mathcal{E}\left(-\sigma^c_\bullet B\right)_T \mathrm{d}\mathbb{P}^t.$$

As long as the utility of consumption streams is evaluated, different conditional minimizers may arise. In general, the new minimizers of $U_t(d)$ for a different consumption stream (d_t) yield a completely different EMM–string.

4.4 Optimality in the Arrow–Debreu World

The asset pricing formulation in Subsection 4.2 is based on the conditional marginal utility. To follow this approach, we show that the conditional directional derivative

$$\nabla U_t^h(c) = \lim_{\varepsilon \to 0} \frac{U_t(c+\varepsilon h) - U_t(c)}{\varepsilon}$$

[14] This results in the correspondence of Girsanov kernels. Since, each $P \in \mathcal{P}^\kappa$ can be identified via $\frac{\mathrm{d}P}{\mathrm{d}\mathbb{P}} = \mathcal{E}(-\vartheta_\bullet B)_T$, where ϑ is the possible drift of the state process. Under the assumption of rectangular priors, the resulting utility process can be formulated as a BSDE. The construction of $\mathcal{P}^\kappa$ parallels the specification of an EMM under Brownian noise.

[15] Notice that the expression below is similar to the characterization of dynamic time-consistent convex risk measures found in Delbaen, Peng, and Rosazza Gianin (2010), where a general penalty term satisfying the so called cocycle property appears instead of $\mathscr{R}_t$.

exists as an $\mathcal{F}_t$-measurable random variable in L_t for a particular feasible direction $h \in \mathbb{L}$, that is $c + \varepsilon h \in \mathbb{L}_+$ for $\varepsilon > 0$ small. Set $\|x\| = E^{\mathbb{P}}[\int_0^T x_t^2 \mathrm{d}t]^{1/2}$. Similarly to Aliprantis (1997) for separable utility functions, $U_0 : \mathbb{L}_+ \to \mathbb{R}$ is concave whenever the utility index is so.

Proposition 3 *Let the Bernoulli utility index $u : \mathbb{R}_+ \to \mathbb{R}$ be concave, continuous, monotone, and differentiable.*

1. *The utility with belief formation in* (21) *or equivalently in* (22) *is concave, $\|\cdot\|$–continuous, monotone, and differentiable with utility gradient*

$$\nabla U_t^h(c) = E_t^{\mathbb{P}_t}\left[\int_t^T u'(c_s)h_s \mathrm{d}s\right]. \tag{28}$$

   Furthermore, the utility process is in general time-inconsistent.

2. *The dynamics of the utility gradient solve the following linear BSVIE:*

$$\nabla U_t^h(c) = \int_t^T \Big(-\vartheta(t,s)\sigma^{\nabla}(t,s) + u'(c_s)h_s\Big)\mathrm{d}s + \int_t^T \sigma^{\nabla}(t,s)\mathrm{d}B_s, \tag{29}$$

   with terminal condition $\nabla U_T^h(c) = 0$ and $s > t$.

The second part of the proposition builds the connection with Subsection 3.2. The proof is postponed to Appendix D. Following Proposition 3, the necessary (and sufficient) first–order conditions for optimal consumption choice deliver the basis for a utility–gradient approach to asset pricing. One major insight points to the new shape of the *state–price density* process. The utility specification in (21) allows for an explicit representation of the utility gradient. At time t in direction h we have

$$\nabla U_t^h(c) = E_t^{\mathbb{P}_t}\left[\int_t^T u'(c_s)h_s \mathrm{d}s\right] = E_t^{\mathbb{P}}\left[\int_t^T \pi(t,s)h_s \mathrm{d}s\right],$$

where the change of measure from $\mathbb{P}_t$ to $\mathbb{P}$ on $[t,T]$ yields the conditional Riesz representation $\nabla U_t^h(c) = \langle \pi(t,\cdot), h\rangle_t$, where

$$\pi(t,\tau) = u'(c_\tau)\cdot \exp\left(-\frac{1}{2}\int_t^\tau \vartheta(t,s)\mathrm{d}s - \int_t^\tau \vartheta(t,s)\mathrm{d}B_s\right),$$

for each $\tau \in [t,T]$. The argument is similar to that of (10).

Fix a strictly positive endowment process $e \in \mathbb{L}_+$. A priori, an Arrow–Debreu state–price density ψ can be any process in $\mathbb{L}_+$ and corresponds to $\tau \mapsto \psi(0,\tau)$ on $[0,T]$. A consumption process $\hat{c}$ is optimal at time $t = 0$ if[16]

$$\hat{c} \in \arg\max \Big\{ U_0(c) : \langle \psi(0,\cdot), c - e\rangle_0 \le 0, c \in \mathbb{L}_+ \Big\}. \tag{30}$$

In view of Proposition 3, optimality can be characterized by the first–order condition, so $\pi = \mu\psi$ in $\mathbb{L}$ for a certain multiplier $\mu > 0$.

[16]Note that the scalar product on $\mathbb{L}$ yields $\langle \psi(0,\cdot), c - e\rangle_0 = E^{\mathbb{P}}[\int_0^T \psi(0,\tau)(c_\tau - e_\tau)\mathrm{d}\tau]$.

5 Conclusion

Recent empirical investigations have explored the maturity dependency in the implied Sharpe ratio. Based on the implicitly revealed incompleteness of the underlying financial market, we present a new dynamic pricing methodology. In contrast to the standard pricing with a given and fixed equivalent martingale measure, we allow for time–to–maturity or for a current–time dependency of the acting pricing measure. The dynamics of pricing give rise to a new type of time–delayed backward stochastic Volterra integral equation.

A Primer on (time–delayed) BSDEs, BSVIEs

For perspective and as a preparation for Appendix B, we recall some notions of backward stochastic equations. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability measure, $(B_t)_{t\geq 0}$ be an m-dimensional Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ be the augmented filtration generated by it. Denote by

$$L^2(0,T;\mathbb{R}^m) = \left\{ \xi : [0,T]\times\Omega \to \mathbb{R}^m : \begin{array}{l} (\xi_t)_{t\in[0,T]} \text{ is progressively} \\ \text{measurable and } E[\int_0^T |\xi_s|^2 ds] < +\infty \end{array} \right\}.$$

$$L^2(0,T;L^2_{\mathcal{F}}(0,T)) = \left\{ Z : [0,T]^2\times\Omega \to \mathbb{R}^{n\times m} : \begin{array}{l} Z \text{ is } \mathcal{B}([0,T]^2)\otimes\mathcal{F}_T\text{-measurable} \\ Z(t,\cdot) \text{ adapted } \forall t \text{ and} \\ E[\int_0^T\int_0^T |Z(t,s)|^2 ds dt] < +\infty \end{array} \right\}.$$

In the following, we briefly introduce several types of backward equations.

A.1 Backward Stochastic Differential Equations

We recall from El Karoui, Peng, and Quenez (1997) that, given a functional $g : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$ and a final condition $X \in L^2(\mathcal{F}_T)$,

$$\begin{cases} -dY_t = & g(t,Y_t,Z_t)dt - Z_t dB_t \\ Y_T = & X \end{cases} \tag{31}$$

or, equivalently $Y_t = X + \int_t^T g(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s$, is called Backward Stochastic Differential Equation (BSDE) with a driver g and a final condition X. A solution to the BSDE (31) (if it exists) is a pair $(Y_t,Z_t)_{t\in[0,T]} \in L^2(0,T;\mathbb{R})\times L^2(0,T;\mathbb{R}^m)$ satisfying (31). The existence and uniqueness of a solution is guaranteed when g is Lipschitz in (y,z), $g(\cdot,y,z)\in L^2(0,T;\mathbb{R})$ and $g(t,y,0)\equiv 0$ (see El Karoui, Peng, and Quenez (1997)).

Notice that the case of a payoff stream $\{x_s\}_{s\in[0,T]}$ (instead of a final condition X) can be embedded in a BSDE by replacing the driver g in (31) with the new driver $\bar{g}(s,y,z) = g(s,y,z) + x_s$ and taking $X = 0$.

For perspective let us recall the comparison theorem for BSDEs: If the drivers satisfy $g^1 \geq g^2$ and the terminal conditions satisfy $X^1 \geq X^2$ then the resulting solutions of the two BSDEs satisfy $Y_t^1 \geq Y_t^2$ for all t.

A.2 Backward Stochastic Volterra Integral Equations

We recall from Yong (2006) that a general Backward Stochastic Volterra Integral Equation (BSVIE) is

$$Y_t = \eta_t + \int_t^T g(t, s, Y_s, Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dB_s, \tag{32}$$

where $g = g(t, s, y, z_1, z_2, \omega) : [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m} \times \Omega \to \mathbb{R}^n$ and $\eta_t \in L^2(\mathcal{F}_T; \mathbb{R}^n)$ (not necessarily $\mathcal{F}_t$-adapted) for any $t \in [0, T]$ are given.

An adapted solution of the BSVIE is a pair of processes $(Y, Z) \in L^2(0, T; \mathbb{R}^n) \times L^2(0, T; L^2_{\mathcal{F}}(0, T))$ satisfying (32). As in the BSDE framework, a Lipschitz condition is often imposed on g. More precisely, g is assumed to satisfy the following conditions:

(a) g is $\mathcal{B}([0, T]^2) \otimes \mathbb{R}^n \otimes \mathbb{R}^{n \times m} \otimes \mathbb{R}^{n \times m} \otimes \mathcal{F}_T$-measurable;

(b) there exists a $C > 0$ such that:

(uniformly bounded) $|g(t, s, 0, 0, 0)| \leq C$

(uniformly Lipschitz): for any $t, s, y, \bar{y}, z_1, \bar{z}_1, z_2, \bar{z}_2$, we have $\mathbb{P}$-a.s.:

$$|g(t, s, y, z_1, z_2) - g(t, s, \bar{y}, \bar{z}_1, \bar{z}_2)| \leq C \left(|y - \bar{y}| + |z_1 - \bar{z}_1| + |z_2 - \bar{z}_2|\right)$$

For any driver g satisfying the conditions above and for any $\eta_t \in L^2(0, T; \mathbb{R}^n)$, the existence and uniqueness of the solution are guaranteed (see Theorem 3.2 of Yong (2006)). When the driver g does not depend on $Z(s, t)$ (briefly $g = g(t, s, y, z)$), that is

$$Y_t = \eta_t + \int_t^T g(t, s, Y_s, Z(t, s))ds - \int_t^T Z(t, s)dB_s, \tag{33}$$

an adapted solution of the BSVIE (33) then corresponds to a pair of processes (Y, Z) satisfying (33) such that, for any $t \in [0, T]$, $(Y_s, Z(t, s))_{s \in [t,T]}$ is adapted. In other words, $Z(t, s)$ is defined uniquely on $\mathbb{T} = \{(t, s) \in [0, T] \times [0, T] : 0 \leq t \leq s \leq T\}$ while for $s < t$ the values of $Z(t, s)$ are not needed.

The notion of a linear BSVIE then corresponds to a driver of the form

$$g(t, s, Y_s, Z(t, s)) = L_1(t, s) \cdot Y_s + L_2(s) \cdot Z(t, s),$$

with $L_1 : \mathbb{T} \times \Omega \to \mathbb{R}^{n \times n}$ and $L_2 : [0, T] \times \Omega \to \mathbb{R}^{n \times n}$.

A weak version of the comparison theorem for BSVIE's holds, for instance, when at least one among the following conditions is satisfied:

1. g is independent of y and satisfies some further assumptions (stated below).

2. $g^1 \leq \bar{g} \leq g^2$ with $\bar{g}$ Lipschitz, nondecreasing in y plus some further assumptions (see Theorem 3.4 of Wang and Yong (2015)).

3. $g(t, s, y, z) = h(t, s, y)+$ (linear term in z) satisfying some further assumptions (see Theorem 3.9 of Wang and Yong (2015)).

Theorem 3 (Prop. 3.3 of Wang and Yong (2015)) *Let g^1, g^2 be two drivers that are independent of y, satisfying the standard conditions, progressive measurability in s and such that there exists a continuous and increasing $k^i : [0, +\infty) \to [0, +\infty)$ with $k^i(0) = 0$ such that (for $i = 1, 2$)*

$$|g^i(t, s, z) - g^i(\bar{t}, s, z)| \leq k^i(|t - \bar{t}|) \quad \forall t, \bar{t}, s, z. \tag{34}$$

If $g^1 \leq g^2$ and $g^i_z(t, s, z)$ exist and are bounded, then for any $\eta^i_t \in L^2(0, T; \mathbb{R}^n)$ with continuous paths and such that $\eta^1_t \leq \eta^2_t$ a.s., for any t, it holds that

$$Y^1_t \leq Y^2_t \quad a.s., \forall t.$$

A.3 Time–Delayed BSDE

As a further preparation for Appendix B, we provide a short overview of time–delayed (TD) BSDEs, introduced by Delong and Imkeller (2010). In our case, the time–delayed aspect is only discussed in the volatility component Z, that is

$$Y_t = X + \int_t^T g(s, Y_s, \{Z_s\})ds - \int_t^T Z_s dB_s, \tag{35}$$

where $\{Z_s\} = \{Z_{s+u} : u \in [-T, 0]\}$ and, by convention, $Z_v = 0$ for $v < 0$. The driver

$$g : \Omega \times [0, T] \times \mathbb{R} \times L^2(0, T; \mathbb{R}^m) \to \mathbb{R}$$

in z now also depends on time-delayed values of (Z_s). As in the case for BSDEs, a solution is a pair $(Y_t, Z_t)_{t \in [0,T]} \in L^2(0, T; \mathbb{R}) \times L^2(0, T; \mathbb{R}^m)$ satisfying (35).

The notion of a linear TD–BSDE then corresponds to a driver of the form

$$g(s, Y_s, \{Z_s\}) = \int_{-T}^0 \theta_{s+u} Z_{s+u} \mathrm{d}u$$

As shown by Delong and Imkeller (2010), the existence and uniqueness of a solution are guaranteed when the generators g are Lipschitz with sufficiently small Lipschitz constants or the time-horizon is also sufficiently small or when g is independent of y and some further assumptions are fulfilled.

B Time–Delayed BSVIE

In the following, we introduce the new class of time–delayed Backward Stochastic Volterra Integral Equations (TD–BSVIE). Loosely speaking, this is a hybrid form of a BSVIE (33) and a time–delayed BSDE (35). To keep the exposition we omit to consider the time–delayed dependency of Y. Note that this special case is sufficient for applications via EMM–strings.

Fix a terminal value $X \in L_T$. A TD–BSVIE is of the following form

$$Y_t = X + \int_t^T g(t, s, Y_s, \{Z(t,s)\})ds - \int_t^T Z(t,s)dB_s, \tag{36}$$

where the driver g is now

$$g : [0,T] \times [0,T] \times \mathbb{R}^n \times L^2\left(0,T; L^2_{\mathcal{F}}(0,T)\right) \times \Omega \to \mathbb{R}^n$$

and $\{Z(t,s)\} = \{Z(s+u, s);\, u \in [(t-s), 0]\}$.

As also noticed previously, the case of a payoff stream $\{x_s\}_{s\in[0,T]}$, instead of a terminal condition X, can be obtained by replacing the driver g in (36) with $\bar{g}(t,s,y,\{z\}) = g(t,s,y,\{z\}) + x_s$ and taking $X = 0$.

In the present application to asset pricing via EMM–strings, we restrict our attention to a linear driver, which is given by

$$g(t,s,y,\mathtt{z},\omega) = g\Big(t,s,Y_s,\{Z(t,s)\}\Big) = \int_{-T}^0 \theta(s+u,s)Z(s+u,s)\mathrm{d}u + Y_s + x_s \tag{37}$$

where $\{x_t\}_{t\in[0,T]} \in L^2(0,T;\mathbb{R}^n)$. For the application to pricing via EMM–strings we assume $\theta(v,s) = 0$ for $v < t$. In this way the time t parameter enters (37) and the driver reads as follows

$$g\Big(t,s,Y_s,\{Z(t,s)\}\Big) = \int_{t-s}^0 \theta(s+u,s)Z(s+u,s)\mathrm{d}u + Y_s + x_s. \tag{38}$$

Note that Y_s enters in neither a Volterra nor a time–delayed manner.

Definition 3 *We say that an adapted solution of the time–delayed BSVIE is a pair of processes* $(Y,Z) \in L^2(0,T;\mathbb{R}^n) \times L^2(0,T;L^2_{\mathcal{F}}(0,T))$ *satisfying* (36).

Lemma 1 *Let* $\theta(t,s)$*, with* $t < s$*, satisfy Assumption 1 and let* T *be sufficiently small. Let* $X = 0$ *and* g *be as in* (37)*. Then there is a unique solution to the corresponding TD–BSVIE in* (36)*.*

Proof: For any pair of processes $(\mathtt{y},\mathtt{z}) \in L^2(0,T;\mathbb{R}^n) \times L^2(0,T;L^2_{\mathcal{F}}(0,T))$, let

$$\eta_t = \int_t^T g\Big(t,s,\mathtt{y}_t,\{\mathtt{z}(t,s)\}\Big)\mathrm{d}s, \quad t \in [0,T].$$

By the assumptions on g and θ, we have

$$\begin{aligned} & E\left[\int_0^T \eta_s \mathrm{d}s\right] \\ = \; & E\left[\int_0^T \int_t^T g\Big(s,t,\mathsf{y}_t,\{\mathsf{z}(t,s)\}\Big)\mathrm{d}s\right] \\ \le \; & kE\left[\int_0^T |x_s|^2\mathrm{d}s + \int_0^T |\mathsf{y}_s|^2 ds + T\int_0^T\int_0^T |\mathsf{z}(t,s)|^2\mathrm{d}s\right] \\ = \; & k'E\left[\int_0^T |x_s|^2 + |\mathsf{y}_s|^2 ds + \int_0^T\int_0^T |\mathsf{z}(t,s)|^2\mathrm{d}s\right] \end{aligned}$$

for some $k, k' > 0$. We therefore obtain $\eta \in L^2(0,T;\mathbb{R}^n)$, apply Theorem 2.2. of Yong (2006) and obtain the unique existence of the following preliminary TD–BSVIE:

$$Y_t = \int_t^T g(t,s,\mathsf{y}_t,\{\mathsf{z}(t,s)\})\mathrm{d}s - \int_t^T Z(t,s)\mathrm{d}B_s.$$

We take another pair $(\mathsf{y}',\mathsf{z}') \in L^2(0,T;\mathbb{R}^n) \times L^2(0,T;L^2_{\mathcal{F}}(0,T))$ and obtain another preliminary solution (Y',Z') when (y,z) is replaced by $(\mathsf{y}',\mathsf{z}')$. This induces a mapping $\Phi(\mathsf{y},\mathsf{z}) = (Y,Z)$.

Another application of Theorem 2.2. (the estimate part) of Yong (2006) gives us

$$\begin{aligned} & E\left[\int_u^T |Y_t - Y_t'|^2\mathrm{d}t\right] + E\left[\int_u^T\int_u^T |Z(t,s) - Z'(t,s)|^2\mathrm{d}t\mathrm{d}s\right] \\ \le \; & cE\left[\int_u^T \left|\int_u^T g(t,s,\mathsf{y}_s,\{\mathsf{z}(t,s)\}) - g(t,s,\mathsf{y}_s',\{\mathsf{z}'(t,s)\})\mathrm{d}s\right|^2 \mathrm{d}t\right] \\ \le \; & c(T-u)^2E\left[\int_u^T |\mathsf{y}_t' - \mathsf{y}_t|^2\,\mathrm{d}t\right] + ck(T-u)E\left[\int_u^T\int_u^T |\mathsf{z}(t,s) - \mathsf{z}'(t,s)|^2\,\mathrm{d}s\mathrm{d}t\right] \end{aligned}$$

for some $c, k > 0$. If u is sufficiently close to T, we can see that Φ is a contraction from $L^2(0,T;\mathbb{R}^n) \times L^2(0,T;L^2_{\mathcal{F}}(0,T))$ onto itself. Hence, there is a unique fixed–point, which is the unique solution of the TD–BSVIE on $[u,T]$. Choosing T such that $T-u>0$ is small, the result follows.

□

Remark 2 To prove the unique existence of a linear TD–BSVIE on an arbitrary interval $[0,T]$, it seems possible to repeat steps 2, 3 and 4 in the proof of Theorem 3.7 of Yong (2008). We give a rough description below:

- step 2 determines the value of Z on $[u,T]\times[s,u]$ with $s<u$ and sufficiently close to s.
- step 3 determines the value of Z on $[s,u]\times[u,T]$, by the unique solution of a forward equation.
- step 4 repeats step 1 to show the unique existence on the interval $[s,u]$ and then on $[s,T]$; we can then continue by induction and obtain unique existence of $[0,T]$.

C Proofs of Section 3

Proof of Proposition 1 Given an EMM–string $\{\mathbb{Q}^r\}_{r\in[0,T]}$ (and the corresponding θ), p_t^* can be written as in (11) and (15) by means of a suitable σ by proceeding similarly as around equation (5). By Theorem 3.2 of Yong (2006), the solutions of the BSVIEs are unique.

We start by proving the properties for random variables and then we pass to streams. Since $p_t^*(X)=X+\int_t^T-\theta(t,s)\sigma(t,s)ds+\int_t^T\sigma(t,s)dB_s$, the pricing scheme satisfies a BSVIE with a linear driver $g(t,s,y,z)=\theta(t,s)z$. g satisfies the Lipschitz condition and, by assumption, the regularity properties in time and thus all the hypotheses of comparison theorem in Proposition 3.3 of Wang and Yong (2015) (see also Theorem 3). Monotonicity follows immediately.

Homogeneity can be checked directly. By definition, indeed, $p_t^*(X)$ and $p_t^*(\lambda X)$ satisfy

$$\begin{aligned} p_t^*(X) &= X+\int_t^T-\theta(t,s)\sigma^X(t,s)ds+\int_t^T\sigma^X(t,s)dB_s \\ p_t^*(\lambda X) &= \lambda X+\int_t^T-\theta(t,s)\sigma^{\lambda X}(t,s)ds+\int_t^T\sigma^{\lambda X}(t,s)dB_s, \end{aligned}$$

respectively. By multiplying the first BSVIE by λ and by the existence and uniqueness of the solution, one immediately has that $p_t^*(\lambda X)=\lambda p_t^*(X)$ for any $\lambda\in\mathbb{R}$.

Conditional homogeneity can be proved similarly to homogeneity (replacing λ with $(\Lambda_t)_t\in L^2(0,T;\mathbb{R}^m)$). The existence and uniqueness of a solution hold indeed true for any terminal condition $\eta_t\in L^2(0,T;\mathbb{R}^m)$ (in our case corresponding to $(\Lambda_t X)_t\in L^2(0,T;\mathbb{R}^m)$). Static linearity: $p_t^*(X)$, $p_t^*(Y)$ and

$p_t^*(X+Y)$ satisfy

$$\begin{aligned} p_t^*(X) &= X + \int_t^T -\theta(t,s)\sigma^X(t,s)ds + \int_t^T \sigma^X(t,s)dB_s \\ p_t^*(Y) &= Y + \int_t^T -\theta(t,s)\sigma^Y(t,s)ds + \int_t^T \sigma^Y(t,s)dB_s \\ p_t^*(X+Y) &= X+Y + \int_t^T -\theta(t,s)\sigma^{X+Y}(t,s)ds + \int_t^T \sigma^{X+Y}(t,s)dB_s, \end{aligned}$$

respectively. Since (by summing the first two BSVIEs)

$$p_t^*(X)+p_t^*(Y) = X+Y+\int_t^T -\theta(t,s)(\sigma^X(t,s)+\sigma^Y(t,s))ds+\int_t^T (\sigma^X(t,s)+\sigma^Y(t,s))dB_s,$$

again with the existence and uniqueness of the solution it follows that $p_t^*(X+Y) = p_t^*(X) + p_t^*(Y)$.

The same properties can be proved similarly to the pricing of streams $\{x_s\}_{s\in[t,T]}$ by replacing X and Y with $\int_t^T x_s ds$ and $\int_t^T y_s ds$. The arguments of BSVIEs applied above indeed hold for any $\mathcal{F}_T$–measurable random variable.

Proof of Proposition 2 Since there are $u, v \in [0,T]$ such that $\mathbb{Q}^u \neq \mathbb{Q}^v$, there is a payoff $C^{u,v}$ such that

$$\mathbb{E}_v^{\mathbb{Q}^v}[C^{u,v}] \neq \mathbb{E}_v^{\mathbb{Q}^u}[C^{u,v}].$$

In particular, we may find a payoff $C^{u,v}$ such that

$$\mathbb{E}_v^{\mathbb{Q}^v}[C^{u,v}] \leq \mathbb{E}_v^{\mathbb{Q}^u}[C^{u,v}], \quad \mathbb{Q}^u\text{-a.s.}$$

and such that the strict inequality holds with a strictly positive probability (evaluated in terms of $\mathbb{Q}^u$ or, equivalently, $\mathbb{P}$).

Assume $u < v$, such that the following computation proves the claim:

$$\begin{aligned} p_u^*(C^{u,v}) &= \mathbb{E}_u^{\mathbb{Q}^u}[C^{u,v}] = \mathbb{E}_u^{\mathbb{Q}^u}\Big[\underbrace{\mathbb{E}_v^{\mathbb{Q}^u}[C^{u,v}]}_{>\mathbb{E}_v^{\mathbb{Q}^v}[C^{u,v}]}\Big] \\ &\neq \mathbb{E}_u^{\mathbb{Q}^u}[p_v^*(C^{u,v})] \\ &= p_u^*\big(p_v^*(C^{u,v})\big), \end{aligned}$$

where we employ the time-consistency under the fixed prior $\mathbb{Q}^u$ in the second equation.

Proof of Corollary 1 This follows directly from Proposition 2, since the EMM–string yields time-inconsistent pricing. Hence we cannot identify the string by an EMM, since each EMM yields time-consistent pricing.

Proof of Theorem 1 This follows directly from Lemma 1 in Appendix B. By the linearity of g in z, the Lipschitz constant k is determined by the range of the SRs assumed - by Assumption 1 - bounded).

D Proof of Section 4

Proof of Proposition 3 1. Concavity, continuity and monotonicity of U_t can be proved similarly to Theorems 3.4-3.5 of Aliprantis (1997).

Time-inconsistency is feasible due to Proposition 2 and the possible time-inconsistency of the solution of a BSVIE.

To show differentiability and the form of the Gateaux derivative, note that

$$\begin{aligned}\int_0^T \mathbb{E}^{\mathbb{P}_t}[u(c_t)]dt &= \int_0^T \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}^{\vartheta t}u(c_t)\right]dt = \mathbb{E}^{\mathbb{P}}\left[\int_0^T \mathcal{E}^{\vartheta t}u(c_t)dt\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T u(t,c_t)dt\right],\end{aligned}$$

where $u(t,c) = \mathcal{E}^{\vartheta t}u(c)$ is a state dependent utility index and $\mathcal{E}^{\vartheta t} = \mathcal{E}\left(\vartheta(t,\cdot)_{\bullet}B\right)$. Using Fubini theorem for conditional expectations and following the same arguments as in Section 3, we obtain

$$\begin{aligned}U_t(c) &= \mathbb{E}_t^{\mathbb{P}_t}\left[\int_t^T u(c_s)ds\right] \\ &= \int_t^T \mathbb{E}_t^{\mathbb{P}_t}\left[u(c_s)\right]ds = \int_t^T \mathbb{E}_t^{\mathbb{P}}\left[\mathcal{E}^{\vartheta t}u(c_s)\right]ds \\ &= \int_t^T \mathbb{E}_t^{\mathbb{P}}\left[u(t,s,c_s)\right]ds = \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T u(t,s,c_s)ds\right], \qquad (39)\end{aligned}$$

where $u(t,s,c) = \mathcal{E}^{\vartheta t}u(c_s)$.

Remember that the Gateaux derivative in the direction h (or gradient of $U_t(c)$) is defined as $\nabla U_t^h(c) \triangleq \lim_{\varepsilon\to 0}\frac{U_t(c+\varepsilon h)-U_t(c)}{\varepsilon}$ (if one exists). Applying equality (39) and standard arguments, including dominated convergence, gives us

$$\nabla U_t^h(c) = \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T u'(t,s,c_s)h_s ds\right], \qquad (40)$$

where $u'(t,s,c)$ denotes the partial derivative of $u(t,s,c)$ in the c-component. Proceeding as in (39), it follows that

$$\nabla U_t^h(c) = \int_t^T \mathbb{E}_t^{\mathbb{P}_t}\left[u'(c_s)h_s\right]ds.$$

2. Finally, (29) can be checked by proceeding similarly to before.

Proof of Theorem 2 *Sufficiency of the optimality condition follows from the concavity of the utility functional of Proposition 3. Suppose for the first part of the proof that the asset pays no intermediate dividends.*

Consider for the optimal pair $(e, 0)$ the following deviation from this no–trade strategy. Let $\Delta > 0$ be an arbitrary small time length and $\varepsilon > 0$.

- *At time $r \geq 0$ buy ε assets.*
- *On $[r, r + \Delta]$ consume $\frac{\varepsilon S_r}{\Delta}$ less.*
- *Sell the asset at time $t > r + \Delta$ with revenue εS_t.*
- *On $[t, t + \Delta]$ consume $\frac{\varepsilon S_t}{\Delta}$ more.*

Denote the resulting consumption rate by $\bar{c}$, which agrees with e outside the $[r, r + \Delta] \cup [t, t + \Delta]$. Since e_t is maximal we have $U(e) \geq U(\bar{c})$:

$$\mathbb{E}_r^{\mathbb{P}_r}\left[\int_r^{r+\Delta}\left[u\Big(c_s - \frac{\varepsilon S_r}{\Delta}\Big) - u(c_s)\right]ds + \int_t^{t+\Delta}\left[u\Big(c_s + \frac{\varepsilon S_t}{\Delta}\Big) - u(c_s)\right]ds\right] \leq 0$$

Dividing by ε and letting $\varepsilon \to 0$ result in

$$\mathbb{E}_r^{\mathbb{P}_r}\left[-\frac{S_r}{\Delta}\int_r^{r+\Delta} u'(c_s)ds + \frac{S_t}{\Delta}\int_t^{t+\Delta} u'(c_s)ds\right] \leq 0.$$

Now letting $\Delta \to 0$ yields

$$\frac{1}{u'(c_r)}\mathbb{E}_r^{\mathbb{P}_r}\left[u'(c_t)S_t\right] \leq S_r$$

By changing the order of buying/selling and consuming more/less with respect to yields, we obtain the reversed inequality:

$$S_r \leq \frac{1}{u'(c_r)}\mathbb{E}_r^{\mathbb{P}_r}\left[u'(c_t)S_t\right]$$

If we repeat the arguments with an asset paying dividends, we have to incorporate the dividend stream into $[r, t]$:

$$S_r = \frac{1}{u'(c_r)}\mathbb{E}_r^{\mathbb{P}_r}\left[\int_r^t u'(c_s)D_s ds + S_t u'(c_t)\right].$$

The thesis follows with $t = T$ and $r = t$ and by taking $S_T = 0 = D_T$ into account.

It remains to show that the conditional formulation solves the linear BSVIE (25). This can be achieved by proceeding as previously (see Sections 2 and 3) by means of the SR.

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