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Yangwei Song (HU Berlin)

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Spokesperson: Prof. Georg Weizsäcker, Ph.D., Humboldt University Berlin, 10117 Berlin, Germany
info@rationality-and-competition.de

Approximate Bayesian Implementation and Exact Maxmin Implementation: An Equivalence*

Yangwei Song†

University of Colorado Boulder

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This paper provides a micro-foundation for approximate incentive compatibility using ambiguity aversion. In particular, we propose a novel notion of approximate interim incentive compatibility, *approximate local incentive compatibility*, and establish an equivalence between approximate local incentive compatibility in a Bayesian environment and exact interim incentive compatibility in the presence of a small degree of ambiguity. We then apply our result to the implementation of efficient allocations. In particular, we identify three economic settings—including ones in which approximately efficient allocations are implementable, ones in which agents are informationally small, and large double auctions—in which efficient allocations are approximately locally implementable when agents are Bayesian. Applying our result to those settings, we conclude that efficient allocations are exactly implementable when agents perceive a small degree of ambiguity.

Keywords: approximate local incentive compatibility, ambiguity aversion, efficiency, informational size, modified VCG mechanism, double auction.

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†Department of Economics, University of Colorado Boulder, USA. E-mail: yangwei.song@colorado.edu.

1 Introduction

The presence of asymmetrically informed agents greatly restricts the set of implementable social choice rules. For example, in a variety of mechanism design contexts, it is impossible to allocate resources in a way which simultaneously gives agents perfect incentives to be truthful and ensures efficient outcomes.[1] In light of the impossibility results, an extensive literature examines if it is possible to maintain some attractive properties a mechanism can satisfy by imposing a weaker incentive criterion—*approximate incentive compatibility*.[2] This literature assumes agents do not misreport when there is a small utility gain. However, if agents are rational, it is not clear why they content themselves with approximately optimal choices. The literature usually takes a "reduced form" approach without explicitly modeling why agents would forgo a small utility gain.

This paper proposes a novel micro-foundation for approximate incentive compatibility using ambiguity aversion. In particular, we propose a notion of approximate interim incentive compatibility, which we call *approximate local incentive compatibility*, and establish an equivalence between approximate local incentive compatibility in a Bayesian environment and exact interim incentive compatibility in the presence of *a small degree of ambiguity*. Following Gilboa and Schmeidler [20], we model ambiguity aversion using the maxmin expected utility model: each agent faces ambiguity about the distribution of the other agents' types and evaluates each action according to the worst-case expected payoff over all possible distributions. Small ambiguity can thus be captured by the size of the set of priors of each agent.

Before defining our notion, recall first the standard notion of approximate incentive compatibility: a mechanism is ε-incentive compatible (ε-IC) if the gain from any lie is at most ε. This notion is normally viewed as only a slight weakening of incentive compatibility when ε is close to zero. We show in Section 4, however,

[1]There are numerous impossibility results in the literature, including Myerson and Satterthwaite [40], Mailath and Postlewaite [34], and Jehiel and Moldovanu [28].
[2]For instance, see Roberts and Postlewaite [42], McLean and Postlewaite [37], and Che and Tercieux [11].

that since the permitted gain from lying can be arbitrarily large relative to the size of the lie, ε-IC does not impose any restriction on local incentive constraints.

In order to restrict local incentive compatibility, we propose the following notion: a mechanism is *ε-locally incentive compatible (ε-LIC)* if the gain from any lie is at most ε times the size of the lie, as measured by the distance between the true type and the reported type. Sometimes approximate incentive compatibility is justified by the cost of lying. Then ε-IC is appropriate if there is a fixed cost of lying, while ε-LIC is appropriate if the cost of lying increases in the size of the lie.[3] While our notion is stronger, many of the mechanisms that are known to be ε-IC are indeed ε-LIC. For example, we find that the competitive equilibrium mechanism is ε-LIC in large double auctions.

Our main result is to show that a social choice rule is *approximately locally implementable* in a Bayesian environment if and only if it is exactly implementable in the presence of small ambiguity. A social choice rule p is ε-locally implementable if there exists a transfer scheme x such that the mechanism (p, x) is ε-LIC and further satisfies two regularity conditions. In a single object allocation problem with private values à la Myerson [39], the regularity conditions reduce to the derivative of each agent's indirect utility function with respect to his type lying in $[0, 1]$.[4]

The basic intuition behind our result is that the presence of ambiguity aversion can weaken incentive constraints. In particular, we show that under a particular class of transfer schemes, ambiguity aversion has no bite when agents report truthfully since an agent's interim payoff from truth-telling does not depend on the distribution of other agents' signals. In contrast, misreporting becomes less attractive because the expected gains are evaluated according to the worst-case beliefs. Therefore, as opposed to the exogenous weakening of incentive constraints un-

[3]Such type of lying costs has been studied in the literature. For example, Kartik [29] studies a model of strategic communication where the informed sender has a lying cost that is proportional to the size of the lie. Gneezy et al. [21] present a model of lying costs and argue that lying cost depends on the size of the lie.

[4]The regularity conditions are trivially satisfied by incentive compatible mechanisms since, by the envelope theorem, the derivative of an agent's indirect utility function associated with any incentive compatible mechanism is his expected probability of obtaining the object, which obviously lies in $[0, 1]$.

der approximate incentive compatibility, the realization of weaker incentive constraints arises endogenously as a result of ambiguity aversion.

Our equivalence result is relevant for two reasons. First, it provides a novel micro-foundation for the use of approximate incentive compatibility, namely, ambiguity aversion, other than the usual justification (e.g., lying costs and bounded rationality). There is both experimental and empirical evidence showing that due to lack of knowledge about the environment, agents typically perceive some degree of ambiguity and moreover, agents desire strategies that are robust to their ambiguity.[5] In addition, Bose and Renou [7] show that ambiguity can be created by a planner deliberately through ambiguous mechanisms,[6] which can be an advantage of this justification since lying costs (or bounded rationality) are relatively more difficult to measure or generate. Second, our equivalence result suggests that we may use ambiguity as a tool to study Bayesian games since it may be more challenging to construct a desirable approximate equilibrium in a Bayesian game than to construct an exact equilibrium in a game with ambiguity.

The leading application for our result is the implementation of efficient allocations. One implication of our result is that the presence of a small degree of ambiguity suffices for efficient implementation whenever efficient allocations are approximately locally implementable in a Bayesian environment. We present three such settings that are widely studied in mechanism design.

First, we show that whenever an approximately efficient social choice rule is Bayesian implementable, the fully efficient social choice rule is approximately locally implementable.[7] A large literature finds that approximately efficient allocations are attainable in a wide array of settings and, hence, our result can be immediately applied.[8] Moreover, this result may be interesting in its own right because

[5]Examples of experimental and empirical evidence include Ellsberg [17], Halevy [25], and Aryal et al. [2].

[6]Song [47] and Kocherlakota and Song [32] present examples that illustrate how to generate a sufficient amount of ambiguity for efficient implementation using their approach.

[7]Even though we restrict our attention to the implementation of efficient and approximate efficient social choice rules, our result extends straightforwardly to the implementation of any two social choice rules that are "close" to each other. We discuss this issue in detail in Section 7.

[8]For instance, see Rustichini et al. [43], Jackson and Manelli [27], and Satterthwaite and Williams

it establishes a connection between the implementation of an approximately efficient social choice rule and the approximate implementation of the efficient social choice rule which are shown to co-exist in many mechanism design settings (e.g., large double auctions[9]).

Our result also applies to settings in which agents are informationally small.[10] Intuitively, agents are informationally small when the incremental information of any single agent given the information of everyone else is small. We present two relevant settings in which the informational size of the agents is small. One instance in which informational smallness arises naturally is when the number of agents is large. More specifically, we consider large double auctions where the relative influence of a single trader's information on the total demand and supply is limited. Gresik and Satterthwaite [22] show that in double auctions, there is no budget-balanced mechanism which implements efficient allocations. We verify that efficient allocations are approximately locally implementable by the competitive equilibrium mechanism. Combining this observation with our main result yields that given any non-trivial degree of ambiguity, efficient allocations are implementable in a budget-feasible way as long as the market is sufficiently large.

In an economy with interdependent valuations and multi-dimensional signals, agents are informationally small if an agent's private information has a small marginal effect on other agents' valuations. We find that in such settings, the efficient social choice rule is approximately locally implementable by a modified Vickrey-Clarke-Groves (VCG) transfer scheme.

The paper is organized as follows. In Section 2, we introduce a simple framework where agents have one-dimensional signals and private values. In Section 3, we define full insurance mechanisms which play a crucial role in our analysis. Section 4 contains our main equivalence result. We then apply our result to the implementation of efficient allocations in Section 5. In Section 6, we show how

[46].

[9]For example, see Carroll [9].

[10]Informational smallness has been studied by Gul and Postlewaite [23] and McLean and Postlewaite [35, 36, 37, 38].

the ideas extend to more general frameworks. We conclude with discussion and related literature in the final section.

2 The Model

Information structure. There are N agents, indexed by $i \in \mathcal{I} \equiv \{1, ..., N\}$. They have to make a collective choice k from a set $\mathcal{K} \equiv \{1, ..., K\}$ of possible social alternatives. Each agent i observes a one-dimensional signal s^i that is drawn from $S^i = [0, 1]$. Let $S \equiv \prod_{i=1}^{N} S^i$ with s as generic element and let $S^{-i} \equiv \prod_{j \neq i} S^j$ with s^{-i} as generic element. Agent i's value in social alternative k is given by $v_k^i(s^i)$.

Let $\mathcal{K}_0^i$ be the set of alternatives on which agent i's own information has no effect, that is,

$$\mathcal{K}_0^i \equiv \{k \in \mathcal{K} | v_k^i(s^i) - v_k^i(t^i) = 0, \forall s^i, t^i \in S^i\}.$$

Let $\mathcal{K}^i \equiv \mathcal{K} \setminus \mathcal{K}_0^i$. To avoid triviality, we assume that $\mathcal{K}^i \neq \varnothing$ for all i. In words, there exists at least one alternative $k \in \mathcal{K}$ such that agent i's value from k depends on his own information. We further assume that v_k^i is twice differentiable and $\frac{dv_k^i(s^i)}{ds^i} > 0$ for all $k \in \mathcal{K}^i$ and $s^i \in S^i$.

Example 1. Consider a canonical mechanism design problem—a single object allocation problem. Let $\mathcal{K} = \{1, ..., N\}$, where $k = i$ represents the object is allocated to agent i. Suppose there is no allocative externality: $v_j^i(s^i) = 0$ for all $s^i \in S^i$, $i \in \mathcal{I}$, and $j \neq i$. Then $\mathcal{K}^i = \{i\}$ for all $i \in \mathcal{I}$.

We assume agents have quasilinear preferences: if alternative k is chosen and agent i obtains a transfer x^i, then his utility is given by $v_k^i(s^i) + x^i$.

Mechanisms. A **social choice rule** (SCR) is a function $p : \mathcal{S} \to \mathbb{R}^K$ such that for every $s \in \mathcal{S}$, $p_k(s) \geq 0$ and $\sum_{k \in \mathcal{K}} p_k(s) = 1$. A **transfer scheme** is a function $x : \mathcal{S} \to \mathbb{R}^N$. A **direct revelation mechanism** is a pair (p, x) where p is a SCR and x is a transfer scheme. For reported signals s, the term $p_k(s)$ is the probability that alternative k is chosen and $x^i(s)$ represents the transfer to agent i.

A SCR p is **(ex post) efficient** if

$$p_k(s) > 0 \Rightarrow k \in \operatorname*{argmax}_{\hat{k}\in\mathcal{K}} \sum_{i=1}^{N} v^i_{\hat{k}}(s^i) \quad \forall s \in S.$$

We use p^* to denote an efficient SCR. For simplicity, we restrict our attention to deterministic SCRs.[11]

Interim utilities. Let Σ^{-i} be the Borel algebra on S^{-i} and $\mathcal{F}^i$ be a set of probability measures on (S^{-i}, Σ^{-i}). This set represents agent i's beliefs about the other agents' signals. A key assumption here is agent i's set of beliefs $\mathcal{F}^i$ is independent of the realization of his signal, which is an analogue of the "independence of signals" assumption from Bayesian settings. We assume that $\mathcal{F}^i$ is weak* compact and convex.

Given a direct mechanism (p, x), agent i's **interim utility** from reporting t^i when his signal is s^i and everyone else reports truthfully is

$$u^i_{(p,x)}(t^i, s^i) \equiv \min_{F^i\in\mathcal{F}^i} \int_{\mathcal{S}^{-i}} \Big(\sum_{k\in\mathcal{K}} p_k(t^i, s^{-i}) v^i_k(s^i) + x^i(t^i, s^{-i}) \Big) dF^i(s^{-i}).$$

The function $\mu^i_{(p,x)} : \mathcal{S}^i \to \mathbb{R}$ defined by $\mu^i_{(p,x)}(s^i) \equiv u^i_{(p,x)}(s^i, s^i)$, is called agent i's **indirect utility function** associated with (p, x).

A direct mechanism (p, x) is **(interim) incentive compatible** (IC) if

$$\mu^i_{(p,x)}(s^i) = u^i_{(p,x)}(s^i, s^i) \geq u^i_{(p,x)}(t^i, s^i) \quad \forall s^i, t^i \in S^i, \forall i \in \mathcal{I}.^{12}$$

A SCR p is **implementable** if there exists a transfer scheme x such that the direct mechanism (p, x) is incentive compatible.

Environment. An environment is a tuple $\langle \mathcal{I}, \mathcal{K}, \big((v^i_k)_{k\in\mathcal{K}}, S^i, \mathcal{F}^i\big)_{i\in\mathcal{I}} \rangle$. We assume that the environment is common knowledge, but the realizations of the signals are private information.

We focus on two special classes of environments. One is a **Bayesian environ-**

[11] A SCR p is deterministic if $p_k(s) = 1$ or 0 for all $s \in S$ and $k \in \mathcal{K}$. All our results extend straightforwardly to SCRs that are deterministic almost everywhere. A possible extension to random SCRs is discussed in Section 7.

[12] Observe that the definition of interim incentive compatibility only invokes pure strategies. This is without loss of generality if either of the following assumptions holds: (i) agents cannot reduce ambiguity by randomizing ex ante; (ii) agents cannot commit to the results of their randomizations. For a more detailed discussion about these assumptions see Saito [44] and Ke and Zhang [30].

ment, denoted by $\mathbf{E}^B$, in which each agent i's set of beliefs is a singleton, $\mathcal{F}^i = \{G^i\}$. We assume that $G^i \in \Delta(S^{-i})$ has a continuous density function $g^i(s^{-i}) > 0$ for all $s^{-i} \in S^{-i}$ and $i \in \mathcal{I}$.[13] For each $\delta \in (0, 1]$, an environment is called a **δ-ambiguity environment**, denoted by $\mathbf{E}^\delta$, if for every $i \in \mathcal{I}$, there exists $H^i \in \Delta(S^{-i})$ such that agent i's set of beliefs $\mathcal{F}^i \supseteq B_\delta(H^i)$, where $B_\delta(H^i) \equiv \{F^i \in \Delta(S^{-i}) | d(F^i, H^i) \leq \delta\}$ and d is the Prokhorov metric.[14] In both environments, we maintain the assumption that each agent's set of beliefs is independent of the realization of his signal.

A pair of environments $(\mathbf{E}^B, \mathbf{E}^\delta)$, where agent i's belief is G^i in $\mathbf{E}^B$ and agent i's set of beliefs is $\mathcal{F}^i$ in $\mathbf{E}^\delta$, is **a corresponding pair of environments** if $B_\delta(G^i) \subseteq \mathcal{F}^i$ for all $i \in \mathcal{I}$ and other components of the two environments are identical.

3 Full Insurance Mechanisms

In this section, we introduce a class of mechanisms, full insurance mechanisms, which is fundamental to our results.[15]

Definition 1. Given a profile of functions $\mu^i : S^i \to \mathbb{R}$, a **full insurance mechanism with** $\{\mu^i\}_{i \in \mathcal{I}}$ is a pair (p, x_F) where p is a SCR and x_F is given by

$$x_F^i(s) \equiv \mu^i(s^i) - \sum_{k \in \mathcal{K}} p_k(s) v_k^i(s^i) \quad \forall i \in \mathcal{I}, \forall s \in S. \tag{1}$$

Two features of full insurance mechanisms greatly facilitate our analysis. An immediate observation from the construction is that if everyone reports truthfully, the ex post utility of agent i who receives signal s^i is independent of the other agents' reports and equal to $\mu^i(s^i)$. Thus, each agent i's interim utility from truth-telling under the full insurance mechanism (p, x_F) with $\{\mu^i\}_{i \in \mathcal{I}}$ is indeed $\mu^i(s^i)$, which is irrespective of his beliefs and, consequently, each agent is fully insured against ambiguity in the interim stage. In contrast, if an agent misreports, his interim utility is evaluated according to a worst-case belief and, hence, interim incentive constraints are weakened.

[13] In a Bayesian environment, we do not require that agents' beliefs be derived from a common prior over S. Moreover, G^i does not need to be interpreted as the true distribution from which s^{-i} are drawn.

[14] We use Prokhorov metric to measure the distance between probability measures and the definition is provided in Appendix A. This particular choice of metric is not crucial for our results.

[15] The class of full insurance mechanisms was first introduced by Bose et al. [8].

Second, this class of mechanisms is robust to ambiguity in the sense that if (p, x_F) is incentive compatible in an environment, with or without ambiguity, then it remains incentive compatible in all environments where agents are more ambiguity averse.[16] Intuitively, this is because each agent's interim utility when he reports truthfully remains the same, but his interim utility when he misreports is lower under a larger set of beliefs. Since truthful revelation is optimal in the original environment, it remains optimal when agents are in fact more ambiguity averse.

4 Approximate Bayesian Implementation and Exact Maxmin Implementation

In this section, we propose a novel notion of approximate interim incentive compatibility and provide conditions under which it is equivalent to exact incentive compatibility in environments with small ambiguity.

4.1 Standard Notion of ε-Incentive Compatibility

We start by presenting the standard notion of approximate incentive compatibility.

Definition 2. For any $\varepsilon > 0$, a mechanism (p, x) is **ε-incentive compatible** (ε-IC) if

$$\mu^i_{(p,x)}(s^i) \geq u^i_{(p,x)}(t^i, s^i) - \varepsilon \quad \forall s^i, t^i \in S^i, \forall i \in \mathcal{I}.$$

A mechanism is approximately incentive compatible if truthful revelation is approximately optimal in the sense that no agent can achieve more than a small utility gain by misreporting. As explained in Section 3, ambiguity aversion weakens agents' incentives to misreport under full insurance mechanisms. One would then expect certain equivalence between approximate Bayesian incentive compatibility and exact incentive compatibility with small ambiguity since incentive constraints are weakened in both cases. However, the next example demonstrates that this intuition is not correct.

[16] Following Ghirardato and Marinacci [19], we say that the agent with the set of priors $\mathcal{F}$ is more ambiguity averse than the agent with the set of priors $\mathcal{F}'$ if $\mathcal{F} \supseteq \mathcal{F}'$.

Example 2. Consider the single object allocation problem from Example 1. Suppose $v_i^i(s^i) = s^i$ for all i. Fix $0 < \varepsilon < 1$. Myerson [39] shows that if a SCR p is implementable, then each agent's expected probability of obtaining the object is nondecreasing in his value. Thus, a necessary condition imposed by incentive compatibility on the SCR p is

$$\int p_i(t^i, s^{-i})dG^i \geq \int p_i(s^i, s^{-i})dG^i \quad \forall t^i > s^i, \forall i \in \mathcal{I}. \tag{2}$$

By similar arguments, a necessary condition imposed by ε-IC on the SCR p is

$$\int p_i(t^i, s^{-i})dG^i + \frac{\varepsilon}{t^i - s^i} \geq \int p_i(s^i, s^{-i})dG^i - \frac{\varepsilon}{t^i - s^i} \quad \forall t^i > s^i, \forall i \in \mathcal{I}. \tag{3}$$

This is clearly a weakening of the monotonicity requirement (2), but is too weak locally: when $t^i - s^i < 2\varepsilon$, the inequality (3) is trivially satisfied. This suggests that the standard notion of ε-IC does not impose any restriction locally on the SCR. On the other hand, Lemma 3.1 in Song [47] provides a necessary condition for p to be implementable in an ambiguity environment:

$$\max_{F^i \in \mathcal{F}^i} \int p_i(t^i, s^{-i})dF^i \geq \min_{F^i \in \mathcal{F}^i} \int p_i(s^i, s^{-i})dF^i \quad \forall t^i > s^i, \forall i \in \mathcal{I}. \tag{4}$$

An inspection of inequalities (3) and (4) indicates that the required degree of ambiguity for maxmin implementation could be bounded away from zero since $\frac{\varepsilon}{t^i - s^i}$ can be large even when ε is close to zero. In Appendix A.1, we present an example that provides an explicit lower bound.

4.2 Notion of ε-Local Incentive Compatibility

In order to impose appropriate restrictions on local incentive constraints, we propose a stronger notion of approximate incentive compatibility.

We see from the inequality in (3) that local incentive constraints are unrestricted when the permitted gain from misreporting ε is large relative to the size of the lie. Therefore, it is intuitive to consider the following notion.

Definition 3. For any $\varepsilon > 0$, a mechanism (p, x) is **ε-locally incentive compatible** (ε-LIC) if

$$\mu^i_{(p,x)}(s^i) \geq u^i_{(p,x)}(t^i, s^i) - \varepsilon |s^i - t^i| \quad \forall s^i, t^i \in S^i, \forall i \in \mathcal{I}.$$[17]

This notion of approximate LIC is more restrictive than the standard one: ε-IC allows an agent to forgo an ε gain regardless of the lie, whereas ε-LIC allows an agent to forgo a gain that is proportional to the size of the lie. To see how ε-LIC restricts local incentive constraints, consider again Example 2. Simple manipulation of ε-LIC constraints implies that

$$\int p_i(t^i, s^{-i})dG^i + \varepsilon \geq \int p_i(s^i, s^{-i})dG^i - \varepsilon \quad \forall t^i > s^i, \forall i \in \mathcal{I},$$

which is clearly weaker than the requirement of incentive compatibility (2) while stronger than the requirement of ε-IC (3).

4.3 ε-Local Implementation

We next define a notion of approximate local implementation which imposes two additional restrictions, *monotonicity and boundedness*, on approximately LIC mechanisms.

The definition of monotonicity is standard. A function $\mu : \mathbb{R}^n \to \mathbb{R}$ is **monotone** if it weakly increases in each argument.

We now define ε-boundedness. For any SCR p, $s^i \in S^i$, and $k \in \mathcal{K}$, define

$$A_k(s^i, p) \equiv \{s^{-i} \in S^{-i} | p_k(s^i, s^{-i}) = 1\}.$$

In words, given $s^i \in S^i$, $A_k(s^i, p)$ is the set of other agents' signals such that alternative k is chosen by the SCR p. Hence, $G^i\big(A_k(s^i, p)\big)$ represents agent i's expected probability for alternative k evaluated according to belief G^i.

Definition 4. For any Bayesian environment $\mathbf{E}^B$ and any $\varepsilon \geq 0$, a mechanism (p, x) is ε-**bounded** if for any $i \in \mathcal{I}$ and $s^i < t^i$, we have

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) \leq \sum_{k \in \mathcal{K}} w_k\big(v^i_k(t^i) - v^i_k(s^i)\big), \tag{5}$$

where $\{w_k\}_{k\in\mathcal{K}}$ satisfies (i) $0 \leq w_k \leq G^i\big(A_k(t^i, p)\big) + \varepsilon$ for all k and (ii) $\sum_{k\in\mathcal{K}} w_k \leq 1$.

To understand this definition, assume for simplicity that there exists $k^* \in \mathcal{K}^i$

[17] Our results do not rely on the permitted gain being linear in the size of the lie. This notion of approximate LIC is equivalent to any notion under which the permitted gain is ε times a function $f(|s^i - t^i|)$ that is strictly increasing, Lipschitz continuous, and $f(0) = 0$.

such that $\frac{dv^i_{k^*}(s^i)}{ds^i} \geq 1$. For any ε-LIC mechanism (p, x) and $s^i < t^i$, we have

$$\begin{aligned}\mu^i_{(p,x)}(s^i) &\geq u^i_{(p,x)}(t^i, s^i) - \varepsilon|s^i - t^i| \\ &= \mu^i_{(p,x)}(t^i) + \sum_{k\in\mathcal{K}} G^i\big(A_k(t^i, p)\big)\big(v^i_k(s^i) - v^i_k(t^i)\big) - \varepsilon(t^i - s^i).\end{aligned}$$

Recall that $v^i_k(s^i) - v^i_k(t^i) = 0$ for all $k \in \mathcal{K}^i_0$. Rearranging inequalities above yields (5) by letting $w_k = 0$ for all $k \in \mathcal{K}^i_0$, $w_k = G^i\big(A_k(t^i, p)\big)$ for all $k \in \mathcal{K}^i \setminus \{k^*\}$, and $w_{k^*} = G^i\big(A_{k^*}(t^i, p)\big) + \varepsilon$. Note that the constructed $\{w_k\}_{k\in\mathcal{K}}$ satisfies constraint (i), so constraint (i) is simply an implication of ε-LIC. However, if $\varepsilon > 0$, it is possible that $\sum_{k\in\mathcal{K}} w_k = \sum_{k\in\mathcal{K}^i} G^i\big(A_k(t^i, p)\big) + \varepsilon > 1$ for some $t^i \in S^i$. Hence, the real restriction imposed by ε-boundedness over ε-LIC is constraint (ii). The two constraints together, roughly speaking, guarantee the existence of a probability measure $\{w_k\}_{k\in\mathcal{K}} \in \Delta(\mathcal{K})$ that is "close" to the probability measure $\{G^i\big(A_k(t^i, p)\big)\}_{k\in\mathcal{K}}$ and satisfies (5).

We are now ready to define ε-local implementation.

Definition 5. In a Bayesian environment, a SCR p is ε-**locally implementable** if there exists a transfer scheme x such that the mechanism (p, x) is ε-LIC and ε-bounded, and $\mu^i_{(p,x)}$ is monotone for all $i \in \mathcal{I}$.

Finally, say a SCR p is **rich** if $\cup_{k\in\mathcal{K}^i_0} A_k(s^i, p) \neq \varnothing$ for all $s^i \in [0, 1)$ and $i \in \mathcal{I}$. In a single object allocation problem, this assumption requires that for each possible type of agent i, other than the highest type, there exist reports of other agents such that agent i is not assigned the object.[18] Intuitively, focusing on rich SCRs ensures the existence of a worst case for almost all types and, consequently, ambiguity can play a role.

4.4 Results

Theorem 4.1. *Fix a Bayesian environment* $\mathbf{E}^B$ *and a SCR* p*. For any* $\varepsilon > 0$*, there exist* $\delta > 0$ *and a corresponding* δ*-ambiguity environment* $\mathbf{E}^\delta$ *such that if* p *is implementable in* $\mathbf{E}^\delta$*, then it is* ε*-locally implementable in* $\mathbf{E}^B$*.*

[18] A weaker condition is for each $s^i \in [0, 1)$ such that $G^i\big(A_k(s^i, p)\big) > 0$ for some $k \in \mathcal{K}^i$, there exists $k' \in \mathcal{K}$ such that $\frac{dv^i_k(s^i)}{ds^i} > \frac{dv^i_{k'}(s^i)}{ds^i}$ and $A_{k'}(s^i, p) \neq \varnothing$.

Theorem 4.1 states that if a SCR is exactly implementable with small ambiguity, then it is approximately locally implementable in a Bayesian environment. The next result is our main result and establishes the converse.

Theorem 4.2. *Fix a Bayesian environment $\mathbf{E}^B$ and a rich SCR p. For any $\delta > 0$, there exists $\varepsilon > 0$ such that if p is ε-locally implementable in $\mathbf{E}^B$, then p is implementable in any corresponding δ-ambiguity environment.*

A combination of the two theorems allows us to conclude that approximate local implementation in Bayesian environments and exact implementation in environments with small ambiguity are equivalent.

We next provide a proof of Theorem 4.2 in the special case of a single object allocation problem (Example 2). The main insight is that ambiguity aversion weakens incentive compatibility constraints under full insurance mechanisms. Fix a Bayesian environment $\mathbf{E}^B$, a rich SCR p, $\delta > 0$, and a corresponding δ-ambiguity environment $\mathbf{E}^\delta$. Take $\varepsilon = \delta$ and suppose p is ε-locally implementable by the transfer scheme x with indirect utility functions $\{\mu^i_{(p,x)}\}_{i\in\mathcal{I}}$ in $\mathbf{E}^B$. We are going to show that the full insurance mechanism (p, x_F) with $\{\mu^i_{(p,x)}\}_i$ is incentive compatible in $\mathbf{E}^\delta$ where, in this special case, x_F simplifies to

$$x^i_F(s) = \mu^i_{(p,x)}(s^i) - p_i(s)s^i \quad \forall s \in S, \forall i \in \mathcal{I}.$$

In words, the transfer scheme is constructed so that the agent who is awarded the object pays his valuation and every agent receives a reward which is solely a function of his report. Fix i and s^i, t^i. By the construction of (p, x_F), incentive compatibility in $\mathbf{E}^\delta$ is equivalent to

$$\mu^i_{(p,x)}(s^i) \geq \mu^i_{(p,x)}(t^i) + \min_{F^i\in\mathcal{F}^i} \int p_i(t^i, s^{-i})(s^i - t^i)dF^i. \tag{6}$$

The left-hand side is the interim utility of agent i who receives and reports s^i while the right-hand side is his utility when he reports t^i. Note that the utility from misreporting is evaluated according to a worst-case belief which minimizes the potential gain and, hence, the incentive constraint is weaker than the one in the Bayesian environment.

Suppose first that $s^i > t^i$. We now show how the ε-LIC of (p, x) in $\mathbf{E}^B$ and the monotonicity of $\mu^i_{(p,x)}$ guarantee (6). The ε-LIC of (p, x) in $\mathbf{E}^B$ implies

$$\mu^i_{(p,x)}(s^i) \geq \mu^i_{(p,x)}(t^i) + \Big(\int p_i(t^i, s^{-i}) dG^i - \varepsilon \Big)(s^i - t^i). \tag{7}$$

An immediate observation is if $\int p_i(t^i, s^{-i}) dG^i > \varepsilon = \delta$, since p is rich, there exists $\hat{F}^i \in B_\delta(G^i) \subseteq \mathcal{F}^i$ such that

$$\int p_i(t^i, s^{-i}) dG^i - \varepsilon \geq \int p_i(t^i, s^{-i}) d\hat{F}^i \geq \min_{F^i \in \mathcal{F}^i} \int p_i(t^i, s^{-i}) dF^i.$$ [19]

We thus can conclude that (7) implies (6), that is, the incentive compatibility of (p, x_F) in $\mathbf{E}^\delta$. If $\int p_i(t^i, s^{-i}) dG^i \leq \varepsilon = \delta$, then $\min_{F^i \in \mathcal{F}^i} \int p_i(t^i, s^{-i}) dF^i = 0$ and, hence, (6) is an immediate consequence of the monotonicity of $\mu^i_{(p,x)}$ and the assumption $s^i > t^i$.

Suppose now $s^i < t^i$. The ε-LIC of (p, x) in $\mathbf{E}^B$ then implies

$$\mu^i_{(p,x)}(t^i) \leq \mu^i_{(p,x)}(s^i) + \Big(\int p_i(t^i, s^{-i}) dG^i + \varepsilon \Big)(t^i - s^i). \tag{8}$$

If $\int p_i(t^i, s^{-i}) dG^i \leq 1 - \varepsilon$, then there exists $\hat{F}^i \in B_\delta(G^i) \subseteq \mathcal{F}^i$ such that

$$\int p_i(t^i, s^{-i}) dG^i + \varepsilon \leq \int p_i(t^i, s^{-i}) d\hat{F}^i \leq \max_{F^i \in \mathcal{F}^i} \int p_i(t^i, s^{-i}) dF^i.$$

Combining this with (8) yields

$$\mu^i_{(p,x)}(t^i) \leq \mu^i_{(p,x)}(s^i) + \max_{F^i \in \mathcal{F}^i} \int p_i(t^i, s^{-i})(t^i - s^i) dF^i, \tag{9}$$

which is exactly (6). If $\int p_i(t^i, s^{-i}) dG^i > 1 - \varepsilon$, then $\max_{F^i \in \mathcal{F}^i} \int p_i(t^i, s^{-i}) dF^i = 1$. The next lemma completes the proof by showing (9) is an immediate consequence of the ε-boundedness of $\mu^i_{(p,x)}$.

Lemma 4.1. *Fix a Bayesian environment. Suppose $v^i_k(s) = s^i$ for all $k \in \mathcal{K}^i$ and $i \in \mathcal{I}$. Then for any $\varepsilon \geq 0$, an ε-LIC mechanism (p, x) is ε-bounded if and only if*

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) \leq t^i - s^i \quad \forall s^i < t^i, \forall i \in \mathcal{I}.$$

[19] To be more precise, $\hat{F}^i$ can be constructed such that $\hat{F}^i(A_i(t^i, p)) = G^i(A_i(t^i, p)) - \varepsilon$ and $\hat{F}^i(\cup_{j \neq i} A_j(t^i, p)) = G^i(\cup_{j \neq i} A_j(t^i, p)) + \varepsilon$. Such construction is feasible because $\cup_{k \in \mathcal{K}^i_0} A_k(t^i, p) = \cup_{j \neq i} A_j(t^i, p) \neq \varnothing$.

4.5 How Restrictive are Monotonicity and ε-Boundedness

To see how restrictive monotonicity and ε-boundedness are, we first demonstrate that these two conditions regulate ε-LIC mechanisms only on a subset of signals and normally become weaker as ε converges to zero. To see this, consider again Example 2. Recall that $G^i\big(A_i(s^i,p)\big)$ is agent i's expected probability of obtaining the object under SCR p. Let $R^i \equiv \{s^i \in S^i | \varepsilon \le G^i\big(A_i(s^i,p)\big) \le 1-\varepsilon\}$. For any ε-LIC mechanism (p,x) and $s^i < t^i$, ε-LIC implies

$$\Big(G^i\big(A_i(s^i,p)\big)-\varepsilon\Big)(t^i-s^i) \le \mu^i_{(p,x)}(t^i)-\mu^i_{(p,x)}(s^i) \le \Big(G^i\big(A_i(t^i,p)\big)+\varepsilon\Big)(t^i-s^i).$$

For all $s^i, t^i \in R^i$ and $s^i < t^i$, it follows from the first inequality that $\mu^i_{(p,x)}(s^i) \le \mu^i_{(p,x)}(t^i)$, that is, $\mu^i_{(p,x)}$ is monotone on R^i; similarly, the second inequality implies ε-boundedness by letting $w_j = 0$ for all $j \neq i$ and $w_i = G^i\big(A_i(t^i,p)\big)+\varepsilon$. Combining these two observations, we can conclude that ε-LIC implies monotonicity and ε-boundedness on R^i. In contrast, on $S^i \setminus R^i$, the requirement of ε-LIC is too weak in the sense that it allows for some $s^i < t^i$,

$$\mu^i_{(p,x)}(t^i)-\mu^i_{(p,x)}(s^i) < 0 \quad \text{or} \quad t^i - s^i < \mu^i_{(p,x)}(t^i)-\mu^i_{(p,x)}(s^i). \tag{10}$$

Monotonicity and ε-boundedness are thus imposed only on $S^i \setminus R^i$ to rule these out. However, observe that this set $S^i \setminus R^i$ shrinks as $\varepsilon \to 0$ under many commonly used SCRs.[20] Therefore, the two restrictions are normally weaker when ε is smaller.

To see the role of monotonicity and ε-boundedness, notice that, similarly to Bayesian implementation, exact maxmin implementation requires that each agent's indirect utility satisfy a generalized envelope formula.[21] For example, in a single object allocation problem, the standard envelope argument implies the derivative of agent i's indirect utility μ^i associated with any incentive compatible mechanism is a probability of getting the object, which implies $\frac{d\mu^i(s^i)}{ds^i} \in [0,1]$. Yet it follows from (10) that an arbitrary ε-LIC mechanism may fail to satisfy this condition. The

[20]For example, if $N=2$, p is the efficient SCR, and G^i is the cdf of the uniform distribution, then $S^i \setminus R^i = [0,\varepsilon) \cup (1-\varepsilon, 1]$.

[21]See Myerson [39] and Jehiel and Moldovanu [28] for a characterization of incentive compatible mechanisms in Bayesian environments, and see Song [47] in environments with ambiguity.

role of monotonicity and boundedness is to ensure the indirect utilities associated with an ε-LIC mechanism satisfy the generalized envelope formula which is necessary for implementation. More fundamentally, ε-LIC is non-equilibrium-based no matter how small ε is, whereas incentive compatibility under ambiguity is an equilibrium concept which requires each agent's interim utility be a sufficiently well-behaved function of his private information and, hence, is stronger than ε-LIC.

Even though monotonicity and ε-boundedness impose additional requirements on ε-LIC mechanisms, they are actually quite permissive. Almost all commonly used ε-IC mechanisms are indeed monotone and bounded (see applications in Sections 5 and 6.2.1).

One instance in which the monotonicity assumption is innocuous is when the SCR satisfies a weak monotonicity condition. More precisely, if $\sum_{k\in\mathcal{K}^i} G^i\left(A_k(s^i,p)\right)$ is nondecreasing in s^i, for any approximately LIC mechanism (p,x) with possibly non-monotone indirect utility functions, we can modify the transfer scheme x so that the new mechanism is approximately LIC and the associated indirect utility functions are monotone. Moreover, the modification preserves the boundedness property.

Proposition 4.1. *Fix a Bayesian environment $\mathbf{E}^B$ and a SCR p such that $\sum_{k\in\mathcal{K}^i} G^i\left(A_k(s^i,p)\right)$ is nondecreasing in s^i for all $i\in\mathcal{I}$. For any $\delta>0$, there is $\varepsilon\in(0,\delta]$ such that if there exists an ε-LIC mechanism (p,x), we can construct a transfer scheme $\hat{x}$ such that $(p,\hat{x})$ is δ-LIC and $\mu^i_{(p,\hat{x})}$ is monotone for all $i\in\mathcal{I}$. Furthermore, if (p,x) is δ-bounded, so is $(p,\hat{x})$.*[22]

While the definition of ε-boundedness is tedious, there are two cases in which one can easily verify whether a mechanism is ε-bounded. The first case is when agents do not have incentives to lie upward. In fact, any mechanism that satisfies upward incentive compatibility is ε-bounded for all $\varepsilon\geq 0$. To see this, observe

[22] We want to point out that the monotonicity condition on p is quite weak. Any efficient SCR satisfies this condition, but not vice versa. Also, in general mechanism design settings, it is different from the usual necessary condition imposed by incentive compatibility.

that given a mechanism (p, x), agent i has no incentive to lie upward if $\mu^i_{(p,x)}(s^i) \geq u^i_{(p,x)}(t^i, s^i)$ for all $s^i < t^i$, which is equivalent to

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) \leq \sum_{k \in \mathcal{K}} G^i\big(A_k(t^i, p)\big)\big(v^i_k(t^i) - v^i_k(s^i)\big) \quad \forall s^i < t^i.$$

Taking $w_k = G^i\big(A_k(t^i, p)\big)$ for all $k \in \mathcal{K}$ immediately shows (p, x) is ε-bounded for all $\varepsilon \geq 0$. For instance, in the competitive equilibrium mechanism, both buyers and sellers only have incentives to lie downward.[23]

The second case is when value functions satisfy a supermodularity condition. Say a value function v^i_k is **supermodular** in (k, s^i) if for any $k', k'' \in \mathcal{K}$, and any $s^i, t^i \in S^i$, $\frac{dv^i_{k'}(s^i)}{ds^i} > \frac{dv^i_{k''}(s^i)}{ds^i}$ implies that $\frac{dv^i_{k'}(t^i)}{dt^i} > \frac{dv^i_{k''}(t^i)}{dt^i}$. This is a sorting condition familiar from mechanism design which allows us to rank social alternatives according to an agent's marginal valuation.[24] It is easily seen that linear models satisfy supermodularity. Lemma 4.1 is a corollary of the next more general result.

Proposition 4.2. *Fix a Bayesian environment. Suppose v^i_k are supermodular in (k, s^i). There exists $c > 0$ such that for any $\varepsilon \geq 0$, an ε-LIC mechanism (p, x) is $c\varepsilon$-bounded if and only if*

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) \leq \max_{k \in \mathcal{K}} \Big(v^i_k(t^i) - v^i_k(s^i)\Big) \quad \forall s^i < t^i, \forall i \in \mathcal{I}. \tag{11}$$

4.6 Ex Ante Revenue

In the analysis above, we did not make any assumption on the mechanism designer's preferences. If the mechanism designer and all agents share a common prior in the Bayesian environment, and the mechanism designer remains ambiguity neutral in any corresponding ambiguity environment, then the mechanism designer's ex ante revenue in any corresponding ambiguity environment is the same as her revenue in the Bayesian environment. To see this, note that the expected social surplus is the same in the two environments since the same SCR is executed.

[23]It is well known that in the competitive equilibrium mechanism, a buyer has incentive to underreport his valuation to induce a lower price, whereas a seller has incentive to overreport his cost to induce a higher price. To see why overreporting cost is actually lying downward, note that for a seller j with cost c^j, his valuation from trading is given by $-c^j$. Given our assumption that each agent's valuation increases in his signal, technically, seller j's signal is $-c^j$.

[24]A similar assumption is adopted in Bergemann and Välimäki [4, Condition (18)].

The construction of the full insurance mechanism further implies expected informational rents are the same. The ex ante revenue, which is the difference between the expected social surplus and expected informational rents, is thus the same.

5 Applications

We next apply our result to the implementation of efficient allocations. An immediate implication of Theorem 4.2 is that a small degree of ambiguity ensures efficient implementation if there exist efficient and approximately LIC mechanisms when agents are Bayesian. Yet when do such mechanisms typically exist? We provide an answer to this question by establishing the existence of such mechanisms in two relevant Bayesian environments.

5.1 Approximate Efficiency and Approximate Implementation

In this section we show that the efficient SCR is approximately locally implementable whenever an approximately efficient SCR is implementable. The literature provides a variety of mechanism design settings in which approximate efficiency is attainable. For instance, Theorem 3.2 in Rustichini et al. [43] shows that in double auctions, the allocation is asymptotically efficient in any Bayesian Nash equilibrium of the competitive equilibrium mechanism.[25] Combining their result with ours yields that efficient allocations are asymptotically locally implementable and, hence, exactly implementable in large double auctions with small ambiguity.

5.1.1 Notions of approximate efficiency

We start with an intuitive notion of approximate efficiency.

Definition 6. For any $\varepsilon > 0$, a SCR p is ε**-ex post efficient** if

$$\max_{k\in\mathcal{K}} \sum_{l=1}^{N} v_k^l(s^l) - \sum_{k\in\mathcal{K}} p_k(s) \sum_{l=1}^{N} v_k^l(s^l) < \varepsilon \quad \forall s \in S.$$

Note that $\max_{k\in\mathcal{K}} \sum_{l=1}^{N} v_k^l(s^l)$ is the maximum welfare and $\sum_{k\in\mathcal{K}} p_k(s) \sum_{l=1}^{N} v_k^l(s^l)$ is the welfare obtained under the SCR p. Thus, a SCR is ε-ex post efficient if the welfare loss is less than ε for all types.

[25] A formal description of large double auction environments and the competitive equilibrium mechanism is given in Section 5.2.

This paper adopts the following weaker notion of approximate efficiency so that our result covers more cases and is hence stronger.

Definition 7. For any $\varepsilon > 0$, a SCR p is **(weakly) ε-efficient** if

$$G^i\left(\{s^{-i} \in S^{-i} | \max_{k\in\mathcal{K}} \sum_{l=1}^{N} v_k^l(s^l) - \sum_{k\in\mathcal{K}} p_k(s) \sum_{l=1}^{N} v_k^l(s^l) < \varepsilon\}\right) > 1-\varepsilon \quad \forall s^i \in S^i, \forall i \in \mathcal{I}.$$

Roughly speaking, a SCR is (weakly) ε-efficient if each agent i assigns probability at least $1-\varepsilon$ to the event that the welfare loss is less than ε.

5.1.2 Pivotality

We make the following assumption on valuations.

Assumption 1 (Pivotality). For any $k \neq k'$, there exist $i \neq j$ such that

$$\frac{\partial\left(\sum_l v_k^l(s^l) - \sum_l v_{k'}^l(s^l)\right)}{\partial s^i} \neq 0 \quad \text{and} \quad \frac{\partial\left(\sum_l v_k^l(s^l) - \sum_l v_{k'}^l(s^l)\right)}{\partial s^j} \neq 0 \quad \forall s \in S.$$

This assumption requires the difference between the social welfare from any two alternatives be affected by the information of at least two agents. Intuitively, Pivotality asserts that at least two agents are potentially pivotal for efficiency considerations. The role of this assumption is discussed in the next section.

Two examples are provided below to help in understanding when Pivotality is satisfied.

Example 3. Consider the single object allocation problem from Example 1. Take any two allocations $i \neq j$. Then $\sum_l v_i^l(s^l) - \sum_l v_j^l(s^l) = v_i^i(s^i) - v_j^j(s^j)$, which clearly depends on i's and j's information. Pivotality is thus always satisfied .

Example 4. Consider the bilateral trade problem of Myerson and Satterthwaite [40]. Let $\mathcal{K} = \{0, 1\}$, where $k = 0$ represents no trade and $k = 1$ represents trade. The difference between the social welfare from $k = 1$ and $k' = 0$ is the difference between the buyer's valuation and the seller's cost which, obviously, depends on the information of both the buyer and the seller. Hence, Pivotality is satisfied. An analogous argument shows that Pivotality is satisfied in the presence of multiple buyers and sellers, namely, in double auctions.

5.1.3 Result

Theorem 5.1. *Fix a Bayesian environment. Assume Pivotality. For any $\varepsilon > 0$, there exists a $\overline{\xi} > 0$ such that whenever a ξ-efficient SCR is implementable, for some $0 < \xi < \overline{\xi}$, the efficient SCR is ε-locally implementable.*

Theorem 5.1 states that the efficient SCR is approximately locally implementable whenever an approximately efficient SCR is implementable. To illustrate the result, we next provide a heuristic proof of Theorem 5.1 in the single object allocation problem (Example 2). As argued in Example 3, Pivotality is trivially satisfied in this case. For any ξ, let p^{ξ} denote a ξ-efficient SCR. Recall that given a SCR p and $s^i \in S^i$, $A_i(s^i, p)$ is the set of the other agents' signals under which agent i is assigned the object by p. A key lemma for our result is that as $\xi \to 0$, the set $A_i(s^i, p^{\xi})$ converges to the set $A_i(s^i, p^*)$ for all $s^i \in S^i$ and $i \in \mathcal{I}$. Intuitively, this means that for small ξ, allocations specified by the ξ-efficient SCR and by the fully efficient SCR coincide for most $s \in S$. Consequently, for any $\varepsilon > 0$, there is a $\overline{\xi} > 0$ such that for any $0 < \xi < \overline{\xi}$ and any ξ-efficient SCR p^{ξ}, we have

$$|G^i\big(A_i(s^i, p^{\xi})\big) - G^i\big(A_i(s^i, p^*)\big)| \leq \varepsilon \quad \forall s^i \in S^i, \forall i \in \mathcal{I}. \tag{12}$$

Fix $\varepsilon > 0$. Suppose there exists $0 < \xi < \overline{\xi}$ such that p^{ξ} is implementable. Let (p^{ξ}, x) denote a ξ-efficient and incentive compatible mechanism with associated indirect utility functions $\{\mu^i_{(p^{\xi},x)}\}_{i\in\mathcal{I}}$. We are going to show that the efficient SCR p^* is ε-locally implementable in the sense of Definition 5 by the full insurance transfer scheme x_F with $\{\mu^i_{(p^{\xi},x)}\}_{i\in\mathcal{I}}$. We first show (p^*, x_F) is ε-LIC. Fix $i \in \mathcal{I}$ and $s^i, t^i \in S^i$. By the construction of the full insurance mechanism, we have $\mu^i_{(p^*,x_F)} = \mu^i_{(p^{\xi},x)}$ and

$$\mu^i_{(p^{\xi},x)}(t^i) = u^i_{(p^*,x_F)}(t^i, s^i) - (s^i - t^i)G^i\big(A_i(t^i, p^*)\big). \tag{13}$$

The incentive compatibility of (p^{ξ}, x) implies

$$\mu^i_{(p^{\xi},x)}(s^i) \geq \mu^i_{(p^{\xi},x)}(t^i) + (s^i - t^i)G^i\big(A_i(t^i, p^{\xi})\big). \tag{14}$$

Plugging (13) into (14) yields

$$
\begin{aligned}
\mu^i_{(p^\xi,x)}(s^i) \geq & u^i_{(p^*,x_F)}(t^i,s^i) - (s^i - t^i)\Big(G^i\big(A_i(t^i,p^*)\big) - G^i\big(A_i(t^i,p^\xi)\big)\Big) \\
\geq & u^i_{(p^*,x_F)}(t^i,s^i) - \varepsilon|s^i - t^i|.
\end{aligned} \tag{15}
$$

The last inequality follows from (12). Since $\mu^i_{(p^*,x_F)}(s^i) = \mu^i_{(p^\xi,x)}(s^i)$, it follows from (15) that

$$
\mu^i_{(p^*,x_F)}(s^i) \geq u^i_{(p^*,x_F)}(t^i,s^i) - \varepsilon|s^i - t^i|,
$$

as desired. Since s^i and t^i were arbitrarily chosen, this shows that ε-LIC is satisfied.

Next we verify monotonicity and ε-boundedness. It follows from the incentive compatibility constraint (14) that $\mu^i_{(p^\xi,x)}(s^i) \geq \mu^i_{(p^\xi,x)}(t^i)$ whenever $s^i > t^i$. Thus, $\mu^i_{(p^*,x_F)}$, which is identical to $\mu^i_{(p^\xi,x)}$, is monotone. Similarly, by the incentive compatibility constraint $\mu^i_{(p^\xi,x)}(t^i) \geq u^i_{(p^\xi,x)}(s^i,t^i)$, we obtain

$$
\mu^i_{(p^\xi,x)}(s^i) - \mu^i_{(p^\xi,x)}(t^i) \leq (s^i - t^i)G^i\big(A_i(s^i,p^\xi)\big) \leq s^i - t^i, \quad \forall s^i > t^i,
$$

which implies $\mu^i_{(p^*,x_F)}(s^i) - \mu^i_{(p^*,x_F)}(t^i) \leq s^i - t^i$. Then it follows from Lemma 4.1 that (p^*, x_F) is ε-bounded.

To see the role of Pivotality, note that analogous to (12), a key step for our proof in a general mechanism design setting is to show $G^i\big(A_k(s^i,p^\xi)\big) \to G^i\big(A_k(s^i,p^*)\big)$ as $\xi \to 0$. In general, the distance between the two sets $A_k(s^i,p^\xi)$ and $A_k(s^i,p^*)$ will vary with the specific choice of the ξ-efficient SCR p^ξ as well as s^i. The role of Pivotality is to guarantee that this convergence is uniform on the set of all ξ-efficient SCRs and on S^i.

5.2 Large Double Auctions

5.2.1 Double Auction Environments and Competitive Equilibrium Mechanism

In this section, we consider one of the most widely studied environments in mechanism design—large double auctions—in which it is well known that under the *competitive equilibrium mechanism*, the gain from biding strategically becomes vanishingly small as the size of the market grows.

We consider the simplest double auction environment. There are M sellers who each has a good to sell, and M buyers who each would like to buy a good. Let $\mathcal{B}$ denote the set of buyers and $\mathcal{S}$ denote the set of sellers. To better identify buyers' and sellers' information, we introduce some new notation: let $v^i \in [0,1]$ denote buyer i's value of the good and $c^i \in [0,1]$ denote seller i's cost. Assume that when there is no trade, an agent's value/cost is zero. Also assume that values and costs are drawn identically and independently according to a continuous distribution function G in the Bayesian environment.[26] Let $G^{2M}(s)$ denote the distribution of $2M$ agents' signals $s = (v^1, ..., v^M, c^1, ..., c^M)$.

Due to the simplicity of the setting, we can simplify the general mechanism to the following mechanism: let $(p^i_B, p^i_S, x^i_B, x^i_S)_{i=1,...,M}$ be a collection of $4M$ functions where $p^i_B, p^i_S : [0,1]^{2M} \to [0,1]$ denote an agent's probability of obtaining a good (i.e., p^i_B is buyer i's probability of receiving a good, and p^i_S is seller i's probability of retaining a good); and $x^i_B, x^i_S : [0,1]^{2M} \to \mathbb{R}$ denote the transfer to a buyer and a seller respectively.

Efficiency requires the goods to be allocated to the agents with the M highest values/costs. Let p_{*B}, p_{*S} denote the efficient SCR. Then

$$v^i > s_{(M)} \Rightarrow p^i_{*B}(s) = 1 \quad \forall i \in \mathcal{B} \quad \text{and} \quad c^i > s_{(M)} \Rightarrow p^i_{*S}(s) = 1 \quad \forall i \in \mathcal{S},$$

where $s_{(M)}$ is the Mth lowest signal among $2M$ reported signals.

In large double auctions, the following two properties are typically imposed on the mechanisms. Say a mechanism $(p^i_B, p^i_S, x^i_B, x^i_S)_{i=1,...,M}$ satisfies **ex ante budget balance** if

$$\int \Big(\sum_{i\in\mathcal{B}} x^i_B(s) + \sum_{i\in\mathcal{S}} x^i_S(s)\Big) dG^{2M}(s) \leq 0 \quad \forall s \in [0,1]^{2M}.$$

A mechanism $(p^i_B, p^i_S, x^i_B, x^i_S)_{i=1,...,M}$ satisfies **ex post individual rationality** if

$$v^i p^i_B(s) + x^i_B(s) \geq 0 \text{ and } -c^i\big(1 - p^i_S(s)\big) + x^i_S(s) \geq 0 \quad \forall s \in [0,1]^{2M}.$$

We now describe the **competitive equilibrium mechanism** (CEM). Sellers and buyers simultaneously submit offers and bids. These offers and bids are arrayed

[26] All results extend to environments in which the numbers of sellers and buyers are different, and sellers' and buyers' signals are drawn from different distributions.

in increasing order and the price p is set at the Mth lowest bid/offer. Trade occurs among sellers whose offers are no more than p and buyers who bid at least p.[27] (If there is a tie at p, ration uniformly at random.[28]) By construction, if everyone bids his true value/cost, the resulting allocation is efficient. Moreover, it is straightforward to see that the CEM satisfies ex post individual rationality and ex ante budget balance.

5.2.2 Results

Carroll [9] shows that the CEM is asymptotically incentive compatible. Theorem 5.2 shows that the CEM is in fact asymptotically LIC.

Theorem 5.2. *Fix a Bayesian double auction environment. For any $\delta > 0$, there exists $\overline{M}$ such that for all $M > \overline{M}$, the efficient SCR is δ-locally implementable by the CEM.*

We apply Theorem 4.2 to double auctions and obtain the following corollary.

Corollary 5.1. *Fix a Bayesian double auction environment. For any $\delta > 0$, if the efficient SCR is δ-locally implementable in the Bayesian environment, then it is implementable in any corresponding δ-ambiguity environment.*[29]

An immediate implication of Theorem 5.2 and Corollary 5.1 is for any $\delta > 0$, if the number of traders is sufficiently large, then there exists an efficient and incentive compatible mechanism in any δ-ambiguity environment. In addition, this mechanism is a full insurance mechanism as constructed in (1) and each agent's interim payoff from truthful reporting under this mechanism is identical to that in the CEM.

We next show that the full insurance mechanism satisfies ex post individual rationality. Recall that in (1), the full insurance transfers are constructed so that each agent's ex post payoffs are equal to his interim payoff from truthful reporting

[27] This mechanism is a special case of k-double auction, which is widely studied in double auction environments. See Rustichini et al. [43, p.1045] for a more detailed description of the mechanism.

[28] This is not important since ties occur with probability zero in the Bayesian environment.

[29] Due to the special structure of double auctions—for an agent, there are essentially only two distinct allocations—the mapping from δ to ε in Theorem 4.2 is independent of the number of traders.

under the CEM in the Bayesian environment, which is nonnegative. Therefore, ex post individual rationality is satisfied in any δ-ambiguity environment.

If the mechanism designer is ambiguity neutral, that is, her belief is given by G^{2M} in any ambiguity environment, then since the CEM satisfies ex ante budget balance, the arguments in Section 4.6 imply that the full insurance mechanism satisfies ex ante budget balance in any ambiguity environment.

To summarize, we have shown:

Theorem 5.3. *Suppose the mechanism designer is ambiguity neutral. For any $\delta > 0$, there exists $\overline{M}$ such that for all $M > \overline{M}$, there exist efficient, incentive compatible, ex ante budget balanced, and ex post individually rational mechanisms in any δ-ambiguity environment.*

6 Extension to Interdependent Values Setting

In this section, we extend Theorem 4.2 to mechanism design settings with interdependent values. Theorem 4.1 also extends to such general settings but the statement and the proof parallel those of Theorem 4.1 and are thus omitted.

6.1 One-dimensional Signals

We first extend Theorem 4.2 to settings with one-dimensional signals and interdependent values. We start with some new notation and definitions. To allow for interdependence in preferences, we use $v_k^i(s^i, s^{-i})$ to denote agent i's valuation from alternative k. We assume that v_k^i is twice differentiable and nondecreasing in s^j for all j.

We distinguish two cases depending on the value functions. The first is when value functions are additively separable. Say $v_k^i(s^i, s^{-i})$ is **additively separable** if there exist functions $f_k^i : S^i \to \mathbb{R}$ and $h_k^i : S^{-i} \to \mathbb{R}$ such that

$$v_k^i(s^i, s^{-i}) = f_k^i(s^i) + h_k^i(s^{-i}) \quad \forall s^i \in S^i, \forall s^{-i} \in S^{-i}.$$

This case is special because agent j's signal does not affect agent i's own marginal valuation, that is, $\frac{\partial^2 v_k^i(s^i, s^{-i})}{\partial s^i \partial s^j} = 0$ for all $j \neq i$ and $s \in S$. The private values model in Section 2 is thus a special case of the interdependent values model with additive separability.

Alternatively, the cross derivatives can be bounded away from zero. For any $k \in \mathcal{K}^i$, $v_k^i(s^i, s^{-i})$ is **nonseparable** if the cross derivatives $\frac{\partial^2 v_k^i(s^i,s^{-i})}{\partial s^i \partial s^j}$ are continuous and $\frac{\partial^2 v_k^i(s^i,s^{-i})}{\partial s^i \partial s^j} \neq 0$ for all $j \neq i$ and $s \in S$.

The definition of ε-boundedness here is the same as Definition 4 with replacing v_k^i by f_k^i and ε-local implementation is defined as follows.

Definition 8. In a Bayesian environment, a SCR p is **ε-locally implementable** if there exists a transfer scheme x such that (i) the mechanism (p, x) is ε-LIC; (ii) $\mu^i_{(p,x)}$ is monotone for all $i \in \mathcal{I}$; and (iii) if values are additively separable, (p, x) is ε-bounded.

Our main result is stated below.

Theorem 6.1. *Fix a Bayesian environment $\mathbf{E}^B$ and a rich SCR p. Assume either v_k^i is nonseparable for all i and k or v_k^i is additively separable for all i and k. For any $\delta > 0$, there exists $\varepsilon > 0$ such that if p is ε-locally implementable in $\mathbf{E}^B$, then p is implementable in any corresponding δ-ambiguity environment.*

Notice that ε-boundedness is only defined under additive separability. Since ε-boundedness is necessary for implementation with small ambiguity (by Theorem 4.1), a natural question is why a counterpart of ε-boundedness is not needed in the nonseparable setting. To answer this, consider the following extension of ε-boundedness to the interdependent values setting: for any Bayesian environment $\mathbf{E}^B$ and any $\varepsilon \geq 0$, a mechanism (p, x) is ε-bounded if for any $i \in \mathcal{I}$ and $s^i < t^i$, we have

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) \leq \sum_{k \in \mathcal{K}} \int w_k(s^{-i}) \big(v_k^i(t^i, s^{-i}) - v_k^i(s^i, s^{-i})\big) ds^{-i}, \tag{16}$$

where $\{w_k(s^{-i})\}_{k \in \mathcal{K}, s^{-i} \in S^{-i}}$ satisfies (i) $0 \leq \int w_k(s^{-i}) ds^{-i} \leq G^i\big(A_k(t^i, p)\big) + \varepsilon$ for all $k \in \mathcal{K}$ and (ii) $\sum_{k \in \mathcal{K}} \int w_k(s^{-i}) ds^{-i} \leq 1$. When values are nonseparable, $v_k^i(t^i, s^{-i}) - v_k^i(s^i, s^{-i})$ varies with s^{-i} for each $k \in \mathcal{K}^i$, which makes inequality (16) easier to satisfy by constructing $\{w_k(s^{-i})\}_{k,s^{-i}}$ properly. In contrast, when values are additively separable, i.e., $v_k^i(t^i, s^{-i}) - v_k^i(s^i, s^{-i})$ is constant in s^{-i}, inequality (16) effectively depends only on $\{\int w_k(s^{-i})ds^{-i}\}_k$ rather than $\{w_k(s^{-i})\}_{k,s^{-i}}$. Therefore, the requirement of ε-boundedness is much weaker in the nonseparable case than in the additively separable case and it is in fact implied by approximate LIC in the former case.

6.2 Multi-dimensional Signals

We next extend the analysis to allow for multi-dimensional signals and interdependent values. This case is interesting because Jehiel and Moldovanu [28] show that in such settings, efficient and Bayesian incentive compatible mechanisms generally do not exist. Our result is then relevant since if we know that the efficient SCR is approximately locally implementable in some Bayesian environment, then by applying our result, we can conclude that the efficient SCR is implementable with small ambiguity. Such an application is provided in Section 6.2.1.

To introduce multi-dimensional signals, we redefine some concepts and notation in this section. Recall N and K are the number of agents and the number of social alternatives, respectively. Assume that agent i's signal s^i is drawn from $S^i \subseteq \mathbb{R}^{K\times N}$. The idea is that coordinate s^i_{kj} of s^i influences the utility of agent j in alternative k. We assume that the signal spaces S^i are compact, convex, and full-dimensional given the usual topology in $\mathbb{R}^{K\times N}$. Agent i's value in social alternative k is given by $v^i_k(s^1_{ki}, ..., s^N_{ki})$. For ease of notation, we will write $v^i_k(s)$ or $v^i_k(s^i, s^{-i})$ instead. As before, we assume that v^i_k is twice differentiable and nondecreasing in s^j_{ki} for all j. Definitions of additive separability and nonseparability are extended in the obvious way.

Our most important concept is ε-LIC. There is no unique way of extending the notion of ε-LIC to multi-dimensional signals. With multi-dimensional signals, some agent possesses information that is relevant to the other agents, but does not directly affect the owner of that information. We call such information **own-payoff irrelevant information**. Theorem 3.1 in Jehiel and Moldovanu [28] shows that under any incentive compatible mechanism, each agent i's equilibrium payoff cannot depend on his own-payoff irrelevant information s^i_{kj}, $j \neq i$. Song [47] extends this result to environments with ambiguity. Motivated by these observations, we propose the following extension. Recall that $\mathcal{K}^i$ is the set of alternatives on which agent i's own information has an effect.

Definition 9 (Multi-dimensional Signals)**.** For any $\varepsilon > 0$, a mechanism (p, x) is **ε-locally incentive compatible** (ε-LIC) if

$$\mu^i_{(p,x)}(s^i) \geq u^i_{(p,x)}(t^i, s^i) - \varepsilon \sum_{k \in \mathcal{K}^i} |s^i_{ki} - t^i_{ki}| \quad \forall s^i, t^i \in S^i, \forall i \in \mathcal{I}.$$

Under this notion of ε-LIC, the indirect utility functions associated with any ε-LIC mechanism are independent of own-payoff irrelevant information.[30]

In Appendix D, we extend the definition of ε-boundedness to accommodate multi-dimensional signals and prove that the statement of Theorem 6.1 remains true.

6.2.1 Efficient Implementation With Small Informational Size

In this section, we show that in any Bayesian environment where agents are informationally small, the efficient SCR is approximately locally implementable by a modified VCG transfer scheme.

Our definition of informational size measures the degree to which one agent's signal can affect the valuations of other agents.[31] Formally, define the **informational size** of agent i as

$$\gamma^i \equiv \max_{j \neq i, k \in \mathcal{K}, s \in S} \frac{\partial v^j_k(s^i, s^{-i})}{\partial s^i_{kj}}.$$

Recall that s^i_{kj} is agent i's information affecting agent j's valuation for alternative k. In the case of private values, the informational size of each agent is 0.

We now define the **modified VCG (MVCG) mechanism**. Define the **general-**

[30]In Section 7, we provide a weaker notion of approximate LIC which allows $\mu^i_{(p,x)}$ to depend on own-payoff irrelevant information.

[31]In our setting, an agent's value depends directly on other agents' types. McLean and Postlewaite [37] study an interdependent values setting in which each agent's value depends indirectly on other agents' types in the sense that each agent's value is a function of the state of nature and other agents' types provide additional information about the state of nature. They adopt a notion of informational size as the degree to which an agent can alter the posterior distribution on the state space given the information of other agents. Both notions capture the "informational" influence of an agent's type on others but accomplish this in two different interdependent values settings.

ized VCG transfer scheme as

$$x^i_{GVCG}(s^i, s^{-i}) \equiv \sum_k p^*_k(s^i, s^{-i}) \sum_{j \neq i} v^j_k(s^i, s^{-i}) - \max_{k \in \mathcal{K}} \sum_{j \neq i} v^j_k(s^i, s^{-i}),$$

which represents the cost that agent i imposes on other agents. For every $i \in \mathcal{I}$ and $s^i \in S^i$, construct $\bar{\varsigma}^i(s^i)$ as follows:

$$\bar{\varsigma}^i_{ki}(s^i) = s^i_{ki} \quad \text{and} \quad \bar{\varsigma}^i_{kj}(s^i) = \max\{t^i_{kj} | t^i \in S^i\}, \quad \forall k \in \mathcal{K}, \forall j \neq i.$$

Note it is possible that the constructed $\bar{\varsigma}^i(s^i) \notin S^i$. However, $v^l_k(\bar{\varsigma}^i(s^i), s^{-i})$ exists and, hence, $p^*(\bar{\varsigma}^i(s^i), s^{-i})$ is well defined.[32] Define the **MVCG transfer scheme** as

$$x^i_{MVCG}(s^i, s^{-i}) \equiv x^i_{GVCG}(\bar{\varsigma}^i(s^i), s^{-i}) + \sum_k \big(p^*_k(\bar{\varsigma}^i(s^i), s^{-i}) - p^*_k(s^i, s^{-i})\big) v^i_k(s^i, s^{-i}).$$

The **MVCG mechanism** is defined by the pair (p^*, x_{MVCG}).

Observe that the MVCG transfer is constructed so that if everyone reports truthfully, agent i's ex post and interim payoffs are independent of his own-payoff irrelevant information $(s^i_{kj})_{j \neq i, k \in \mathcal{K}}$. The next lemma shows the MVCG mechanism is ε-LIC when agents are sufficiently informationally small.

Theorem 6.2. *Fix a Bayesian environment. For any $\varepsilon > 0$, there exists $\gamma > 0$ such that if $\gamma^i < \gamma$ for all $i \in \mathcal{I}$, the efficient SCR p^* is ε-locally implementable by the MVCG transfer scheme x_{MVCG}.*

To explain why the MVCG mechanism is ε-LIC, consider a single object allocation problem where values are additively separable. That is, $v^i_i(s) = f^i_i(s^i) + h^i_i(s^{-i})$. By the construction of the MVCG transfer scheme, we can derive an upper bound on the gain to agent i when his true signal is s^i but he reports t^i and others report truthfully:

$$\big(f^i_i(s^i) - f^i_i(t^i)\big)\Big(G^i\big(A_i(t^i, p^*)\big) - G^i\big(A_i(\bar{\varsigma}^i(t^i), p^*)\big)\Big), \tag{17}$$

where $G^i\big(A_i(\tilde{s}^i, p^*)\big)$ is agent i's probability of getting the object given $\tilde{s}^i$. When agent i's informational size is small, his own-payoff irrelevant information has little effect on the values of the other agents and, hence, on the determination of the

[32]For every $k \in \mathcal{K}$, we have $p^*_k(\bar{\varsigma}^i(s^i), s^{-i}) = 1$ if $\sum_l v^l_k(\bar{\varsigma}^i_{kl}(s^i), s^{-i}_{kl}) > \sum_l v^l_{k'}(\bar{\varsigma}^i_{k'l}(s^i), s^{-i}_{k'l})$ for all $k' \neq k$.

efficient allocation. Since $\overline{\varsigma}^i(t^i)$ and t^i only differ in the own-payoff irrelevant information, for most of the realizations of s^{-i}, the efficient allocation under $(\overline{\varsigma}^i(t^i), s^{-i})$ is identical to that under (t^i, s^{-i}). This means that as γ^i converges to 0, the set $A_i(\overline{\varsigma}^i(t^i), p^*)$ converges to the set $A_i(t^i, p^*)$. Thus, for any $\varepsilon > 0$, if γ^i is sufficiently small, the upper bound on the gain from misreporting (17) is less than $\varepsilon|s^i_{ii} - t^i_{ii}|$. Thus, the MVCG mechanism is ε-LIC.

7 Discussion and Related Literature

An alternative notion of ε-LIC. In Definition 9, we formulate a notion of ε-LIC in multi-dimensional environments. A more intuitive and also weaker notion is:

Definition 10. For any $\varepsilon > 0$, a mechanism (p, x) is **weakly ε-locally incentive compatible** if

$$\mu^i_{(p,x)}(s^i) \geq u^i_{(p,x)}(t^i, s^i) - \varepsilon \parallel s^i - t^i \parallel_\infty \quad \forall s^i, t^i \in S^i, \forall i \in \mathcal{I}.$$

The two notions, ε-LIC and weak ε-LIC, coincide in one-dimensional environments, whereas ε-LIC is stronger in multi-dimensional environments. The difference lies in whether $\mu^i_{(p,x)}$ is allowed to depend on own-payoff irrelevant information. For example, the generalized VCG mechanism is not ε-LIC but is weakly ε-LIC when agents are sufficiently informationally small. In Appendix F, we show that the two notions are equivalent when the own-payoff irrelevant information of the agents has little influence on the determination of allocations.

Extension to general social choice rules. Theorem 5.1 can be extended to any two SCRs that are "close", suitably defined, to each other. That is, under appropriately reformulated assumptions on valuations, if a SCR p is implementable, then any SCR that is "close" to p is approximately (locally) implementable. We next identify a precise sense in which two SCRs are close.

For any SCR p, $s^i \in S^i$, and $k \in \mathcal{K}$, recall that $A_k(s^i, p) = \{s^{-i} \in S^{-i} | p_k(s^i, s^{-i}) = 1\}$. The distance between any two SCRs p and p' is defined by

$$d_G(p, p') \equiv \sup_{i \in \mathcal{I}, s^i \in S^i, k \in \mathcal{K}} \max\{G^i(A_k(s^i, p) \setminus A_k(s^i, p')), G^i(A_k(s^i, p') \setminus A_k(s^i, p))\}.$$[33]

Indeed, the proof of Theorem 5.1 does not rely on (ε-)efficiency but rather on the observation that for any ε-efficient SCR p^ε, we have $d_G(p^\varepsilon, p^*) \to 0$ as $\varepsilon \to 0$.

Even though we restrict our attention to deterministic SCRs, our result can be extended to a restrictive class of random SCRs. To illustrate, let p' be a deterministic SCR and $\hat{p}$ be any (possibly random) SCR. For any $\varepsilon > 0$, define a SCR $p = (1-\varepsilon)p' + \varepsilon\hat{p}$ as follows: given the report signals, pick the outcome of p' with probability $1-\varepsilon$ and pick the outcome of $\hat{p}$ with probability ε. Such random SCRs are used in the literature of virtual implementation (e.g., Duggan [16]). With slight modifications to our current proofs, we can show that for any small ε, if p is implementable, then p' is approximately locally implementable and, hence, exactly implementable in the presence of small ambiguity.

Large markets. This paper is closely related to the literature on mechanism design in large markets. This literature establishes asymptotic efficiency or asymptotic incentive compatibility of specific mechanisms. For example, see Satterthwaite and Williams [45], Kojima and Yamashita [33], Williams [48], and Andreyanov and Sadzik [1]. Our results suggest that in some of the settings where approximate mechanisms have been established, a small degree of ambiguity may be used to obtain exact results. Azevedo and Budish [3] and Hatfield et al. [26] propose two new notions of approximate incentive compatibility, strategy proofness in the large and strategy proofness within ε in expectation respectively, which lie between the standard notion of approximate ex post incentive compatibility and approximate interim incentive compatibility. Yet these two notions do not restrict local incentive constraints. Our paper proposes a stronger notion of approximate interim incentive compatibility which imposes proper restrictions on local incentive constraints.

Independence assumption. We assume that each agent's set of beliefs is independent of the realization of his signal. Since our approach is essentially based on a generalization of the standard Myersonian approach, our results do not extend straightforwardly to settings with correlated information. In particular, given

[33] A more intuitive metric is $d(p,p') \equiv \sup_{i\in\mathcal{I},s^i\in S^i,k\in\mathcal{K}} d_H^i\big((A_k(s^i,p), A_k(s^i,p')\big)$, where d_H^i is the Hausdorff metric on S^{-i}. Since G^i is assumed to be absolutely continuous, we have $d_G(p,p') \to 0$ whenever $d(p,p') \to 0$, but not vice versa. Therefore, our definition is more permissive.

an ε-LIC mechanism (p, x) in a Bayesian environment with correlated information, whether p is implementable with small ambiguity hinges on (i) how agent i's belief G^i depends on s^i in the Bayesian environment (e.g., whether types are affiliated), (ii) how agent i's set of beliefs is constructed in the corresponding ambiguity environments (e.g., whether ambiguity is on a common prior or on the conditional beliefs directly), and (iii) properties of the mechanism (p, x) (e.g., whether $p_k(s)$ is monotone in s^j). For example, if (p, x) is a full insurance mechanism, then Theorem 4.2 continues to hold with a suitable definition of the corresponding ambiguity environments, regardless of how types are correlated in the Bayesian environment. Another complication that arises in models with correlated types is the possibility of using lottery mechanisms, proposed by Cremer and McLean [12, 13]. It is well-known that using lottery mechanisms can greatly enlarge the set of implementable SCRs. A straightforward application of lottery mechanisms however has its limitations in ambiguity environments as the belief used to evaluate a lottery is endogenously determined and, hence, it is difficult to construct a lottery for each type with the desired property.[34] The connection between approximate Bayesian implementation and exact maxmin implementation when types are correlated will be the subject of future work.

Mechanism design with maxmin preferences. This paper adopts the maxmin expected utility model of Gilboa and Schmeidler [20] to model ambiguity aversion, which is one of the most commonly adopted model of robust decision-making under uncertainty in mechanism design. For example, Bose et al. [8], Bose and Daripa [6], Bodoh-Creed [5], and Carroll [10] study revenue maximization with maxmin agents.[35] Wolitzky [49], Song [47], de Castro and Yannelis [14], and Kocherlakota and Song [32] address the possibility of implementing efficient allocations with maxmin agents in a variety of economic applications. By comparison, this paper considers a general implementation problem in a social choice setting that allows for interdependent valuations and multi-dimensional signals and explores a con-

[34] See Renou [41] and Song [47] for an analysis of mechanism design problems with ambiguity averse agents and correlated information.

[35] Di Tillio et al. [15] and Guo [24] study the effects of introducing ambiguity in mechanisms.

nection between approximate Bayesian implementation and maxmin implementation under a small degree of ambiguity.

Another commonly used model of ambiguity in the literature is the smooth ambiguity model of Klibanoff et al. [31].[36] In the smooth ambiguity model, agents' utilities in general are not quasilinear in transfers and the qualitative implications of the smooth ambiguity model are similar to those of risk aversion. Moreover, in the smooth ambiguity model, agents are locally ambiguity neutral and, consequently, smooth ambiguity cannot weaken local incentive constraints as in ε-LIC. Thus, our results do not extend to the smooth ambiguity model.

The closest work to the current paper is Song [47]. There are two main differences. First, Song [47] focuses on overturning the impossibility result of Jehiel and Moldovanu [28] whereas the primary focus of this paper is to establish an equivalence between maxmin implementation with small ambiguity and approximate implementation. Second, in terms of methodology, Song [47] extends the Myersonian first order approach to environments with maxmin agents. The Myersonian approach is feasible only if the SCR, value functions, and the sets of priors satisfy certain conditions.[37] In contrast, we do not impose conditions under which the Myersonian approach applies. Instead, we modify an existing ε-LIC mechanism to obtain an incentive compatible mechanism in environments with ambiguity. As a result, we are able to study more general mechanism design problems whereas Song [47] is limited to single object allocation problems.

[36] Epstein and Schneider [18] review different models of ambiguity aversion and illustrate differences in behavior implied by those models.

[37] For example, in order to apply the Myersonian approach in one-dimensional environments, we need the following monotonicity constraint on the SCR p: for every $i \in \mathcal{I}$, every $s^i, t^i, \hat{s}^i \in S^i$ such that $s^i < t^i$, and every $s^{-i} \in S^{-i}$, we have $\sum_{k\in\mathcal{K}} p_k(t^i, s^{-i})\frac{\partial v_k^i(\hat{s}^i, s^{-i})}{\partial \hat{s}^i} \geq \sum_{k\in\mathcal{K}} p_k(s^i, s^{-i})\frac{\partial v_k^i(\hat{s}^i, s^{-i})}{\partial \hat{s}^i}$. In a linear setting, the weak congruence condition (5.1) in Jehiel and Moldovanu [28] implies this monotonicity constraint.

Appendix

A Appendix for Section 4

For any two probability measures $F, H \in \Delta(S^{-i})$, the **Prokhorov metric** is

$$d(F,H) \equiv \inf\{\varepsilon > 0 | F(A) \leq H(\mathcal{B}_\varepsilon(A)) + \varepsilon, \forall A \in \Sigma^{-i}\},$$

where $\mathcal{B}_\varepsilon(A) \equiv \{s^{-i} \in S^{-i} | \inf_{t^{-i} \in A} \parallel s^{-i} - t^{-i} \parallel_\infty \leq \varepsilon\}$ and $\parallel \cdot \parallel_\infty$ denotes the uniform metric on S^{-i}.

A.1 Example: ε-Incentive Compatibility

In this section, we present a simple example to demonstrate the necessity of using the notion of ε-LIC rather than ε-IC in Theorem 4.2. More precisely, for any $\varepsilon > 0$, we construct explicitly a mechanism that is ε-IC in the Bayesian environment. We then show that as ε converges to 0, the necessary amount of ambiguity for exact implementation is bounded away from zero.

Consider Example 2 with $N = 2$. Let $G^i(s^j) = s^j$ for all $i \in \mathcal{I}$ and $j \neq i$. Fix $0 < \varepsilon < 1$. Consider the following mechanism (p, x):

$$p_1(s^1, s^2) = \begin{cases} 0 & \text{if } s^1 \neq 1 - \varepsilon,\ s^2 \in [0.5, 1], \\ 1 & \text{otherwise,} \end{cases} \qquad p_2(s^1, s^2) = 1 - p_1(s^1, s^2),$$

and

$$x^1(s^1, s^2) = -p_1(s^1, s^2)s^1 + 0.5s^1, \qquad x^2(s^1, s^2) = -p_2(s^1, s^2)s^2 + \max\{s^2 - 0.5, 0\}.$$

In the Bayesian environment, under p, agent 1's expected probability of obtaining the object is 0.5 if $s^1 \neq 1 - \varepsilon$ and is 1 if $s^1 = 1 - \varepsilon$. Thus, (p, x) is not incentive compatible as agent 1's expected probability of obtaining the object is not increasing in his valuation. However, it is easy to verify that this mechanism is ε-IC.

Suppose $\mathcal{F}^1 = B_\delta(G^1)$ in the δ-ambiguity environment. We next show that this SCR p is implementable only when δ is bounded away from 0 regardless of ε. Take any $t^1 \in (1 - \varepsilon, 1]$ and $s^1 = 1 - \varepsilon$. The necessary condition (4) for p to be implementable implies $\max_{F^1 \in \mathcal{F}^1} F^1(\{s^2 \in [0, 0.5]\}) \geq 1$. By the definition of the Prokhorov metric, this inequality holds only if $\delta \geq 0.25$.

A.2 Proof of Theorem 4.1

Define $\overline{m} \equiv \max_{i\in\mathcal{I},k\in\mathcal{K},s^i\in S^i} \frac{dv_k^i(s^i)}{ds^i}$. The next lemma follows immediately from the Mean Value Theorem and the definition of $\overline{m}$.

Lemma A1. *$|v_k^i(s^i) - v_k^i(t^i)| \le \overline{m}|s^i - t^i|$ for all $s^i, t^i \in S^i$, $i \in \mathcal{I}$, and $k \in \mathcal{K}$.*

Fix a Bayesian environment $\mathbf{E}^B$ and a SCR p. Take $\varepsilon > 0$. Recall that for any $A \subseteq S^{-i}$ and $\delta > 0$, $\mathcal{B}_\delta(A) = \{s^{-i} \in S^{-i} | \inf_{t^{-i}\in A} \parallel s^{-i} - t^{-i} \parallel_\infty \le \delta\}$. Let $\phi(\delta) \equiv \sup_{i,A\in\Sigma^{-i}} G^i(\mathcal{B}_\delta(A)) - G^i(A)$. Since G^i is absolutely continuous with respect to the Lebesgue measure, we obtain $\lim_{\delta\to 0}\phi(\delta) = 0$. Take $0 < \delta \le 1$ such that $\phi(\delta) + \delta \le \min\{\varepsilon, \frac{\varepsilon}{\overline{m}}\}$, and a δ-ambiguity environment $\mathbf{E}^\delta$ in which $\mathcal{F}^i = B_\delta(G^i)$ for all i. Suppose that p is implementable by the transfer scheme x with associated indirect utility functions $\mu^i_{(p,x)}$ in $\mathbf{E}^\delta$. We are going to show p is ε-locally implementable by the full insurance transfer scheme x_F with $\{\mu^i_{(p,x)}\}_i$ in $\mathbf{E}^B$. By construction, $\mu^i_{(p,x)}$ is also agent i's indirect utility function associated with (p, x_F) in $\mathbf{E}^B$. We thus need to show $\mu^i_{(p,x)}$ is monotone, and (p, x_F) is ε-bounded and ε-LIC.

Fix i and s^i, t^i. Since (p, x) is incentive compatible in $\mathbf{E}^\delta$, we have

$$\begin{aligned}\mu^i_{(p,x)}(s^i) &\ge \min_{F^i\in\mathcal{F}^i}\int_{\mathcal{S}^{-i}} \Big(\sum_{k\in\mathcal{K}} p_k(t^i, s^{-i})v_k^i(s^i) + x^i(t^i, s^{-i})\Big)dF^i(s^{-i}) \\ &\ge \mu^i_{(p,x)}(t^i) + \min_{F^i\in\mathcal{F}^i}\int_{\mathcal{S}^{-i}} \sum_{k\in\mathcal{K}} p_k(t^i, s^{-i})\big(v_k^i(s^i) - v_k^i(t^i)\big)dF^i(s^{-i}).\end{aligned} \tag{A1}$$

An immediate observation is if $s^i > t^i$, we have $v_k^i(s^i) - v_k^i(t^i) \ge 0$ for all k and, hence, $\mu^i_{(p,x)}(s^i) \ge \mu^i_{(p,x)}(t^i)$. That is, $\mu^i_{(p,x)}$ is monotone, as desired. If $s^i < t^i$, (A1) implies there exists $\hat{F}^i \in \mathcal{F}^i = B_\delta(G^i)$ such that

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) \le \sum_{k\in\mathcal{K}} \hat{F}^i\big(A_k(t^i, p)\big)(v_k^i(t^i) - v_k^i(s^i)).$$

Since $\hat{F}^i \in B_\delta(G^i)$ and $\phi(\delta) + \delta \le \varepsilon$, the definition of the Prokhorov metric implies that $0 \le \hat{F}^i\big(A_k(t^i, p)\big) \le G^i\big(A_k(t^i, p)\big) + \varepsilon$ for all k. Since $\hat{F}^i$ is a probability distribution, $\sum_{k\in\mathcal{K}} \hat{F}^i\big(A_k(t^i, p)\big) = 1$. Taking $w_k = \hat{F}^i\big(A_k(t^i, p)\big)$ for all $k \in \mathcal{K}$ yields that (p, x_F) is ε-bounded according to Definition 4.

We now show (p, x_F) is ε-LIC. Since $\mathcal{F}^i = B_\delta(G^i)$, we have

$$\min_{F^i\in\mathcal{F}^i}\int_{\mathcal{S}^{-i}} \sum_{k\in\mathcal{K}} p_k(t^i, s^{-i})(v_k^i(s^i) - v_k^i(t^i))dF^i(s^{-i})$$

$$\geq \int_{\mathcal{S}^{-i}} \sum_{k\in\mathcal{K}} p_k(t^i, s^{-i})(v_k^i(s^i) - v_k^i(t^i))dG^i(s^{-i}) - \overline{m}(\phi(\delta)+\delta)|s^i - t^i|$$

$$\geq \int_{\mathcal{S}^{-i}} \sum_{k\in\mathcal{K}} p_k(t^i, s^{-i})\big(v_k^i(s^i) - v_k^i(t^i)\big)dG^i(s^{-i}) - \varepsilon|s^i - t^i|.$$

The first inequality follows from Lemma A1 and the last inequality follows from $\phi(\delta)+\delta \leq \frac{\varepsilon}{\overline{m}}$. Combining this with (A1) yields

$$\mu^i_{(p,x)}(s^i) \geq \mu^i_{(p,x)}(t^i) + \int_{\mathcal{S}^{-i}} \sum_{k\in\mathcal{K}} p_k(t^i, s^{-i})\big(v_k^i(s^i) - v_k^i(t^i)\big)dG^i(s^{-i}) - \varepsilon|s^i - t^i|$$

$$= u^i_{(p,x_F)}(t^i, s^i) - \varepsilon|s^i - t^i|,$$

where $u^i_{(p,x_F)}$ is i's payoff under the full insurance mechanism (p, x_F) in $\mathbf{E}^B$. This completes the proof.

A.3 Proof of Theorem 4.2

We start with some notation. Let $K^i \equiv |\mathcal{K}^i|$. Define $\underline{m} \equiv \min_{i\in\mathcal{I},k\in\mathcal{K}^i,s^i\in S^i} \frac{dv_k^i(s^i)}{ds^i}$. Since v_k^i is continuously differentiable and $\frac{dv_k^i(s^i)}{ds^i} > 0$ for all $k \in \mathcal{K}^i$ and $s^i \in S^i$, the compactness of the signal space implies $\underline{m} > 0$. Let $A_0(s^i, p) \equiv \cup_{k\in\mathcal{K}_0^i} A_k(s^i, p)$.

Fix a Bayesian environment $\mathbf{E}^B$ and a rich SCR p. Take $0 < \delta \leq 1$ and a corresponding δ-ambiguity environment $\mathbf{E}^\delta$. Take $\varepsilon > 0$ such that $\max\{\frac{\varepsilon}{\underline{m}}, \varepsilon\} \leq \delta$. Suppose that p is ε-locally implementable by the transfer scheme x with associated indirect utility functions $\mu^i_{(p,x)}$ in $\mathbf{E}^B$. We are going to show that p is implementable by the full insurance transfer scheme x_F with $\{\mu^i_{(p,x)}\}_i$ in $\mathbf{E}^\delta$. Recall from Section 3 that by construction, $\mu^i_{(p,x)}$ is also agent i's indirect utility function associated with (p, x_F) in $\mathbf{E}^\delta$. We thus only need to show for all $s^i, t^i \in S^i$,

$$\begin{aligned}\mu^i_{(p,x)}(s^i) &\geq u^i_{(p,x_F)}(t^i, s^i)\\ &= \mu^i_{(p,x)}(t^i) + \min_{F^i\in\mathcal{F}^i} \sum_{k\in\mathcal{K}} (v_k^i(s^i) - v_k^i(t^i))F^i(A_k(t^i, p)),\end{aligned} \tag{A2}$$

where $u^i_{(p,x_F)}$ is i's interim payoff under (p, x_F) in $\mathbf{E}^\delta$.

Fix $i \in \mathcal{I}$ and $s^i, t^i \in S^i$. Consider first the case $s^i > t^i$. If $\sum_{k\in\mathcal{K}^i} G^i\big(A_k(t^i, p)\big) < \frac{\varepsilon}{\underline{m}}$, construct $\hat{F}^i$ as follows: $\hat{F}^i\big(\cup_{k\in\mathcal{K}^i} A_k(t^i, p)\big) = 0$ and $\hat{F}^i\big(A_0(t^i, p)\big) = 1$. Since $\delta \geq \frac{\varepsilon}{\underline{m}}$, we have $\hat{F}^i \in B_\delta(G^i) \subseteq \mathcal{F}^i$. Since $\mu^i_{(p,x)}$ is increasing, we have

$$\mu^i_{(p,x)}(s^i) \geq \mu^i_{(p,x)}(t^i) + \sum_{k\in\mathcal{K}} (v_k^i(s^i) - v_k^i(t^i))\hat{F}^i(A_k(t^i, p))$$

$$\geq \mu^i_{(p,x)}(t^i) + \min_{F^i \in \mathcal{F}^i} \sum_{k \in \mathcal{K}} (v^i_k(s^i) - v^i_k(t^i)) F^i(A_k(t^i, p)),$$

as desired. Suppose now $\sum_{k \in \mathcal{K}^i} G^i\big(A_k(t^i, p)\big) \geq \frac{\varepsilon}{m}$. By the ε-LIC of (p, x),

$$\mu^i_{(p,x)}(s^i) \geq \mu^i_{(p,x)}(t^i) + \sum_{k \in \mathcal{K}} (v^i_k(s^i) - v^i_k(t^i)) G^i(A_k(t^i, p)) - \varepsilon |s^i - t^i|. \tag{A3}$$

Combining (A2) and (A3) indicates it suffices to show for some $\hat{F}^i \in \mathcal{F}^i$,

$$\begin{aligned} &\sum_{k \in \mathcal{K}} \big(v^i_k(s^i) - v^i_k(t^i)\big) G^i(A_k(t^i, p)) - \varepsilon |s^i - t^i| \\ &\qquad\qquad \geq \sum_{k \in \mathcal{K}} \big(v^i_k(s^i) - v^i_k(t^i)\big) \hat{F}^i(A_k(t^i, p)). \end{aligned} \tag{A4}$$

Since $\sum_{k \in \mathcal{K}^i} G^i\big(A_k(t^i, p)\big) \geq \frac{\varepsilon}{m}$, we can construct $\hat{F}^i$ as follows: $\hat{F}^i\big(A_0(t^i, p)\big) = G^i\big(A_0(t^i, p)\big) + \frac{\varepsilon}{m}$ and $\hat{F}^i\big(\cup_{k \in \mathcal{K}^i} A_k(t^i, p)\big) = G^i\big(\cup_{k \in \mathcal{K}^i} A_k(t^i, p)\big) - \frac{\varepsilon}{m}$. By assumption, $t^i < s^i \leq 1$ and, hence, $A_0(t^i, p) \neq \varnothing$. Thus, $\hat{F}^i$ is well-defined. Since $\delta \geq \frac{\varepsilon}{m}$, we have $\hat{F}^i \in B_\delta(G^i) \subseteq \mathcal{F}^i$. It is straightforward to verify that the constructed $\hat{F}^i$ satisfies (A4), as desired.

Suppose $s^i < t^i$. If $G^i\big(A_0(t^i, p)\big) \geq \frac{\varepsilon}{m}$, construct $\hat{F}^i$ such that $\hat{F}^i\big(A_0(t^i, p)\big) = G^i\big(A_0(t^i, p)\big) - \frac{\varepsilon}{m}$ and $\hat{F}^i\big(\cup_{k \in \mathcal{K}^i} A_k(t^i, p)\big) = G^i\big(\cup_{k \in \mathcal{K}^i} A_k(t^i, p)\big) + \frac{\varepsilon}{m}$. Since $\delta \geq \frac{\varepsilon}{m}$, we have $\hat{F}^i \in B_\delta(G^i) \subseteq \mathcal{F}^i$. The construction of $\hat{F}^i$ yields

$$\begin{aligned} &\sum_{k \in \mathcal{K}} \big(v^i_k(t^i) - v^i_k(s^i)\big) G^i(A_k(t^i, p)) + \varepsilon(t^i - s^i) \\ &\qquad\qquad \leq \sum_{k \in \mathcal{K}} \big(v^i_k(t^i) - v^i_k(s^i)\big) \hat{F}^i(A_k(t^i, p)). \end{aligned}$$

Combining this with the ε-LIC constraint (A3) yields (A2), as desired. Suppose now that $G^i\big(A_0(t^i, p)\big) < \frac{\varepsilon}{m}$. Since $\delta \geq \frac{\varepsilon}{m}$,

$$\sum_{k \in \mathcal{K}^i} G^i\big(A_k(t^i, p)\big) + \delta \geq \sum_{k \in \mathcal{K}^i} G^i\big(A_k(t^i, p)\big) + \frac{\varepsilon}{m} > 1. \tag{A5}$$

Without loss of generality, we can relabel the indexes of alternatives in $\mathcal{K}^i$ so that

$$v^i_{K^i}(t^i) - v^i_{K^i}(s^i) \geq \cdots \geq v^i_1(t^i) - v^i_1(s^i) > 0. \tag{A6}$$

If $G^i\big(A_{K^i}(t^i, p)\big) + \delta \geq 1$, there exists $\hat{F}^i \in \mathcal{F}^i$ so that $\hat{F}^i\big(A_{K^i}(t^i, p)\big) = 1$. Since (p, x) is ε-bounded,

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) \leq v^i_{K^i}(t^i) - v^i_{K^i}(s^i) = \hat{F}^i\big(A_{K^i}(t^i, p)\big)\big(v^i_{K^i}(t^i) - v^i_{K^i}(s^i)\big).$$

Since $\hat{F}^i \in \mathcal{F}^i$, the above inequalities imply the desired inequalities in (A2). If

$G^i\big(A_{K^i}(t^i,p)\big)+\delta<1$, let $l\in\{2,...,K^i\}$ be such that

$$\sum_{k=l}^{K^i} G^i\big(A_k(t^i,p)\big)+\delta<1 \quad \text{and} \quad \sum_{k=l-1}^{K^i} G^i\big(A_k(t^i,p)+\delta\geq 1. \tag{A7}$$

Such l exists by the inequalities in (A5). Construct $\hat{F}^i$ as follows:

$$\begin{aligned}
&\hat{F}^i\big(A_k(t^i,p)\big)=0 \quad \forall k=0,...,l-2,\\
&\hat{F}^i\big(A_{l-1}(t^i,p)\big)=\sum_{k=0}^{l-1} G^i\big(A_k(t^i,p)\big)-\delta,\\
&\hat{F}^i\big(A_k(t^i,p)\big)=G^i\big(A_k(t^i,p)\big) \quad \forall k=l,...,K^i-1,\\
&\hat{F}^i\big(A_{K^i}(t^i,p)\big)=G^i\big(A_{K^i}(t^i,p)\big)+\delta.
\end{aligned}$$

By construction, $\hat{F}^i\in B_\delta(G^i)\subseteq\mathcal{F}^i$. Since (p,x) is ε-bounded, there exists $w_k\leq G^i\big(A_k(t^i,p)\big)+\varepsilon\leq G^i\big(A_k(t^i,p)\big)+\delta$ and $\sum_{k\in\mathcal{K}} w_k\leq 1$ such that

$$\mu^i_{(p,x)}(t^i)-\mu^i_{(p,x)}(s^i)\leq\sum_{k\in\mathcal{K}} w_k\big(v^i_k(t^i)-v^i_k(s^i)\big)\leq\sum_{k\in\mathcal{K}}\hat{F}^i\big(A_k(t^i,p)\big)\big(v^i_k(t^i)-v^i_k(s^i)\big).$$

The last inequality follows from (A6) and the construction of $\hat{F}^i$. Since $\hat{F}^i\in\mathcal{F}^i$, this implies (A2) and completes the proof.

A.4 Proof of Proposition 4.1

Fix a Bayesian environment $\mathbf{E}^B$ and a SCR p such that $\sum_{k\in\mathcal{K}^i} G^i\big(A_k(s^i,p)\big)$ is nondecreasing in s^i for all $i\in\mathcal{I}$. Take $\delta>0$ and $\varepsilon=\frac{\underline{m}}{\overline{m}}\delta$. Let (p,x) be an ε-LIC mechanism with indirect utility functions $\mu^i_{(p,x)}$. Fix $i\in\mathcal{I}$. Define

$$R^i_-=\{s^i\in S^i|\sum_{k\in\mathcal{K}^i} G^i\big(A_k(s^i,p)\big)<\frac{\varepsilon}{\underline{m}}\} \quad \text{and} \quad R^i_+=S^i\setminus R^i_-.$$

The assumption that $\sum_{k\in\mathcal{K}^i} G^i\big(A_k(s^i,p)\big)$ is nondecreasing implies R^i_+ lies to the right of R^i_-.

Lemma A2. *The indirect utility function $\mu^i_{(p,x)}$ is monotone on R^i_+.*

Proof. Take $s^i,t^i\in R^i_+$ with $s^i>t^i$. The definition of $\underline{m}$ and the ε-LIC constraint (A3) imply

$$\mu^i_{(p,x)}(s^i)\geq\mu^i_{(p,x)}(t^i)+(s^i-t^i)\Big(\underline{m}\sum_{k\in\mathcal{K}^i} G^i(A_k(t^i,p))-\varepsilon\Big)\geq\mu^i_{(p,x)}(t^i).$$

The second inequality follows from $t^i\in R^i_+$. □

Lemma A3. *The indirect utility function $\mu^i_{(p,x)}$ is Lipschitz continuous on S^i.*

Proof. For any $s^i, t^i \in S^i$, the ε-LIC of (p, x) implies

$$\sum_{k\in\mathcal{K}} (v^i_k(s^i) - v^i_k(t^i))G^i(A_k(t^i, p)) - \varepsilon|s^i - t^i| \le \mu^i_{(p,x)}(s^i) - \mu^i_{(p,x)}(t^i)$$
$$\le \sum_{k\in\mathcal{K}} (v^i_k(s^i) - v^i_k(t^i))G^i(A_k(s^i, p)) + \varepsilon|s^i - t^i|.$$

Combining these two inequalities with Lemma A1 yields

$$|\mu^i_{(p,x)}(s^i) - \mu^i_{(p,x)}(t^i)| \le (\overline{m} + \varepsilon)|s^i - t^i|,$$

as desired. □

We are now ready to prove Proposition 4.1. We will construct a transfer scheme $\hat{x}$ such that $(p, \hat{x})$ is δ-LIC and $\mu^i_{(p,\hat{x})}$ is monotone. If $R^i_- = \varnothing$, then Lemma A2 implies $\mu^i_{(p,x)}$ is monotone. Since $\overline{m} \ge \underline{m}$, we know that $\delta \ge \varepsilon$. Taking $\hat{x} = x$ completes the proof. If $R^i_+ = \varnothing$, fix a constant $\alpha \in \mathbb{R}$. Construct $\hat{x}$ as follows:

$$\hat{x}^i(s) = \alpha - \sum_{k\in\mathcal{K}} p_k(s)v^i_k(s^i) \quad \forall s \in S, \forall i \in \mathcal{I}.$$

By construction, $\mu^i_{(p,\hat{x})}(s^i) = \alpha$ for all s^i, which is trivially monotone and δ-bounded. We now show $(p, \hat{x})$ is δ-LIC. Fix $s^i, t^i \in S^i = R^i_-$. It follows from Lemma A1 and $t^i \in R^i_-$ that

$$\sum_{k\in\mathcal{K}} (v^i_k(s^i) - v^i_k(t^i))G^i(A_k(t^i, p)) \le \overline{m}|s^i - t^i| \sum_{k\in\mathcal{K}^i} G^i(A_k(t^i, p)) \le \delta|s^i - t^i|. \quad \text{(A8)}$$

By (A8) and $\mu^i_{(p,\hat{x})}(t^i) = \mu^i_{(p,\hat{x})}(s^i) = \alpha$, we obtain

$$\mu^i_{(p,\hat{x})}(s^i) \ge \mu^i_{(p,\hat{x})}(t^i) + \sum_{k\in\mathcal{K}} (v^i_k(s^i) - v^i_k(t^i))G^i(A_k(t^i, p)) - \delta|s^i - t^i|$$
$$= u^i_{(p,\hat{x})}(t^i, s^i) - \delta|s^i - t^i|,$$

as desired.

We now consider the last case where $R^i_- \ne \varnothing$ and $R^i_+ \ne \varnothing$. Define $\underline{\mu}^i \equiv \inf_{s^i\in R^i_+} \mu^i_{(p,x)}(s^i)$. Construct $\hat{x}$ as follows:

$$\hat{x}^i(s) = \underline{\mu}^i - \sum_{k\in\mathcal{K}} p_k(s)v^i_k(s^i) \quad \forall s^i \in R^i_-$$
$$\hat{x}^i(s) = \mu^i_{(p,x)}(s^i) - \sum_{k\in\mathcal{K}} p_k(s)v^i_k(s^i) \quad \forall s^i \in R^i_+.$$

We first show $\mu^i_{(p,\hat{x})}$ is monotone. Take $s^i > t^i$. If $s^i, t^i \in R^i_+$, the construction of $\hat{x}$ and Lemma A2 imply that $\mu^i_{(p,\hat{x})}(s^i) = \mu^i_{(p,x)}(s^i) \geq \mu^i_{(p,x)}(t^i) = \mu^i_{(p,\hat{x})}(t^i)$. If $s^i, t^i \in R^i_-$, the construction of $\hat{x}$ implies $\mu^i_{(p,\hat{x})}(s^i) = \mu^i_{(p,\hat{x})}(t^i) = \underline{\mu}^i$ and, hence, is trivially monotone. Note that it is impossible that $s^i \in R^i_-$ and $t^i \in R^i_+$ since R^i_+ lies to the right of R^i_-. Thus, the last possible case is $s^i \in R^i_+$ and $t^i \in R^i_-$. By the definition of $\underline{\mu}^i$, we have $\mu^i_{(p,\hat{x})}(s^i) = \mu^i_{(p,x)}(s^i) \geq \underline{\mu}^i = \mu^i_{(p,\hat{x})}(t^i)$. Thus, $\mu^i_{(p,\hat{x})}$ is monotone.

We next show $(p, \hat{x})$ is δ-LIC. Fix $s^i, t^i \in S^i$. If $s^i, t^i \in R^i_+$, the ε-LIC of (p, x) implies the δ-LIC of $(p, \hat{x})$ as $\delta \geq \varepsilon$. If $s^i, t^i \in R^i_-$, the proof follows from the same argument as in the case of $R^i_+ = \varnothing$. Consider now $s^i \in R^i_+$ and $t^i \in R^i_-$. As argued above, $s^i > t^i$. The construction of $\hat{x}$ and (A8) together imply

$$
\begin{aligned}
u^i_{(p,\hat{x})}(t^i, s^i) - \delta|s^i - t^i| &= \underline{\mu}^i + \sum_{k \in \mathcal{K}^i} (v^i_k(s^i) - v^i_k(t^i)) G^i(A_k(t^i, p)) - \delta|s^i - t^i| \\
&\leq \underline{\mu}^i \leq \mu^i_{(p,\hat{x})}(s^i),
\end{aligned}
$$

as desired. Suppose $t^i \in R^i_+$ and $s^i \in R^i_-$. Then $t^i > s^i$. Let $r^i \equiv \min\{\tilde{s}^i \in cl(R^i_+) | \mu^i_{(p,x)}(\tilde{s}^i) = \underline{\mu}^i\}$, where $cl(R^i_+)$ is the closure of R^i_+. The existence of r^i follows from Lemma A3. Since $\sum_{k \in \mathcal{K}^i} G^i(A_k(\cdot, p))$ is nondecreasing, we have $s^i \leq r^i \leq t^i$. By the ε-LIC of (p, x), we have

$$
\begin{aligned}
\underline{\mu}^i = \mu^i_{(p,x)}(r^i) &\geq \mu^i_{(p,x)}(t^i) + \sum_{k \in \mathcal{K}^i} (v^i_k(r^i) - v^i_k(t^i)) G^i(A_k(t^i, p)) - \varepsilon|r^i - t^i| \\
&\geq \mu^i_{(p,x)}(t^i) + \sum_{k \in \mathcal{K}^i} (v^i_k(s^i) - v^i_k(t^i)) G^i(A_k(t^i, p)) - \varepsilon|s^i - t^i| \qquad \text{(A9)} \\
&\geq u^i_{(p,\hat{x})}(t^i, s^i) - \delta|s^i - t^i|.
\end{aligned}
$$

The second inequality follows from $r^i \geq s^i$ and the last inequality follows from the construction of $\hat{x}$ and $\delta \geq \varepsilon$. Then since $\mu^i_{(p,\hat{x})}(s^i) = \underline{\mu}^i$, (A9) implies $\mu^i_{(p,\hat{x})}(s^i) \geq u^i_{(p,\hat{x})}(t^i, s^i) - \delta|s^i - t^i|$, as desired.

Finally, we show if (p, x) is δ-bounded, so is $(p, \hat{x})$. Take $s^i, t^i \in S^i$. The construction of $\hat{x}$ implies that the requirements in Definition 4 are satisfied if $s^i, t^i \in R^i_+$ or $s^i, t^i \in R^i_-$. Now consider $s^i \in R^i_-$, $t^i \in R^i_+$ and $s^i < t^i$. Since (p, x) is δ-bounded, there exists $\{w_k\}_{k \in \mathcal{K}}$ such that $0 \leq w_k \leq G^i(A_k(t^i, p)) + \delta$ for all k, $\sum_{k \in \mathcal{K}} w_k \leq 1$, and

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(r^i) \le \sum_{k\in\mathcal{K}} w_k\big(v^i_k(t^i) - v^i_k(r^i)\big). \tag{A10}$$

Since $s^i \in R^i_{\underline{\ }}$, we have $\mu^i_{(p,\hat{x})}(s^i) = \underline{\mu}^i = \mu^i_{(p,x)}(r^i)$ and $s^i \le r^i$. Combining these two observations with (A10) yields

$$\mu^i_{(p,\hat{x})}(t^i) - \mu^i_{(p,\hat{x})}(s^i) \le \sum_{k\in\mathcal{K}} w_k\big(v^i_k(t^i) - v^i_k(s^i)\big).$$

This completes the proof.

A.5 Proof of Lemma 4.1 and Proposition 4.2

We prove Proposition 4.2 and the proof of Lemma 4.1 is immediate.

By supermodularity, for each i, we can relabel the indexes of social alternatives $\{1, ..., K^i\}$ so that

$$0 = \frac{dv^i_k(s^i)}{ds^i} < \frac{dv^i_1(s^i)}{ds^i} \le \cdots \le \frac{dv^i_{K^i}(s^i)}{ds^i} \quad \forall k \in \mathcal{K}^i_0, \forall s^i \in S^i. \tag{A11}$$

Let $\underline{K}^i \equiv \{k \in \mathcal{K}^i | \frac{dv^i_k(s^i)}{ds^i} < \frac{dv^i_{K^i}(s^i)}{ds^i} \; \forall s^i \in S^i\}$. If $\underline{K}^i \neq \varnothing$, let $\frac{1}{\varrho^i} \equiv \min_{k\in\underline{K}^i, \tilde{s}^i \in S^i}\Big(\frac{dv^i_{K^i}(\tilde{s}^i)}{d\tilde{s}^i} - \frac{dv^i_k(\tilde{s}^i)}{d\tilde{s}^i}\Big)$. Supermodularity implies $\varrho^i < \infty$. If $\underline{K}^i = \varnothing$, let $\varrho^i \equiv 0$. Take $c^i \equiv \max\{\varrho^i, \frac{1}{\underline{m}}\}$ and $c \equiv \max_i c^i$. Let $w_0 = \sum_{k\in\mathcal{K}^i_0} w_k$.

The result is trivially true when $\varepsilon = 0$. Fix $\varepsilon > 0$, an ε-LIC mechanism (p, x), and i. We first show (11) implies $c\varepsilon$-boundedness. Fix $s^i < t^i$. It follows from (A11) that (11) is equivalent to

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) \le v^i_{K^i}(t^i) - v^i_{K^i}(s^i). \tag{A12}$$

Thus, we only need to show (A12) implies $c\varepsilon$-boundedness. If $c\varepsilon \ge 1$, then taking $w_{K^i} = 1$ and $w_k = 0$ for all $k \neq K^i$ yields the desired result trivially. Thus, suppose $c\varepsilon < 1$. Consider first the case $\underline{K}^i = \varnothing$. Since (p, x) is ε-LIC, we have

$$\begin{aligned}\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) &\le \int \sum_k p_k(t^i, s^{-i})\big(v^i_k(t^i) - v^i_k(s^i)\big)dG^i + \varepsilon(t^i - s^i) \\ &= \sum_{k\in\mathcal{K}} \big(v^i_k(t^i) - v^i_k(s^i)\big)G^i\big(A_k(t^i, p)\big) + \varepsilon(t^i - s^i).\end{aligned} \tag{A13}$$

If $\sum_{k\in\mathcal{K}^i} G^i\big(A_k(t^i, p)\big) \le 1 - c\varepsilon$, then take $w_k = G^i\big(A_k(t^i, p)\big)$ for all $0 < k < K^i$, $w_{K^i} = G^i\big(A_{K^i}(t^i, p)\big) + c\varepsilon$, and $w_0 = 1 - \sum_{k>0} w_k$. By construction, conditions (i) and (ii) in Definition 4 are satisfied. Since $c \ge \frac{1}{\underline{m}}$, we obtain that $c\big(v^i_{K^i}(t^i) -$

$v^i_{K^i}(s^i)) \geq c\underline{m}(t^i - s^i) \geq t^i - s^i$. Then by the construction of $\{w_k\}_k$, we obtain

$$\sum_{k\in\mathcal{K}} \big(v^i_k(t^i) - v^i_k(s^i)\big)G^i\big(A_k(t^i,p)\big) + \varepsilon(t^i - s^i) \leq \sum_{k\in\mathcal{K}} w_k\big(v^i_k(t^i) - v^i_k(s^i)\big).$$

Combining this inequality with (A13) yields inequality (5), as desired.

If $\sum_{k\in\mathcal{K}^i} G^i\big(A_k(t^i,p)\big) > 1 - c\varepsilon$, then take $w_0 = 0$, $w_k = G^i\big(A_k(t^i,p)\big)$ for all $0 < k < K^i$, and $w_{K^i} = 1 - \sum_{k<K^i} w_k$. Since $\sum_{k\in\mathcal{K}^i} G^i\big(A_k(t^i,p)\big) > 1 - c\varepsilon$, we obtain $w_{K^i} < G^i\big(A_{K^i}(t^i,p)\big) + c\varepsilon$. Thus, conditions (i) and (ii) are satisfied. To verify inequality (5), note that $\underline{K}^i = \varnothing$ implies $v^i_{K^i}(t^i) - v^i_{K^i}(s^i) = v^i_k(t^i) - v^i_k(s^i)$ for all $k \in \mathcal{K}^i$. Thus, $\sum_{k\in\mathcal{K}} w_k\big(v^i_k(t^i) - v^i_k(s^i)\big) = v^i_{K^i}(t^i) - v^i_{K^i}(s^i)$. Then inequality (5) is a direct implication of (A12), as desired.

Suppose now $\underline{K}^i \neq \varnothing$. Let $l \in \{0, 1, ..., K^i - 1\}$ be such that

$$G^i\big(A_0(t^i,p)\big) + \sum_{k=1}^{l} G^i\big(A_k(t^i,p)\big) < c\varepsilon \quad \text{and} \quad G^i\big(A_0(t^i,p)\big) + \sum_{k=1}^{l+1} G^i\big(A_k(t^i,p)\big) \geq c\varepsilon.$$

If $G^i\big(A_0(t^i,p)\big) \geq c\varepsilon$, then let $l = -1$. If $\frac{dv^i_{l+1}(\tilde{s}^i)}{d\tilde{s}^i} = \frac{dv^i_{K^i}(\tilde{s}^i)}{d\tilde{s}^i}$ for all $\tilde{s}^i \in S^i$, take $w_k = 0$ for all $k \leq l$, $w_k = G^i\big(A_k(t^i,p)\big)$ for all $l < k < K^i$, and $w_{K^i} = 1 - \sum_{k<K^i} w_k$. By the choice of l, we have $w_{K^i} < G^i\big(A_{K^i}(t^i,p)\big) + c\varepsilon$. By (A12),

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) \leq v^i_{K^i}(t^i) - v^i_{K^i}(s^i) = \sum_{k\in\mathcal{K}} w_k\big(v^i_k(t^i) - v^i_k(s^i)\big).$$

Thus, (p,x) is $c\varepsilon$-bounded. If $\frac{dv^i_{l+1}(\tilde{s}^i)}{d\tilde{s}^i} < \frac{dv^i_{K^i}(\tilde{s}^i)}{d\tilde{s}^i}$, supermodularity implies $\frac{dv^i_{K^i}(\tilde{s}^i)}{d\tilde{s}^i} - \frac{dv^i_{l+1}(\tilde{s}^i)}{d\tilde{s}^i} \geq \frac{1}{\varrho^i}$ for all $\tilde{s}^i$. Thus,

$$v^i_{K^i}(t^i) - v^i_{K^i}(s^i) - \big(v^i_{l+1}(t^i) - v^i_{l+1}(s^i)\big) = \int_{s^i}^{t^i} \Big(\frac{dv^i_{K^i}(\tilde{s}^i)}{d\tilde{s}^i} - \frac{dv^i_{l+1}(\tilde{s}^i)}{d\tilde{s}^i}\Big)d\tilde{s}^i \geq \frac{1}{\varrho^i}(t^i - s^i). \quad \text{(A14)}$$

Take $w_k = 0$ for all $k \leq l$, $w_{l+1} = \sum_{k=0}^{l+1} G^i\big(A_k(t^i,p)\big) - c\varepsilon$, $w_k = G^i\big(A_k(t^i,p)\big)$ for all $l + 1 < k < K^i$, and $w_{K^i} = 1 - \sum_{k<K^i} w_k = G^i\big(A_{K^i}(t^i,p)\big) + c\varepsilon$. By construction, conditions (i) and (ii) are satisfied. To see why inequality (5) is satisfied, note that by ε-LIC,

$$\begin{aligned}\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) &\leq \int \sum_k p_k(t^i, s^{-i})\big(v^i_k(t^i) - v^i_k(s^i)\big)dG^i + \varepsilon(t^i - s^i)\\ &= \sum_k G^i\big(A_k(t^i,p)\big)\big(v^i_k(t^i) - v^i_k(s^i)\big) + \varepsilon(t^i - s^i)\\ &\leq \sum_{k\in\mathcal{K}} w_k\big(v^i_k(t^i) - v^i_k(s^i)\big).\end{aligned}$$

The last inequality follows from (A14) and $c \geq \varrho^i$. This completes the proof.

Next we show that if (p, x) is $c\varepsilon$-bounded, then it satisfies (11). By $c\varepsilon$-boundedness,

$$\mu^i_{(p,x)}(t^i) - \mu^i_{(p,x)}(s^i) \leq \sum_{k\in\mathcal{K}} w_k\big(v^i_k(t^i) - v^i_k(s^i)\big) \leq \max_k \left(v^i_k(t^i) - v^i_k(s^i)\right) \quad \forall s^i < t^i,$$

where the last inequality follows from $w_k \geq 0$ for all k and $\sum_k w_k \leq 1$.

B Proof of Theorem 5.1

For simplicity, denote $\mathcal{E}_k(s^i) \equiv A_k(s^i, p^*)$ for all s^i, k, and i. Lemmas B4 to B8 below establish a key step of the proof of Theorem 5.1: for all k, s^i, and ξ-efficient SCR p^ξ, the set $A_k(s^i, p^\xi)$ converges to the set $\mathcal{E}_k(s^i)$ as $\xi \to 0$.

Fix $\xi > 0$ and a ξ-efficient SCR p^ξ. For each $i \in \mathcal{I}$ and $s^i \in S^i$, define

$$O(s^i, p^\xi) \equiv \{s^{-i} \in S^{-i} | \max_{k\in\mathcal{K}} \sum_{l=1}^{N} v^l_k(s^l) - \sum_{k\in\mathcal{K}} p^\xi_k(s) \sum_{l=1}^{N} v^l_k(s^l) \geq \xi\}.$$

For each s^i and each $k \in \mathcal{K}$, let $A^+_k(s^i, p^\xi) \equiv A_k(s^i, p^\xi) \cap O(s^i, p^\xi)$ and $A^-_k(s^i, p^\xi) \equiv A_k(s^i, p^\xi) \setminus A^+_k(s^i, p^\xi)$. For each $i \in \mathcal{I}$, $s^i \in S^i$, and $k \in \mathcal{K}$, define

$$\underline{A}^\xi_k(s^i) \equiv \{s^{-i} \in S^{-i} | \max_{k'\neq k} \sum_{l=1}^{N} v^l_{k'}(s^l) - \sum_{l=1}^{N} v^l_k(s^l) \leq -\xi\},$$

$$\overline{A}^\xi_k(s^i) \equiv \{s^{-i} \in S^{-i} | \max_{k'\neq k} \sum_{l=1}^{N} v^l_{k'}(s^l) - \sum_{l=1}^{N} v^l_k(s^l) < \xi\}.$$

For any $s^{-i} \in \mathcal{E}_k(s^i)$, $\max_{k'\neq k} \sum_{l=1}^N v^l_{k'}(s^l) - \sum_{l=1}^N v^l_k(s^l) \leq 0$. Thus,

$$\underline{A}^\xi_k(s^i) \subseteq \mathcal{E}_k(s^i) \subseteq \overline{A}^\xi_k(s^i) \quad \forall s^i \in S^i, \forall i \in \mathcal{I}, \forall k \in \mathcal{K}. \tag{B15}$$

Lemma B4. *For every $i \in \mathcal{I}$, $s^i \in S^i$, and $k \in \mathcal{K}$, we have $A^+_k(s^i, p^\xi) \cap \mathcal{E}_k(s^i) = \varnothing$.*

Proof. Note that $A^+_k(s^i, p^\xi) \subseteq \{s^{-i} \in S^{-i} | \max_{k'\in\mathcal{K}} \sum_{l=1}^N v^l_{k'}(s^l) - \sum_{l=1}^N v^l_k(s^l) \geq \xi\}$. It follows from the definition of $\mathcal{E}_k(s^i)$ that $s^{-i} \notin A^+_k(s^i, p^\xi)$ for any $s^{-i} \in \mathcal{E}_k(s^i)$. □

Lemma B5. $\underline{A}^\xi_k(s^i) \setminus O(s^i, p^\xi) \subseteq A^-_k(s^i, p^\xi) \subseteq \overline{A}^\xi_k(s^i)$ *for all $k \in \mathcal{K}$, $s^i \in S^i$, and $i \in \mathcal{I}$.*

Proof. Fix $k \in \mathcal{K}$, $i \in \mathcal{I}$, and $s^i \in S^i$. We first prove $\underline{A}^\xi_k(s^i) \setminus O(s^i, p^\xi) \subseteq A^-_k(s^i, p^\xi)$ by way of contradiction. Take $s^{-i} \in \underline{A}^\xi_k(s^i) \setminus O(s^i, p^\xi)$. Suppose that $s^{-i} \notin A^-_k(s^i, p^\xi)$. Since $A^+_k(s^i, p^\xi) \subseteq O(s^i, p^\xi)$, we know that $s^{-i} \notin A^+_k(s^i, p^\xi)$. Then there exists $\tilde{k} \neq k$ such that $s^{-i} \in A_{\tilde{k}}(s^i, p^\xi)$. Since $s^{-i} \in \underline{A}^\xi_k(s^i)$, we have

$$\sum_l v^l_k(s^l) \geq \max_{k'\neq k} \sum_l v^l_{k'}(s^l) + \xi \geq \sum_l v^l_{\tilde{k}}(s^l) + \xi.$$

Moreover, $s^{-i} \in \underline{A}_k^{\xi}(s^i)$ implies k is efficient given (s^i, s^{-i}) and, hence,

$$\max_{k'} \sum_l v_{k'}^l(s^l) - \sum_l v_{\tilde{k}}^l(s^l) = \sum_l v_k^l(s^l) - \sum_l v_{\tilde{k}}^l(s^l) \geq \xi.$$

Thus, $s^{-i} \in A_{\tilde{k}}^+(s^i, p^{\xi}) \subseteq O(s^i, p^{\xi})$, which contradicts that $s^{-i} \notin O(s^i, p^{\xi})$.

We now prove $A_k^-(s^i, p^{\xi}) \subseteq \overline{A}_k^{\xi}(s^i)$. Take $s^{-i} \in A_k^-(s^i, p^{\xi})$. If $s^{-i} \in \mathcal{E}_k(s^i)$, then (B15) implies $s^{-i} \in \overline{A}_k^{\xi}(s^i)$. If k is not efficient, the choice of s^{-i} implies that

$$\max_{k' \in \mathcal{K}} \sum_l v_{k'}^l(s^l) - \sum_l v_k^l(s^l) = \max_{k' \neq k} \sum_l v_{k'}^l(s^l) - \sum_l v_k^l(s^l) < \xi.$$

Thus, $s^{-i} \in \overline{A}_k^{\xi}(s^i)$. This completes the proof. □

Since v_k^l are continuously differentiable and signal spaces are convex and compact, Pivotality implies there is $b > 0$ such that for any $i \in \mathcal{I}$ and $k \neq k'$, there exists $j \neq i$ such that either $\frac{\partial\left(\sum_l v_k^l(s^l) - \sum_l v_{k'}^l(s^l)\right)}{\partial s^j} \geq b$ for all $s \in S$ or $\frac{\partial\left(\sum_l v_k^l(s^l) - \sum_l v_{k'}^l(s^l)\right)}{\partial s^j} \leq -b$ for all $s \in S$.

Lemma B6. *Assume Pivotality. Then $S^{-i} \setminus \underline{A}_k^{\xi}(s^i) \subseteq \mathcal{B}_{\frac{2\xi}{b}}\left(S^{-i} \setminus \overline{A}_k^{\xi}(s^i)\right)$ for all $s^i \in S^i$, $k \in \mathcal{K}$, and $i \in \mathcal{I}$.*

Proof. Fix $i \in \mathcal{I}$, $s^i \in S^i$, and $k \in \mathcal{K}$. Take $s^{-i} \in \overline{A}_k^{\xi}(s^i) \setminus \underline{A}_k^{\xi}(s^i)$. Since $S^{-i} \setminus \overline{A}_k^{\xi}(s^i) \subseteq S^{-i} \setminus \underline{A}_k^{\xi}(s^i)$, we only need to show there exists $t^{-i} \in S^{-i} \setminus \overline{A}_k^{\xi}(s^i)$ such that $\| s^{-i} - t^{-i} \|_\infty \leq \frac{2\xi}{b}$. Let $\hat{k} \in \operatorname{argmax}_{k' \neq k} \sum_l v_{k'}^l(s^i, s^{-i})$. Suppose first there exists $j \neq i$ such that $\frac{\partial\left(\sum_l v_k^l(s^l) - \sum_l v_{\hat{k}}^l(s^l)\right)}{\partial s^j} \geq b$ for all $s \in S$. Construct a t such that $t^j = s^j - \frac{2\xi}{b}$ and $t^l = s^l$ for all $l \neq j$. By construction, $\| s^{-i} - t^{-i} \|_\infty = \frac{2\xi}{b}$. We are going to show that $t^{-i} \in S^{-i} \setminus \overline{A}_k^{\xi}(s^i)$. Observe that

$$\max_{k' \neq k} \sum_l v_{k'}^l(t^l) - \sum_l v_k^l(t^l) \geq \sum_l v_{\hat{k}}^l(t^l) - \sum_l v_k^l(t^l) \geq \sum_l v_{\hat{k}}^l(s^l) - \sum_l v_k^l(s^l) + 2\xi > \xi,$$

as desired. Suppose now there exists $j \neq i$ such that $\frac{\partial\left(\sum_l v_k^l(s^l) - \sum_l v_{\hat{k}}^l(s^l)\right)}{\partial s^j} \leq -b$ for all $s \in S$. Construct a t such that $t^j = s^j + \frac{2\xi}{b}$ and $t^l = s^l$ for all $l \neq j$. By construction, $\| s^{-i} - t^{-i} \|_\infty = \frac{2\xi}{b}$. Since

$$\max_{k' \neq k} \sum_l v_{k'}^l(t^l) - \sum_l v_k^l(t^l) \geq \sum_l v_{\hat{k}}^l(t^l) - \sum_l v_k^l(t^l) \geq \sum_l v_{\hat{k}}^l(s^l) - \sum_l v_k^l(s^l) + 2\xi > \xi,$$

we obtain $t^{-i} \in S^{-i} \setminus \overline{A}_k^{\xi}(s^i)$. This completes the proof.[38] □

[38]If the constructed $t^{-i} \notin S^{-i}$, we can enlarge the set of signals to include t^{-i} and extend agent i's

For any $\xi > 0$, let $\beta_\xi \equiv \max_{s^i \in S^i, k \in \mathcal{K}, i \in \mathcal{I}} \left(G^i\big(\overline{A}_k^\xi(s^i)\big) - G^i\big(\underline{A}_k^\xi(s^i)\big) \right)$.

Lemma B7. *Assume Pivotality. Then* $\lim_{\xi \to 0} \beta_\xi = 0$.

Proof. Lemma B6 implies that for every $i \in \mathcal{I}$, $s^i \in S^i$, and $k \in \mathcal{K}$,

$$\begin{aligned} G^i\big(\overline{A}_k^\xi(s^i)\big) - G^i\big(\underline{A}_k^\xi(s^i)\big) &= G^i\big(S^{-i} \setminus \underline{A}_k^\xi(s^i) \setminus \big(S^{-i} \setminus \overline{A}_k^\xi(s^i)\big)\big) \\ &\le G^i\big(\mathcal{B}_{\frac{2\xi}{b}}\big(S^{-i} \setminus \overline{A}_k^\xi(s^i)\big) \setminus \big(S^{-i} \setminus \overline{A}_k^\xi(s^i)\big)\big). \end{aligned}$$

Since G^i is absolutely continuous with respect to the Lebesgue measure, we obtain $\lim_{\xi \to 0} \beta_\xi = 0$. □

Recall that the distance between any two SCRs p and p' is

$$d_G(p, p') = \sup_{i \in \mathcal{I}, s^i \in S^i, k \in \mathcal{K}} \max\{G^i\big(A_k(s^i, p) \setminus A_k(s^i, p')\big), G^i\big(A_k(s^i, p') \setminus A_k(s^i, p)\big)\}.$$

The next lemma states that $d_G(p^\xi, p^*) \to 0$ as $\xi \to 0$.

Lemma B8. *Assume Pivotality. For any $\varepsilon > 0$, there exists a $\overline{\xi} > 0$ such that $d_G(p^\xi, p^*) \le \varepsilon$ for all ξ-efficient SCR p^ξ and $0 < \xi < \overline{\xi}$.*

Proof. Take $\varepsilon > 0$. Let $\overline{\xi} > 0$ be such that $\beta_{\overline{\xi}} + \overline{\xi} \le \varepsilon$. The existence of such $\overline{\xi}$ follows from Lemma B7. Fix $0 < \xi < \overline{\xi}$, ξ-efficient SCR p^ξ, $i \in \mathcal{I}$, $s^i \in S^i$, and $k \in \mathcal{K}$. We first show that $G^i\big(\mathcal{E}_k(s^i) \setminus A_k(s^i, p^\xi)\big) \le \varepsilon$. Observe that

$$\begin{aligned} &G^i\big(\mathcal{E}_k(s^i) \setminus A_k(s^i, p^\xi)\big) = G^i\big(\mathcal{E}_k(s^i) \setminus A_k^-(s^i, p^\xi)\big) \le G^i\big(\overline{A}_k^\xi(s^i) \setminus A_k^-(s^i, p^\xi)\big) \\ \le &G^i\big(\overline{A}_k^\xi(s^i)\big) - G^i\big(\underline{A}_k^\xi(s^i)\big) + G^i\big(O(s^i, p^\xi)\big) \le \beta_\xi + \xi \le \varepsilon. \end{aligned}$$

The first equality follows from $A_k(s^i, p^\xi) = A_k^+(s^i, p^\xi) \cup A_k^-(s^i, p^\xi)$ and Lemma B4; the first inequality follows from (B15); the second inequality follows from Lemma B5; the third inequality follows from the definition of β_ξ and $G^i\big(O(s^i, p^\xi)\big) \le \xi$.

We next show $G^i\big(A_k(s^i, p^\xi) \setminus \mathcal{E}_k(s^i)\big) \le \varepsilon$. Since $A_k(s^i, p^\xi) = A_k^+(s^i, p^\xi) \cup A_k^-(s^i, p^\xi)$, Lemma B4 implies that

$$\begin{aligned} G^i\big(A_k(s^i, p^\xi) \setminus \mathcal{E}_k(s^i)\big) &= G^i\big(A_k^+(s^i, p^\xi) \cup A_k^-(s^i, p^\xi) \setminus \mathcal{E}_k(s^i)\big) \\ &= G^i\big(A_k^+(s^i, p^\xi)\big) + G^i\big(A_k^-(s^i, p^\xi) \setminus \mathcal{E}_k(s^i)\big). \end{aligned} \tag{B16}$$

beliefs to this larger domain with $F^i(\{t^{-i}\}) = 0$ for all $F^i \in \mathcal{F}^i$. Then all our results remain valid.

Observe that

$$G^i\big(A_k^-(s^i,p^\xi)\setminus\mathcal{E}_k(s^i)\big)\le G^i\big(\overline{A}_k^\xi(s^i)\setminus\mathcal{E}_k(s^i)\big)\le G^i\big(\overline{A}_k^\xi(s^i)\big)-G^i\big(\underline{A}_k^\xi(s^i)\big)\le\beta_\xi. \quad \text{(B17)}$$

The first inequality follows from Lemma B5; the second inequality follows from (B15); the last inequality follows from the definition of β_ξ. Since $A_k^+(s^i,p^\xi)\subseteq O(s^i,p^\xi)$ and p^ξ is ξ-efficient, we have $G^i\big(A_k^+(s^i,p^\xi)\big)\le G^i\big(O(s^i,p^\xi)\big)\le\xi$ for every $k\in\mathcal{K}$. Combining this observation with (B16) and (B17) yields the desired result. □

We are now ready to prove Theorem 5.1. Define $\kappa\equiv\max_i K^i$. Fix $\varepsilon>0$. By Lemma B8, there exists a $\overline{\xi}>0$ such that $d_G(p^\xi,p^*)\le\min\{\varepsilon,\frac{\varepsilon}{\overline{m}\kappa}\}$ for all ξ-efficient SCR p^ξ and $0<\xi<\overline{\xi}$. Suppose there exist $0<\xi<\overline{\xi}$ and ξ-efficient SCR p^ξ such that p^ξ is implementable by the transfer scheme x. We are going to show that p^* is ε-locally implementable by x_F with $\{\mu^i_{(p^\xi,x)}\}_{i\in\mathcal{I}}$.

Fix $i\in\mathcal{I}$ and $s^i,t^i\in S^i$. We first show that (p^*,x_F) is ε-LIC. By the construction of (p^*,x_F), we know that $\mu^i_{(p^*,x_F)}=\mu^i_{(p^\xi,x)}$ and

$$\begin{aligned}u^i_{(p^*,x_F)}(t^i,s^i)&=\mu^i_{(p^\xi,x)}(t^i)+\int\sum_k p_k^*(t^i,s^{-i})\big(v_k^i(s^i)-v_k^i(t^i)\big)dG^i\\&=\mu^i_{(p^\xi,x)}(t^i)+\sum_k\int_{\mathcal{E}_k(t^i)}\big(v_k^i(s^i)-v_k^i(t^i)\big)dG^i.\end{aligned}$$

Then (p^*,x_F) is ε-LIC if

$$\mu^i_{(p^\xi,x)}(s^i)\ge\mu^i_{(p^\xi,x)}(t^i)+\sum_k\int_{\mathcal{E}_k(t^i)}\big(v_k^i(s^i)-v_k^i(t^i)\big)dG^i-\varepsilon|s^i-t^i|. \quad \text{(B18)}$$

Since (p^ξ,x) is IC, we have $\mu^i_{(p^\xi,x)}(s^i)\ge u^i_{(p^\xi,x)}(t^i,s^i)$, which implies

$$\mu^i_{(p^\xi,x)}(s^i)-\mu^i_{(p^\xi,x)}(t^i)\ge\sum_k\int_{A_k(t^i,p^\xi)}\big(v_k^i(s^i)-v_k^i(t^i)\big)dG^i. \quad \text{(B19)}$$

Combining this inequality with (B18), we obtain that (p^*,x_F) is ε-LIC if

$$\sum_{k\in\mathcal{K}^i}\int_{A_k(t^i,p^\xi)}\big(v_k^i(s^i)-v_k^i(t^i)\big)dG^i\ge\sum_{k\in\mathcal{K}^i}\int_{\mathcal{E}_k(t^i)}\big(v_k^i(s^i)-v_k^i(t^i)\big)dG^i-\varepsilon|s^i-t^i|.$$

Thus, we only need to show this inequality holds. If $s^i>t^i$, by Lemma A1 and the choice of ξ, we obtain that

$$\sum_{k\in\mathcal{K}^i}\Big(\int_{\mathcal{E}_k(t^i)}\big(v_k^i(s^i)-v_k^i(t^i)\big)dG^i-\int_{A_k(t^i,p^\xi)}\big(v_k^i(s^i)-v_k^i(t^i)\big)dG^i\Big)$$

$$\leq \sum_{k\in\mathcal{K}^i} \int_{\mathcal{E}_k(t^i)\setminus A_k(t^i,p^\xi)} \left(v_k^i(s^i) - v_k^i(t^i)\right) dG^i \leq \sum_{k\in\mathcal{K}^i} \overline{m}(s^i - t^i) G^i(\mathcal{E}_k(t^i) \setminus A_k(t^i, p^\xi))$$
$$\leq \kappa\overline{m}(s^i - t^i) d_G(p^\xi, p^*) \leq \varepsilon|s^i - t^i|,$$

as desired. The case in which $s^i < t^i$ can be dealt with analogously.

We next show $\mu^i_{(p^*,x_F)}$ is monotone, which is equivalent to showing $\mu^i_{(p^\xi,x)}$ is monotone. Note that for any $s^i > t^i$, we have $v_k^i(s^i) - v_k^i(t^i) \geq 0$ for all k. Then it follows immediately from the incentive constraint (B19) that $\mu^i_{(p^\xi,x)}(s^i) \geq \mu^i_{(p^\xi,x)}(t^i)$, as desired.

Finally, we show that (p^*, x_F) is ε-bounded. For any $s^i > t^i$, the incentive compatibility of (p^ξ, x) implies

$$\mu^i_{(p^\xi,x)}(s^i) - \mu^i_{(p^\xi,x)}(t^i) \leq \sum_{k\in\mathcal{K}} G^i\left(A_k(s^i, p^\xi)\right)\left(v_k^i(s^i) - v_k^i(t^i)\right).$$

Since G^i is a probability distribution, $\sum_{k\in\mathcal{K}} G^i(A_k(s^i, p^\xi)) = 1$. Moreover, the choice of ξ implies that for every $k \in \mathcal{K}$,

$$G^i\left(A_k(s^i, p^\xi)\right) - G^i\left(\mathcal{E}_k(s^i)\right) \leq d_G(p^\xi, p^*) \leq \varepsilon.$$

Taking $w_k = G^i(A_k(s^i, p^\xi))$ for all $k \in \mathcal{K}$ completes the proof.

C Proof of Theorem 5.2

The proof consists of two steps: the first step is to show the CEM is asymptotically LIC; the second step is to show the CEM is asymptotically bounded and indirect utility functions are monotone. Fix $\delta > 0$. We first show that there exists $\overline{M}$ such that if $M > \overline{M}$, the CEM is δ-LIC. In the proofs below, we focus on sellers and similar arguments apply to buyers. Suppose seller j with cost c^j bids $\hat{c}^j$. The other buyers' and sellers' offers/bids are arrayed in increasing order $s_{(1)} \leq ... \leq s_{(2M-1)}$. There are three cases to consider. If $\hat{c}^j \leq s_{(M-1)}$, the price is $s_{(M-1)}$ and seller j trades; if $s_{(M-1)} < \hat{c}^j \leq s_{(M)}$, the price is $\hat{c}^j$ and seller j trades; if $s_{(M)} < \hat{c}^j$, seller j does not trade. Let $F_{(M-1)}$ denote the distribution of $s_{(M-1)}$ and $\Pr(s_{(M-1)} < \hat{c}^j \leq s_{(M)})$ denote the probability that $\hat{c}^j$ lies between $s_{(M-1)}$ and $s_{(M)}$. Seller j's utility under the CEM is given by

$$u^j(\hat{c}^j, c^j) = \int_{\hat{c}^j}^1 (s_{(M-1)} - c^j) dF_{(M-1)} + (\hat{c}^j - c^j)\Pr(s_{(M-1)} < \hat{c}^j \leq s_{(M)}).$$

Observe that for any $\hat{c}^j < c^j$,

$$u^j(c^j, c^j) - u^j(\hat{c}^j, c^j) = (c^j - \hat{c}^j)\Pr(s_{(M-1)} < \hat{c}^j \le s_{(M)}) + \int_{\hat{c}^j}^{c^j} (c^j - s_{(M-1)})dF_{(M-1)} > 0.$$

Thus, a seller will not underreport his cost. Suppose now that $\hat{c}^j > c^j$. Note that

$$\begin{aligned} u^j(\hat{c}^j, c^j) - u^j(c^j, c^j) &< (\hat{c}^j - c^j)\Pr(s_{(M-1)} < \hat{c}^j \le s_{(M)}) \\ &= (\hat{c}^j - c^j)\frac{(2M-1)!}{(M-1)!M!}G(\hat{c}^j)^{M-1}\big(1 - G(\hat{c}^j)\big)^M. \end{aligned}$$

By Stirling's Approximation, we have $\frac{(2M-2)!}{(M-1)!(M-1)!} \le \frac{4^{M-1}}{\sqrt{\pi(M-1)}}$. Also, $G(\hat{c}^j)\big(1 - G(\hat{c}^j)\big) \le \frac{1}{4}$. Thus, there exists $\overline{M}$ such that for all $M > \overline{M}$,

$$\frac{(2M-1)!}{(M-1)!M!}G(\hat{c}^j)^{M-1}\big(1 - G(\hat{c}^j)\big)^M \le \frac{2M-1}{M}\frac{4^{M-1}}{\sqrt{\pi(M-1)}}\frac{1}{4^{M-1}} = \frac{2M-1}{M\sqrt{\pi(M-1)}} < \delta.$$

Hence, $u^j(c^j, c^j) > u^j(\hat{c}^j, c^j) - \delta(\hat{c}^j - c^j)$ for all $M > \overline{M}$, as desired.

We next show CEM is δ-bounded and indirect utility functions associated with the CEM, denoted by μ^i, are monotone. Notice that for a seller j, his valuation from trading is $-c^j$. Thus, by Lemma 4.1, we only need to show

$$0 \le \mu^j(c^j) - \mu^j(\hat{c}^j) \le \hat{c}^j - c^j \quad \forall c^j < \hat{c}^j. \tag{C20}$$

Take a seller j and $c^j < \hat{c}^j$. By construction of the mechanism,

$$\begin{aligned} \mu^j(c^j) - \mu^j(\hat{c}^j) &= \int_{c^j}^1 (s_{(M-1)} - c^j)dF_{(M-1)} - \int_{\hat{c}^j}^1 (s_{(M-1)} - \hat{c}^j)dF_{(M-1)} \\ &= \int_{c^j}^{\hat{c}^j} \big(1 - F_{(M-1)}(x)\big)dx. \end{aligned}$$

Clearly, the inequalities in (C20) are satisfied.

D Proof of Theorem 6.1

Theorem 6.1 holds in both one- and multi-dimensional environments. We here provide a proof of Theorem 6.1 in the case of multi-dimensional signals. The proof for one-dimensional signals can be derived as a special case and, hence, omitted.

With interdependent values, a full insurance transfer scheme is constructed as in (1) with replacing $v_k^i(s^i)$ by $v_k^i(s)$. The definition of ε-boundedness also needs to be modified when signals are multi-dimensional. Let

$$L^+(s^i, t^i) \equiv \{k \in \mathcal{K}^i | s_{ki}^i \ge t_{ki}^i\} \text{ and } L^-(s^i, t^i) \equiv \mathcal{K}^i \setminus L^+(s^i, t^i) \quad \forall s^i, t^i \in S^i, \forall i \in \mathcal{I}.$$

Definition 11 (Multi-dimensional Signals)**.** For any Bayesian environment $\mathbf{E}^B$ with additively separable value functions and any $\varepsilon \geq 0$, a mechanism (p,x) is **ε-bounded** if for any $i \in \mathcal{I}$ and $s^i, t^i \in S^i$ with $L^+(s^i,t^i) \neq \varnothing$, we have

$$\mu^i_{(p,x)}(s^i) - \mu^i_{(p,x)}(t^i) \leq \sum_{k \in L^+(s^i,t^i)} w_k \big(f^i_k(s^i) - f^i_k(t^i)\big), \tag{D21}$$

where $\{w_k\}_{k \in L^+(s^i,t^i)}$ satisfies (i) $0 \leq w_k \leq G^i\big(A_k(t^i,p)\big) + \varepsilon$ for all $k \in L^+(s^i,t^i)$ and (ii) $\sum_{k \in L^+(s^i,t^i)} w_k \leq 1$.

It is easily seen that, with the suitably modified definitions, all our results and proofs in the case of private values directly extend to the case of interdependent and additively separable values. We next prove Theorem 6.1 with nonseparable values.

We start with some notation and preliminary lemmas. Let $D^i_k(s^i,t^i,s^{-i}) \equiv v^i_k(s^i,s^{-i}) - v^i_k(t^i,s^{-i})$. Also define

$$\overline{m} \equiv \max_{i \in \mathcal{I}, k \in \mathcal{K}^i, s \in S} \frac{\partial v^i_k(s^i,s^{-i})}{\partial s^i_{ki}} \quad \text{and} \quad \underline{m} \equiv \min_{i \in \mathcal{I}, k \in \mathcal{K}^i, s \in S} \frac{\partial v^i_k(s^i,s^{-i})}{\partial s^i_{ki}}$$

Lemma D9. *Fix a Bayesian environment and $\varepsilon > 0$. Let (p,x) be an ε-LIC mechanism with monotone indirect utility functions. Then for all $i \in \mathcal{I}$, $s^i, t^i \in S^i$, and $L' \subseteq L^+(s^i,t^i)$,*

$$\mu^i_{(p,x)}(s^i) - \mu^i_{(p,x)}(t^i) \geq \sum_{k \in L' \cup L^-(s^i,t^i)} \left(\int_{A_k(t^i,p)} D^i_k(s^i,t^i,s^{-i}) dG^i - \varepsilon |s^i_{ki} - t^i_{ki}| \right).$$

Proof. Fix $i \in \mathcal{I}$, $s^i, t^i \in S^i$, and $L' \subseteq L^+(s^i,t^i)$. Construct $\tilde{s}^i$ as follows:

$$\tilde{s}^i_{ki} = s^i_{ki} \quad \forall k \in L' \cup L^-(s^i,t^i), \quad \text{and} \quad \tilde{s}^i_{ki} = t^i_{ki} \quad \forall k \in L^+(s^i,t^i) \setminus L'.$$

By definition, we have $s^i_{ki} \geq t^i_{ki}$ for all $k \in L^+(s^i,t^i)$ and, hence, the construction of $\tilde{s}^i$ implies $s^i_{ki} \geq \tilde{s}^i_{ki}$ for all k. That is, $L^-(s^i,\tilde{s}^i) = \varnothing$. Since $\mu^i_{(p,x)}$ is monotone, we obtain $\mu^i_{(p,x)}(s^i) \geq \mu^i_{(p,x)}(\tilde{s}^i)$. Thus,

$$\begin{aligned} \mu^i_{(p,x)}(s^i) - \mu^i_{(p,x)}(t^i) &\geq \mu^i_{(p,x)}(\tilde{s}^i) - \mu^i_{(p,x)}(t^i) \\ &\geq \sum_{k \in L' \cup L^-(s^i,t^i)} \left(\int_{A_k(t^i,p)} D^i_k(s^i,t^i,s^{-i}) dG^i - \varepsilon |s^i_{ki} - t^i_{ki}| \right). \end{aligned}$$

The second inequality follows from ε-LIC and the construction of $\tilde{s}^i$. □

For every $s^i \in S^i$, let $e(s^i) \equiv \{t^i \in S^i | s^i_{ki} = t^i_{ki}, \forall k \in \mathcal{K}\}$. Any two signals in $e(s^i)$ only differ in own-payoff irrelevant information.

Lemma D10. *Fix $\varepsilon \geq 0$. For any ε-LIC mechanism (p,x), $\mu^i_{(p,x)}(s^i) = \mu^i_{(p,x)}(t^i)$ for all $s^i \in S^i$, $t^i \in e(s^i)$, and $i \in \mathcal{I}$.*

Proof. Fix $i \in \mathcal{I}$, $s^i \in S^i$, and $t^i \in e(s^i)$. Since $t^i \in e(s^i)$, we have $s^i_{ki} = t^i_{ki}$ and, hence, $v^i_k(s^i, s^{-i}) = v^i_k(t^i, s^{-i})$ for all $k \in \mathcal{K}$ and $s^{-i} \in S^{-i}$. Since (p,x) is ε-LIC,

$$\mu^i_{(p,x)}(s^i) \geq \mu^i_{(p,x)}(t^i) + \int \sum p_k(t^i, s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i - \varepsilon \sum_{k \in \mathcal{K}^i} |s^i_{ki} - t^i_{ki}| = \mu^i_{(p,x)}(t^i).$$

Reversing the roles of s^i and t^i yields $\mu^i_{(p,x)}(t^i) \geq \mu^i_{(p,x)}(s^i)$. Thus, $\mu^i_{(p,x)}(s^i) = \mu^i_{(p,x)}(t^i)$. $\square$

For every SCR p, $k \in \mathcal{K}$, and $s^i, t^i \in S^i$, let $\overline{D}^i_{k,p}(s^i, t^i) \equiv \sup_{s^{-i} \in A_k(s^i,p)} D^i_k(s^i, t^i, s^{-i})$ and $C^i_{k,p}(s^i, t^i) \equiv \{s^{-i} \in S^{-i} | \overline{D}^i_{k,p}(s^i, t^i) = D^i_k(s^i, t^i, s^{-i})\}$. By the definition of $\overline{D}^i_{k,p}(s^i, t^i)$ and the compactness of S^{-i}, we have $C^i_{k,p}(s^i, t^i) \neq \varnothing$. For every SCR p, $\xi > 0$, $k \in \mathcal{K}$, and $s^i, t^i \in S^i$, define

$$\tilde{S}^{-i}_{\xi,k,p}(s^i, t^i) \equiv \{s^{-i} \in A_k(s^i, p) | \overline{D}^i_{k,p}(s^i, t^i) - \xi(s^i_{ki} - t^i_{ki}) \geq D^i_k(s^i, t^i, s^{-i})\}.$$

Recall that $K^i = |\mathcal{K}^i|$, $\kappa = \max_i K^i$, and $A_0(s^i, p) \equiv \cup_{k \in \mathcal{K}^i_0} A_k(s^i, p)$. For any SCR p, $i \in \mathcal{I}$, and $s^i \in S^i$, define $\tilde{\mathcal{K}}^i_p(s^i) \equiv \{k \in \mathcal{K} | G^i\big(A_k(s^i, p)\big) > \frac{1}{K^i+1}\}$. Let $\tilde{S}^i_p \equiv \{s^i \in S^i | G^i\big(A_0(s^i, p)\big) < \frac{1}{K^i+1}\}$. Finally, for any $s^{-i} \in S^{-i}$ and s^j_{kl}, let $s^{-i} \setminus s^j_{kl}$ denote the coordinates of s^{-i} other than s^j_{kl}.

Lemma D11. *There exist $0 < \xi < \underline{m}$ and $0 < \eta < \frac{1}{\kappa+1}$ such that $G^i\big(\tilde{S}^{-i}_{\xi,k,p}(s^i, t^i)\big) \geq \eta$ for all SCR p, $s^i \in \tilde{S}^i_p$, $t^i \in S^i$, $k \in \tilde{\mathcal{K}}^i_p(s^i)$, and $i \in \mathcal{I}$.*

Proof. Fix $0 < \eta < \frac{1}{\kappa+1}$. By nonseparability, there exists $\tau > 0$ such that $|\frac{\partial^2 v^i_k(s^i, s^{-i})}{\partial s^i_{ki} \partial s^j_{ki}}| \geq \tau$ for all $k \in \mathcal{K}^i$, $j \neq i$, $i \in \mathcal{I}$ and $s \in S$. This implies that given any $s^{-i} \in C^i_{k,p}(s^i, t^i)$, for any signal $\tilde{s}^{-i}$ so that $\tilde{s}^j_{ki} \neq s^j_{ki}$ for some $j \neq i$ and $\tilde{s}^{-i} \setminus \tilde{s}^j_{ki} = s^{-i} \setminus s^j_{ki}$, we have $\tilde{s}^{-i} \notin C^i_{k,p}(s^i, t^i)$. Thus, the set $C^i_{k,p}(s^i, t^i)$ has an empty interior. Since g^i is continuous and S^{-i} is compact, there exists $C > 0$ such that $g^i(s^{-i}) < C$ for all $s^{-i} \in S^{-i}$ and $i \in \mathcal{I}$. Then there exists $\delta > 0$ such that $G^i\big(\mathcal{B}_\delta(C^i_{k,p}(s^i, t^i))\big) < \frac{1}{\kappa+1} - \eta$ for all $s^i, t^i \in S^i$, $k \in \mathcal{K}^i$, $i \in \mathcal{I}$, and SCR p. Take $0 < \xi < \min\{\underline{m}, \tau\delta\}$.

Fix a SCR p, $i \in \mathcal{I}$, $s^i \in \tilde{S}^i_p$, and $t^i \in S^i$. Since $s^i \in \tilde{S}^i_p$, we know that $\tilde{\mathcal{K}}^i_p(s^i) \neq \varnothing$. Otherwise, we have $\sum_{\hat{k} \in \mathcal{K}} G^i\big(A_{\hat{k}}(s^i, p)\big) < 1$, a contradiction. Take $k \in \tilde{\mathcal{K}}^i_p(s^i)$.

Assume that $\frac{\partial^2 v_k^i(s^i,s^{-i})}{\partial s_{ki}^i \partial s_{ki}^j} > 0$ for all $j \neq i$ and $s^{-i} \in S^{-i}$. The other cases follow from analogous arguments. Note that if $s_{ki}^i \leq t_{ki}^i$, then $\tilde{S}^{-i}_{\xi,k,p}(s^i,t^i) = A_k(s^i,p)$. Since $k \in \tilde{\mathcal{K}}_p^i(s^i)$, we have $G^i\big(\tilde{S}^{-i}_{\xi,k,p}(s^i,t^i)\big) = G^i\big(A_k(s^i,p)\big) > \frac{1}{\kappa+1} > \eta$, as desired. Thus, assume that $s_{ki}^i > t_{ki}^i$. Take any $\hat{s}^{-i} \in A_k(s^i,p) \setminus \mathcal{B}_\delta\big(C_{k,p}^i(s^i,t^i)\big)$. We next show that there exists $t^{-i} \in C_{k,p}^i(s^i,t^i)$ such that $t_{ki}^l - \hat{s}_{ki}^l \geq 0$ for all $l \neq i$. Take any $\tilde{s}^{-i} \in C_{k,p}^i(s^i,t^i)$ and by way of contradiction, suppose that there exists $j' \neq i$ such that $\tilde{s}_{ki}^{j'} - \hat{s}_{ki}^{j'} < 0$. Construct $\bar{s}^{-i}$ such that $\bar{s}_{ki}^l = \max\{\hat{s}_{ki}^l, \tilde{s}_{ki}^l\}$ for all $l \neq i$. For the other dimensions, if $\bar{s}_{ki}^l = \hat{s}_{ki}^l$, then $\bar{s}_{k'l'}^l = \hat{s}_{k'l'}^l$; similarly, if $\bar{s}_{ki}^l = \tilde{s}_{ki}^l$, then $\bar{s}_{k'l'}^l = \tilde{s}_{k'l'}^l$. By construction, $\bar{s}^{-i} \in S^{-i}$ where $\bar{s}_{ki}^l \geq \tilde{s}_{ki}^l$ for all $l \neq i$ and $\bar{s}_{ki}^{j'} = \hat{s}_{ki}^{j'} > \tilde{s}_{ki}^{j'}$. Since $\frac{\partial^2 v_k^i(s^i,s^{-i})}{\partial s_{ki}^i \partial s_{ki}^j} > 0$ for all $j \neq i$, we obtain

$$D_k^i(s^i,t^i,\bar{s}^{-i}) > D_k^i(s^i,t^i,\tilde{s}^{-i}) = \overline{D}_{k,p}^i(s^i,t^i). \tag{D22}$$

It follows from the definition of $\overline{D}_{k,p}^i(s^i,t^i)$ and the choice of $\hat{s}^{-i}$ that $\overline{D}_{k,p}^i(s^i,t^i) > D_k^i(s^i,t^i,\hat{s}^{-i})$. Combining the latter inequality with (D22) yields $D_k^i(s^i,t^i,\bar{s}^{-i}) > D_k^i(s^i,t^i,\hat{s}^{-i})$. Then by continuity, there exists $t^{-i} \in S^{-i}$ such that $\hat{s}_{ki}^l \leq t_{ki}^l \leq \bar{s}_{ki}^l$ for all $l \neq i$ and $D_k^i(s^i,t^i,t^{-i}) = \overline{D}_{k,p}^i(s^i,t^i)$, that is, $t^{-i} \in C_{k,p}^i(s^i,t^i)$.

Since $\hat{s}^{-i} \notin \mathcal{B}_\delta\big(C_{k,p}^i(s^i,t^i)\big)$, there exists $j \neq i$ such that $t_{ki}^j - \hat{s}_{ki}^j > \delta$. Therefore,

$$\begin{aligned}\overline{D}_{k,p}^i(s^i,t^i) - D_k^i(s^i,t^i,\hat{s}^{-i}) &= D_k^i(s^i,t^i,t^{-i}) - D_k^i(s^i,t^i,\hat{s}^{-i})\\ = \int_{t_{ki}^i}^{s_{ki}^i}\Big(\frac{\partial v_k^i(\theta,t_{ki}^{-i})}{\partial\theta} - \frac{\partial v_k^i(\theta,\hat{s}_{ki}^{-i})}{\partial\theta}\Big)d\theta &\geq \tau\delta(s_{ki}^i - t_{ki}^i) > \xi(s_{ki}^i - t_{ki}^i).\end{aligned}$$

Thus, $\tilde{S}^{-i}_{\xi,k,p}(s^i,t^i) \supseteq A_k(s^i,p) \setminus \mathcal{B}_\delta\big(C_{k,p}^i(s^i,t^i)\big)$. By the choice of δ, we conclude that $G^i\big(\tilde{S}^{-i}_{\xi,k,p}(s^i,t^i)\big) \geq G^i\big(A_k(s^i,p) \setminus \mathcal{B}_\delta(C_{k,p}^i(s^i,t^i))\big) > \eta$, as desired. □

We are now ready to prove Theorem 6.1. Fix a Bayesian environment $\mathbf{E}^B$ and a rich SCR p. Take $0 < \delta \leq 1$ and a δ-ambiguity environment $\mathbf{E}^\delta$. Lemma D11 implies that there exist $0 < \xi < \underline{m}$ and $0 < \eta < \frac{1}{\kappa+1}$ such that $G^i\big(\tilde{S}^{-i}_{\xi,k,p}(s^i,t^i)\big) \geq \eta$ for all $s^i \in \tilde{S}_p^i$, $t^i \in S^i$, $k \in \tilde{\mathcal{K}}_p^i(s^i)$, and $i \in \mathcal{I}$. Take $\varepsilon > 0$ such that $\frac{\varepsilon\kappa}{\underline{m}} + \frac{\varepsilon\kappa(\overline{m}-\xi)}{\xi(\underline{m}-\xi)} \leq \delta$ and $\frac{\varepsilon\kappa(\overline{m}-\xi)}{\xi(\underline{m}-\xi)} \leq \eta$. Suppose that p is ε-locally implementable by the transfer scheme x with associated indirect utility functions $\mu^i_{(p,x)}$ in $\mathbf{E}^B$. We are going to show that p is implementable by the full insurance transfer scheme x_F with $\{\mu^i_{(p,x)}\}_i$ in $\mathbf{E}^\delta$. Recall from Section 3 that by construction, $\mu^i_{(p,x)}$ is also agent i's indirect utility function associated with (p, x_F) in $\mathbf{E}^\delta$. We thus only need to show for all $s^i, t^i \in S^i$,

$$\mu^i_{(p,x)}(s^i) \geq u^i_{(p,x_F)}(t^i, s^i) = \mu^i_{(p,x)}(t^i) + \min_{F^i \in \mathcal{F}^i} \int \sum p_k(t^i, s^{-i}) D^i_k(s^i, t^i, s^{-i}) dF^i, \quad \text{(D23)}$$

where $u^i_{(p,x_F)}$ is i's interim payoff under (p, x_F) in $\mathbf{E}^\delta$.

Fix $i \in \mathcal{I}$ and $s^i, t^i \in S^i$. If $t^i \in e(s^i)$, Lemma D10 implies $\mu^i_{(p,x)}(s^i) = \mu^i_{(p,x)}(t^i)$. Since $t^i \in e(s^i)$, we have $D^i_k(s^i, t^i, s^{-i}) = 0$ for all s^{-i}. Therefore,

$$\mu^i_{(p,x)}(s^i) = \mu^i_{(p,x)}(t^i) = \mu^i_{(p,x)}(t^i) + \min_{F^i \in \mathcal{F}^i} \int \sum p_k(t^i, s^{-i}) D^i_k(s^i, t^i, s^{-i}) dF^i,$$

as desired. Now suppose $t^i \notin e(s^i)$. That is, there exists $k \in \mathcal{K}^i$ such that $|s^i_{ki} - t^i_{ki}| \neq 0$. Since s^i and t^i are fixed, we write L^+ and L^- in place of $L^+(s^i, t^i)$ and $L^-(s^i, t^i)$ respectively. Let $L^{++} \equiv \{k \in L^+ | G^i(A_k(t^i, p)) \geq \frac{\varepsilon}{m}\}$. Lemma D9 implies

$$\mu^i_{(p,x)}(s^i) \geq \mu^i_{(p,x)}(t^i) + \sum_{k \in L^{++} \cup L^-} \Big(\int_{A_k(t^i,p)} D^i_k(s^i, t^i, s^{-i}) dG^i - \varepsilon |s^i_{ki} - t^i_{ki}| \Big). \quad \text{(D24)}$$

Combining (D24) and (D23) indicates that it suffices to show that there exists a $\hat{F}^i$ such that $\hat{F}^i \in \mathcal{F}^i$ and

$$\begin{aligned} \sum_{k \in L^{++} \cup L^-} \Big(\int_{A_k(t^i,p)} D^i_k(s^i, t^i, s^{-i}) dG^i - \varepsilon |s^i_{ki} - t^i_{ki}| \Big) & \\ \geq \int \sum p_k(t^i, s^{-i}) D^i_k(s^i, t^i, s^{-i}) d\hat{F}^i. & \end{aligned} \quad \text{(D25)}$$

The rest of the proof is to construct such a $\hat{F}^i$ explicitly in all possible cases.

Suppose first that $L^- = \varnothing$. Construct $\hat{F}^i$ as follows:

$$[\hat{F}^i(A_k(t^i, p)) = G^i(A_k(t^i, p)) - \frac{\varepsilon}{m} \quad \forall k \in L^{++}], \quad [\hat{F}^i(A_k(t^i, p)) = 0 \quad \forall k \in \mathcal{K}^i \setminus L^{++}],$$

$$\hat{F}^i(A_0(t^i, p)) = G^i(A_0(t^i, p)) + \sum_{k=1}^{K^i} \big(G^i(A_k(t^i, p)) - \hat{F}^i(A_k(t^i, p)) \big).$$

By construction, $\sum_{k=1}^{K^i} \big(G^i(A_k(t^i, p)) - \hat{F}^i(A_k(t^i, p)) \big) \leq \frac{K^i \varepsilon}{m}$. Since $\delta \geq \frac{K^i \varepsilon}{m}$, $\hat{F}^i \in B_\delta(G^i) \subseteq \mathcal{F}^i$. The construction of $\hat{F}^i$ yields

$$\sum_{k \in L^{++}} \int_{A_k(t^i,p)} D^i_k(s^i, t^i, s^{-i}) dG^i - \varepsilon \sum_{k \in L^{++}} |s^i_{ki} - t^i_{ki}| \geq \sum_{k \in \mathcal{K}^i} \int_{A_k(t^i,p)} D^i_k(s^i, t^i, s^{-i}) d\hat{F}^i.$$

Thus, (D25) is satisfied. Suppose now $L^- \neq \varnothing$. Suppose also that

$$G^i(A_0(t^i, p)) + \sum_{k \in L^+ \setminus L^{++}} G^i(A_k(t^i, p)) + |L^{++}| \frac{\varepsilon}{m} \geq |L^-| \frac{\varepsilon}{m}. \quad \text{(D26)}$$

Then construct $\hat{F}^i$ as follows:

$$[\hat{F}^i(A_k(t^i, p)) = G^i(A_k(t^i, p)) + \frac{\varepsilon}{m} \quad \forall k \in L^-], \quad [\hat{F}^i(A_k(t^i, p)) = 0 \quad \forall k \in L^+ \setminus L^{++}],$$

$$\hat{F}^i\big(A_k(t^i,p)\big) = G^i\big(A_k(t^i,p)\big) - \frac{\varepsilon}{\underline{m}} \quad \forall k \in L^{++},$$
$$\hat{F}^i\big(A_0(t^i,p)\big) = G^i\big(A_0(t^i,p)\big) + \sum_{k\in L^+\setminus L^{++}} G^i\big(A_k(t^i,p)\big) + (|L^{++}| - |L^-|)\frac{\varepsilon}{\underline{m}}.$$

By (D26), $\hat{F}^i$ is well defined. Since $\delta \geq \frac{K^i\varepsilon}{\underline{m}}$, we have $\hat{F}^i \in B_\delta(G^i) \subseteq \mathcal{F}^i$. It is straightforward to verify that the construction of $\hat{F}^i$ implies (D25). Suppose now

$$\begin{aligned} &G^i\big(A_0(t^i,p)\big) + \sum_{k\in L^+} G^i\big(A_k(t^i,p)\big) \\ \geq &|L^-|\frac{\varepsilon}{\underline{m}} > G^i\big(A_0(t^i,p)\big) + \sum_{k\in L^+\setminus L^{++}} G^i\big(A_k(t^i,p)\big) + |L^{++}|\frac{\varepsilon}{\underline{m}}. \end{aligned} \tag{D27}$$

Then construct $\hat{F}^i$ as follows:

$$[\hat{F}^i\big(A_k(t^i,p)\big) = G^i\big(A_k(t^i,p)\big) + \frac{\varepsilon}{\underline{m}} \quad \forall k \in L^-], \quad [\hat{F}^i\big(A_k(t^i,p)\big) = 0 \quad \forall k \in L^+ \setminus L^{++}],$$
$$\hat{F}^i\big(A_0(t^i,p)\big) = 0, \quad \sum_{k\in L^{++}} \hat{F}^i\big(A_k(t^i,p)\big) = G^i\big(A_0(t^i,p)\big) + \sum_{k\in L^+} G^i\big(A_k(t^i,p)\big) - |L^-|\frac{\varepsilon}{\underline{m}},$$
$$G^i\big(A_k(t^i,p)\big) - |L^-|\frac{\varepsilon}{\underline{m}} \leq \hat{F}^i\big(A_k(t^i,p)\big) \leq G^i\big(A_k(t^i,p)\big) - \frac{\varepsilon}{\underline{m}} \quad \forall k \in L^{++}.$$

Notice that $\hat{F}^i$ may not be unique but such $\hat{F}^i$ exists due to (D27). Since $\delta \geq \frac{K^i\varepsilon}{\underline{m}}$, we have $\hat{F}^i \in B_\delta(G^i) \subseteq \mathcal{F}^i$. It is straightforward to verify that (D25) follows directly from the construction of $\hat{F}^i$. Finally, suppose

$$G^i\big(A_0(t^i,p)\big) + \sum_{k\in L^+} G^i\big(A_k(t^i,p)\big) < |L^-|\frac{\varepsilon}{\underline{m}}. \tag{D28}$$

For every $k \in L^-$, define

$$\bar{s}_k^{-i} \in \underset{s^{-i}\in A_k(t^i,p)}{\operatorname{argmax}} \frac{v_k^i(t^i,s^{-i}) - v_k^i(s^i,s^{-i})}{t_{ki}^i - s_{ki}^i} \quad \text{and} \quad \overline{m}_k^i = \frac{v_k^i(t_{ki}^i,\bar{s}_k^{-i}) - v_k^i(s_{ki}^i,\bar{s}_k^{-i})}{t_{ki}^i - s_{ki}^i}.$$ [39]

Also, let $k^* \in \operatorname{argmax}_{k\in L^-} D_k^i(t^i,s^i,s^{-i})$ and $\bar{k} \in \operatorname{argmax}_{k\in L^-} \big(t_{ki}^i - s_{ki}^i\big)$. By the choice of ε, we have $\frac{\varepsilon}{\underline{m}} < \frac{1}{K^i(K^i+1)}$. Then (D28) implies

$$G^i\big(A_0(t^i,p)\big) + \sum_{k\in L^+} G^i\big(A_k(t^i,p)\big) < |L^-|\frac{\varepsilon}{\underline{m}} \leq \frac{1}{K^i+1}.$$

Hence, $t^i \in \tilde{S}_p^i$ and $\tilde{\mathcal{K}}_p^i(t^i) \neq \varnothing$. If $k^* \notin \tilde{\mathcal{K}}_p^i(t^i) \cap L^-$, take $\tilde{k} \in \tilde{\mathcal{K}}_p^i(t^i) \cap L^-$; other-

[39] Since $A_k(t^i,p)$ might be an open set, it is possible that $\bar{s}_k^{-i}$ does not exist. Then we can take a signal $\bar{s}_k^{-i} \in A_k(t^i,p)$ such that $\overline{m}_k^i > \sup_{t^{-i}\in A_k(t^i,p)} \frac{v_k^i(t_{ki}^i,t_{ki}^{-i}) - v_k^i(s_{ki}^i,t_{ki}^{-i})}{t_{ki}^i - s_{ki}^i} - \epsilon$ for some small $\epsilon > 0$.

wise, take $\tilde{k} = k^*$. Let $\lambda = \frac{K^i \varepsilon(\overline{m} - \xi)}{\xi(\underline{m} - \xi)} \geq \frac{K^i \varepsilon}{\xi}$. Construct $\hat{F}^i$ as follows:[40]

$$\hat{F}^i\big(A_0(t^i,p)\big) + \sum_{k \in L^+} \hat{F}^i\big(A_k(t^i,p)\big) = 0, \quad \hat{F}^i\big(\tilde{S}^{-i}_{\xi,\tilde{k},p}(t^i,s^i)\big) = G^i\big(\tilde{S}^{-i}_{\xi,\tilde{k},p}(t^i,s^i)\big) - \lambda,$$

$$\hat{F}^i\big(A_{\tilde{k}}(t^i,p) \setminus \tilde{S}^{-i}_{\xi,\tilde{k},p}(t^i,s^i)\big) = G^i\big(A_{\tilde{k}}(t^i,p) \setminus \tilde{S}^{-i}_{\xi,\tilde{k},p}(t^i,s^i)\big),$$

$$\hat{F}^i(\overline{s}^{-i}_{k^*}) = \lambda + G^i\big(A_0(t^i,p)\big) + \sum_{k \in L^+} G^i\big(A_k(t^i,p)\big), \quad \hat{F}^i(A_{k^*}(t^i,p) \setminus \{\overline{s}^{-i}_{k^*}\}) = G^i\big(A_{k^*}(t^i,p)\big),$$

$$\hat{F}^i\big(A_k(t^i,p)\big) = G^i\big(A_k(t^i,p)\big) \quad \forall k \in L^- \setminus \{k^*, \tilde{k}\}.$$

By the choice of ε, we have $\lambda \leq \eta \leq G^i\big(\tilde{S}^{-i}_{\xi,\tilde{k},p}(t^i,s^i)\big)$. Thus, the constructed $\hat{F}^i$ is feasible. Since $\delta \geq \frac{K^i \varepsilon(\overline{m} - \xi)}{\xi(\underline{m} - \xi)} + \frac{K^i \varepsilon}{\underline{m}}$, we know that $\hat{F}^i \in B_\delta(G^i) \subseteq \mathcal{F}^i$. By the construction of $\hat{F}^i$, we obtain

$$\begin{aligned} &\sum_{k \in L^-} \int_{A_k(t^i,p)} D^i_k(s^i,t^i,s^{-i}) dG^i - \sum_{k \in \mathcal{K}} \int_{A_k(t^i,p)} D^i_k(s^i,t^i,s^{-i}) d\hat{F}^i \\ \geq& \lambda \big(\overline{m}^i_{k^*}(t^i_{k^*i} - s^i_{k^*i}) - (\overline{m}^i_{\tilde{k}} - \xi)(t^i_{\tilde{k}i} - s^i_{\tilde{k}i})\big). \end{aligned} \tag{D29}$$

By the definition of L^{++}, we obtain

$$\int_{A_k(t^i,p)} D^i_k(s^i,t^i,s^{-i}) dG^i \geq \underline{m}(s^i_{ki} - t^i_{ki}) G^i\big(A_k(t^i,p)\big) \geq \varepsilon(s^i_{ki} - t^i_{ki}) \quad \forall k \in L^{++}. \tag{D30}$$

It follows from (D30) and (D29) that (D25) is satisfied if

$$\overline{m}^i_{k^*}(t^i_{k^*i} - s^i_{k^*i}) - (\overline{m}^i_{\tilde{k}} - \xi)(t^i_{\tilde{k}i} - s^i_{\tilde{k}i}) \geq \frac{\varepsilon}{\lambda} \sum_{k \in L^-} (t^i_{ki} - s^i_{ki}).$$

If $\tilde{k} = \overline{k}$, then

$$\begin{aligned} &\overline{m}^i_{k^*}(t^i_{k^*i} - s^i_{k^*i}) - (\overline{m}^i_{\tilde{k}} - \xi)(t^i_{\tilde{k}i} - s^i_{\tilde{k}i}) \geq \overline{m}^i_{\tilde{k}}(t^i_{\tilde{k}i} - s^i_{\tilde{k}i}) - (\overline{m}^i_{\tilde{k}} - \xi)(t^i_{\tilde{k}i} - s^i_{\tilde{k}i}) \\ =& \xi(t^i_{\tilde{k}i} - s^i_{\tilde{k}i}) \geq \frac{\varepsilon}{\lambda} K^i (t^i_{\tilde{k}i} - s^i_{\tilde{k}i}) \geq \frac{\varepsilon}{\lambda} \sum_{k \in L^-} (t^i_{ki} - s^i_{ki}). \end{aligned}$$

The first inequality follows from the definition of k^*; the second inequality follows from the assumption that $\tilde{k} = \overline{k}$ and $\lambda \geq \frac{\varepsilon}{\xi} K^i$. Suppose now $\tilde{k} \neq \overline{k}$. If

$$(\overline{m}^i_{\tilde{k}} - \xi)(t^i_{\tilde{k}i} - s^i_{\tilde{k}i}) \leq (\overline{m}^i_{\overline{k}} - \xi)(t^i_{\overline{k}i} - s^i_{\overline{k}i}), \tag{D31}$$

then

$$\begin{aligned} &\overline{m}^i_{k^*}(t^i_{k^*i} - s^i_{k^*i}) - (\overline{m}^i_{\tilde{k}} - \xi)(t^i_{\tilde{k}i} - s^i_{\tilde{k}i}) \geq \overline{m}^i_{\overline{k}}(t^i_{\overline{k}i} - s^i_{\overline{k}i}) - (\overline{m}^i_{\overline{k}} - \xi)(t^i_{\overline{k}i} - s^i_{\overline{k}i}) \\ =& \xi(t^i_{\overline{k}i} - s^i_{\overline{k}i}) \geq \frac{\varepsilon}{\lambda} K^i (t^i_{\overline{k}i} - s^i_{\overline{k}i}) \geq \frac{\varepsilon}{\lambda} \sum_{k \in L^-} (t^i_{ki} - s^i_{ki}). \end{aligned}$$

[40]When $\tilde{k} = k^*$, let $\hat{F}^i\big(A_{k^*}(t^i,p) \setminus \big(\{\overline{s}^{-i}_{k^*}\} \cup \tilde{S}^{-i}_{\xi,k^*,p}(t^i,s^i)\big)\big) = G^i\big(A_{k^*}(t^i,p) \setminus \tilde{S}^{-i}_{\xi,k^*,p}(t^i,s^i)\big)$.

If the inequality in (D31) is reversed, then

$$\begin{aligned}&\overline{m}^i_{k^*}(t^i_{k^*i}-s^i_{k^*i})-(\overline{m}^i_{\tilde{k}}-\xi)(t^i_{\tilde{k}i}-s^i_{\tilde{k}i})\geq\overline{m}^i_{\tilde{k}}(t^i_{\tilde{k}i}-s^i_{\tilde{k}i})-(\overline{m}^i_{\tilde{k}}-\xi)(t^i_{\tilde{k}i}-s^i_{\tilde{k}i})\\=&\xi(t^i_{\tilde{k}i}-s^i_{\tilde{k}i})>\xi\frac{\overline{m}^i_{\bar{k}}-\xi}{\overline{m}^i_{\tilde{k}}-\xi}(t^i_{\bar{k}i}-s^i_{\bar{k}i})\geq\frac{\varepsilon}{\lambda}K^i(t^i_{\bar{k}i}-s^i_{\bar{k}i})\geq\frac{\varepsilon}{\lambda}\sum_{k\in L^-}(t^i_{ki}-s^i_{ki}).\end{aligned}$$

The second inequality follows from the violation of (D31); the third inequality follows from $\lambda=\frac{K^i\varepsilon(\overline{m}-\xi)}{\xi(\underline{m}-\xi)}\geq\frac{K^i\varepsilon(\overline{m}^i_{\bar{k}}-\xi)}{\xi(\overline{m}^i_{\tilde{k}}-\xi)}$.

E Proof of Theorem 6.2

We start with some notation and a preliminary lemma. For every $i\in\mathcal{I}$ and $s^i\in S^i$, let $\bar{e}(s^i)\equiv\{t^i|t^i_{ki}=s^i_{ki},t^i_{kj}\in S^i_{kj},\forall k\in\mathcal{K},\forall j\neq i\}$. Note that, by definition, $\overline{\varsigma}^i(s^i)\in\bar{e}(s^i)$. It is possible that $t^i\notin S^i$ for some $t^i\in\bar{e}(s^i)$ but $\mathcal{E}_k(t^i)$ is well defined for all $t^i\in\bar{e}(s^i)$. Let $\tilde{d}\equiv\max_{i\in\mathcal{I},j\neq i,k\in\mathcal{K}}(\max_{s^j\in S^j}s^j_{ki}-\min_{s^j\in S^j}s^j_{ki})$.

Lemma E12. *For any $\varepsilon>0$, there exists $\gamma>0$ such that if $\gamma^i<\gamma$ for all $i\in\mathcal{I}$, then $G^i\big(\mathcal{E}_k(s^i)\setminus\mathcal{E}_k(t^i)\big)\leq\varepsilon$ for all $s^i,t^i\in\bar{e}(\tilde{s}^i)$, $\tilde{s}^i\in S^i$, $k\in\mathcal{K}$, and $i\in\mathcal{I}$.*

Proof. Take $\gamma>0$ such that

$$\sup_{i\in\mathcal{I},\tilde{s}^i\in S^i,t^i\in\bar{e}(\tilde{s}^i)}G^i\big(\mathcal{B}_{\frac{\gamma\tilde{d}}{\underline{m}}}(\mathcal{E}_k(t^i))\big)-G^i\big(\mathcal{E}_k(t^i)\big)\leq\varepsilon.\tag{E32}$$

Such γ exists since G^i is absolutely continuous with respect to the Lebesgue measure.

Fix $i\in\mathcal{I}$, $\tilde{s}^i\in S^i$, $s^i,t^i\in\bar{e}(\tilde{s}^i)$, and $k\in\mathcal{K}$. We first show that $\mathcal{E}_k(s^i)\subseteq\mathcal{B}_{\frac{\gamma^i\tilde{d}}{\underline{m}}}(\mathcal{E}_k(t^i))$. Take $s^{-i}\in\mathcal{E}_k(s^i)\setminus\mathcal{E}_k(t^i)$. It suffices to show there exists $t^{-i}\in\mathcal{E}_k(t^i)$ such that $\parallel s^{-i}-t^{-i}\parallel_\infty\leq\frac{\gamma^i\tilde{d}}{\underline{m}}$. We are going to construct one: let t^j for $j\neq i$ be such that

$$t^j_{kj}=s^j_{kj}+\frac{\gamma^i\tilde{d}}{\underline{m}},\qquad t^j_{k'j}=s^j_{k'j}-\frac{\gamma^i\tilde{d}}{\underline{m}},\quad\forall k'\neq k,\qquad t^j_{\hat{k}l}=s^j_{\hat{k}l}\quad\forall l\neq j,\forall\hat{k}\in\mathcal{K}.$$

By construction, $\parallel s^{-i}-t^{-i}\parallel_\infty=\frac{\gamma^i\tilde{d}}{\underline{m}}$. We now show that $t^{-i}\in\mathcal{E}_k(t^i)$.[41] By construction, we have $t^j_{kj}=s^j_{kj}+\frac{\gamma^i\tilde{d}}{\underline{m}}$ and $t^l_{kj}=s^l_{kj}$ for all $j\neq i$, $l\neq j,l\neq i$. Since

[41] If $t^{-i}\notin S^{-i}$, we can enlarge the set of signals to include t^{-i} and extend agent i's belief to this larger domain with $g^i(t^{-i})=0$. Then all our results remain valid.

$s^i_{kj} - t^i_{kj} \leq \tilde{d}$, we obtain

$$\begin{aligned} v^j_k(t^i_{kj}, t^{-i}_{kj}) - v^j_k(s^i_{kj}, s^{-i}_{kj}) &= v^j_k(t^i_{kj}, s^j_{kj} + \frac{\gamma^i \tilde{d}}{\underline{m}}, s^{-i-j}_{kj}) - v^j_k(s^i_{kj}, s^j_{kj}, s^{-i-j}_{kj}) \\ &\geq \underline{m}\frac{\gamma^i \tilde{d}}{\underline{m}} - \gamma^i \tilde{d} = 0 \quad \forall j \neq i, \end{aligned} \tag{E33}$$

where $s^{-i-j}_{kj} = (s^l_{kj})_{l \neq i, l \neq j}$. Since $s^i_{ki} = t^i_{ki}$ and $s^j_{ki} = t^j_{ki}$, we have $v^i_k(t^i_{ki}, t^{-i}_{ki}) = v^i_k(s^i_{ki}, s^{-i}_{ki})$. Combining this observation with (E33) yields

$$\sum_l v^l_k(t^i_{kl}, t^{-i}_{kl}) \geq \sum_l v^l_k(s^i_{kl}, s^{-i}_{kl}). \tag{E34}$$

Similarly, by the construction of t^{-i}, we have $t^j_{k'j} = s^j_{k'j} - \frac{\gamma^i \tilde{d}}{\underline{m}}$ and $t^l_{k'j} = s^l_{k'j}$ for all $j \neq i, l \neq j, l \neq i$. Since $t^i_{k'j} - s^i_{k'j} \leq \tilde{d}$, we obtain

$$\begin{aligned} v^j_{k'}(s^i_{k'j}, s^{-i}_{k'j}) - v^j_{k'}(t^i_{k'j}, t^{-i}_{k'j}) &= v^j_{k'}(s^i_{k'j}, s^j_{k'j}, s^{-i-j}_{k'j}) - v^j_{k'}(t^i_{k'j}, s^j_{k'j} - \frac{\gamma^i \tilde{d}}{\underline{m}}, s^{-i-j}_{k'j}) \\ &\geq \underline{m}\frac{\gamma^i \tilde{d}}{\underline{m}} - \gamma^i \tilde{d} = 0 \quad \forall j \neq i, \forall k' \neq k. \end{aligned} \tag{E35}$$

Since $s^i_{k'i} = t^i_{k'i}$ and $s^j_{k'i} = t^j_{k'i}$, we have $v^i_{k'}(s^i_{k'i}, s^{-i}_{k'i}) = v^i_{k'}(t^i_{k'i}, t^{-i}_{k'i})$ for all $k' \neq k$. Combining this observation with (E35) yields

$$\sum_l v^l_{k'}(s^i_{k'l}, s^{-i}_{k'l}) \geq \sum_l v^l_{k'}(t^i_{k'l}, t^{-i}_{k'l}) \quad \forall k' \neq k. \tag{E36}$$

Since $s^{-i} \in \mathcal{E}_k(s^i)$, we have $\sum_l v^l_k(s^i_{kl}, s^{-i}_{kl}) \geq \sum_l v^l_{k'}(s^i_{k'l}, s^{-i}_{k'l})$ for all $k' \neq k$. Combining this inequality with (E34) and (E36) yields $\sum_l v^l_k(t^i_{kl}, t^{-i}_{kl}) \geq \sum_l v^l_{k'}(t^i_{k'l}, t^{-i}_{k'l})$ for all $k' \neq k$. That is, $t^{-i} \in \mathcal{E}_k(t^i)$. Therefore, $\mathcal{E}_k(s^i) \subseteq \mathcal{B}_{\frac{\gamma^i \tilde{d}}{\underline{m}}}(\mathcal{E}_k(t^i))$.

Since $\gamma^i < \gamma$, (E32) implies

$$G^i(\mathcal{E}_k(s^i) \setminus \mathcal{E}_k(t^i)) \leq G^i(\mathcal{B}_{\frac{\gamma \tilde{d}}{\underline{m}}}(\mathcal{E}_k(t^i))) - G^i(\mathcal{E}_k(t^i)) \leq \varepsilon,$$

as desired. □

We now prove Theorem 6.2. Fix $\gamma > 0$ such that if $\gamma^i < \gamma$ for all $i \in \mathcal{I}$, then $G^i(\mathcal{E}_k(s^i) \setminus \mathcal{E}_k(t^i)) \leq \min\{\varepsilon, \frac{\varepsilon}{\overline{m}}\}$ for all $s^i, t^i \in \bar{e}(\tilde{s}^i)$, $\tilde{s}^i \in S^i$, $k \in \mathcal{K}$, and $i \in \mathcal{I}$. By Lemma E12, such γ exists. We are going to show that the MVCG mechanism satisfies (i)–(iii) in Definition 8. Fix $i \in \mathcal{I}$ and $s^i, t^i \in S^i$. Since p^* is efficient,

$$\sum_k p^*_k(\bar{\varsigma}^i(s^i), s^{-i}) \sum_j v^j_k(\bar{\varsigma}^i_{kj}(s^i), s^{-i}_{kj}) - \sum_k p^*_k(\bar{\varsigma}^i(t^i), s^{-i}) \sum_j v^j_k(\bar{\varsigma}^i_{kj}(s^i), s^{-i}_{kj}) \geq 0. \tag{E37}$$

By construction, $\overline{\varsigma}^i_{kj}(s^i) = \overline{\varsigma}^i_{kj}(t^i)$ for all $k \in \mathcal{K}$ and all $j \neq i$. Thus,

$$\begin{aligned} &\max_k \sum_{j \neq i} v^j_k(\overline{\varsigma}^i_{kj}(t^i), s^{-i}_{kj}) - \max_k \sum_{j \neq i} v^j_k(\overline{\varsigma}^i_{kj}(s^i), s^{-i}_{kj}) = 0 \quad \forall s^{-i} \in S^{-i} \\ &\sum_{j \neq i} \left(v^j_k(\overline{\varsigma}^i_{kj}(s^i), s^{-i}_{kj}) - v^j_k(\overline{\varsigma}^i_{kj}(t^i), s^{-i}_{kj}) \right) = 0 \quad \forall s^{-i} \in S^{-i}, \forall k \in \mathcal{K}. \end{aligned} \tag{E38}$$

Let μ^i_{MVCG} denote the indirect utility function of agent i associated with the MVCG mechanism. By construction, we know that

$$\begin{aligned} &\mu^i_{MVCG}(s^i) - \mu^i_{MVCG}(t^i) \\ &= \int \Bigg(\sum_k p^*_k(\overline{\varsigma}^i(s^i), s^{-i}) \sum_j v^j_k(\overline{\varsigma}^i_{kj}(s^i), s^{-i}_{kj}) - \sum_k p^*_k(\overline{\varsigma}^i(t^i), s^{-i}) \sum_j v^j_k(\overline{\varsigma}^i_{kj}(s^i), s^{-i}_{kj}) \\ &\quad + \max_k \sum_{j \neq i} v^j_k(\overline{\varsigma}^i_{kj}(t^i), s^{-i}_{kj}) - \max_k \sum_{j \neq i} v^j_k(\overline{\varsigma}^i_{kj}(s^i), s^{-i}_{kj}) + \sum_k p^*_k(\overline{\varsigma}^i(t^i), s^{-i}) D^i_k(s^i, t^i, s^{-i}) \\ &\quad + \sum_k p^*_k(\overline{\varsigma}^i(t^i), s^{-i}) \sum_{j \neq i} \left(v^j_k(\overline{\varsigma}^i_{kj}(s^i), s^{-i}_{kj}) - v^j_k(\overline{\varsigma}^i_{kj}(t^i), s^{-i}_{kj}) \right) \Bigg) dG^i. \end{aligned}$$

Plugging (E37) and (E38) into the equality above yields

$$\mu^i_{MVCG}(s^i) - \mu^i_{MVCG}(t^i) \geq \int \sum_k p^*_k(\overline{\varsigma}^i(t^i), s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i. \tag{E39}$$

Clearly, if $L^-(s^i, t^i) = \varnothing$, the inequality above implies $\mu^i_{MVCG}(s^i) - \mu^i_{MVCG}(t^i) \geq 0$. Thus, μ^i_{MVCG} is monotone. We next show that the MVCG mechanism is ε-LIC, that is,

$$\mu^i_{MVCG}(s^i) \geq \mu^i_{MVCG}(t^i) + \int \sum_k p^*_k(t^i, s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i - \varepsilon \sum_{k \in \mathcal{K}^i} |s^i_{ki} - t^i_{ki}|.$$

Combining this with (E39), we can see that a sufficient condition for ε-LIC is

$$\int_{\mathcal{E}_k(\overline{\varsigma}^i(t^i))} D^i_k(s^i, t^i, s^{-i}) dG^i \geq \int_{\mathcal{E}_k(t^i)} D^i_k(s^i, t^i, s^{-i}) dG^i - \varepsilon |s^i_{ki} - t^i_{ki}| \quad \forall k \in \mathcal{K}^i.$$

Since $\overline{\varsigma}^i(t^i) \in \overline{e}(t^i)$, the definition of $\overline{m}$ and the choice of γ imply for any $k \in L^+(s^i, t^i)$,

$$\begin{aligned} &\int_{\mathcal{E}_k(t^i)} D^i_k(s^i, t^i, s^{-i}) dG^i - \int_{\mathcal{E}_k(\overline{\varsigma}^i(t^i))} D^i_k(s^i, t^i, s^{-i}) dG^i \leq \int_{\mathcal{E}_k(t^i) \setminus \mathcal{E}_k(\overline{\varsigma}^i(t^i))} D^i_k(s^i, t^i, s^{-i}) dG^i \\ &\leq \overline{m} |s^i_{ki} - t^i_{ki}| G^i\left(\mathcal{E}_k(t^i) \setminus \mathcal{E}_k(\overline{\varsigma}^i(t^i))\right) \leq \overline{m} |s^i_{ki} - t^i_{ki}| \frac{\varepsilon}{\overline{m}} = \varepsilon |s^i_{ki} - t^i_{ki}|, \end{aligned}$$

as desired. The proof for $k \in L^-(s^i, t^i)$ follows analogous arguments. Thus, the MVCG mechanism is ε-LIC.

Finally, we show that if values are additively separable, the MVCG mechanism is ε-bounded. Note that (E39) is equivalent to

$$\begin{aligned}\mu^i_{MVCG}(t^i) - \mu^i_{MVCG}(s^i) &\leq \sum_k G^i\big(\mathcal{E}_k(\overline{\varsigma}^i(t^i))\big)\big(f^i_k(t^i) - f^i_k(s^i)\big) \\ &\leq \sum_{k\in L^+(t^i,s^i)} G^i\big(\mathcal{E}_k(\overline{\varsigma}^i(t^i))\big)\big(f^i_k(t^i) - f^i_k(s^i)\big).\end{aligned}$$

Since $G^i\big(\mathcal{E}_k(\overline{\varsigma}^i(t^i))\big)$ is the probability that k is chosen, $\sum_{k\in L^+(t^i,s^i)} G^i\big(\mathcal{E}_k(\overline{\varsigma}^i(t^i))\big) \leq 1$. Moreover, since $\gamma^i < \gamma$ and $\overline{\varsigma}^i(t^i) \in \overline{e}(t^i)$, it follows from Lemma E12 that

$$G^i\big(\mathcal{E}_k(\overline{\varsigma}^i(t^i))\big) \leq G^i\big(\mathcal{E}_k(t^i)\big) + \varepsilon \quad \forall k \in L^+(t^i,s^i).$$

Taking $w_k = G^i\big(\mathcal{E}_k(\overline{\varsigma}^i(t^i))\big)$ for all $k \in L^+(t^i,s^i)$ completes the proof.

F An Alternative Notion of Approximate Local Incentive Compatibility

This section provides conditions under which ε-LIC and weak ε-LIC are equivalent in multi-dimensional environments.

Definition 12. For any $\varepsilon > 0$, a SCR p is **ε-robust to own-payoff irrelevant information** (ε-robust) if $G^i\big(A_k(s^i,p) \setminus A_k(t^i,p)\big) \leq \varepsilon$ for all $s^i \in S^i$, $t^i \in e(s^i)$, $k \in \mathcal{K}$, and $i \in \mathcal{I}$.

In words, a SCR p is ε-robust if the expected probability assignments do not vary much as own-payoff irrelevant information varies. In the case of private values and the case of one-dimensional signals, any SCR is ε-robust for all $\varepsilon \geq 0$, as the set $e(s^i)$ is a singleton. Another instance in which a SCR p is ε-robust is when p is solely a function of valuations v^i_k and the marginal effect of agent j's information on v^i_k is relatively small for all $j \neq i$. By Lemma E12, for any $\varepsilon > 0$, the efficient SCR is ε-robust if agents are sufficiently informationally small.

We next impose a restriction on the signal spaces. We assume that the correspondence $e(\cdot)$ **admits a Lipschitz selection**: there exists a selection $\varsigma^i(s^i) \in e(s^i)$ such that $\varsigma^i(s^i)$ is Lipschitz continuous in $(s^i_{ki})_{k\in\mathcal{K}^i}$.

The next lemma presents the equivalence result.

Lemma F13. *For any $\varepsilon > 0$, there exists $\xi > 0$ such that for any ξ-robust SCR p and any weakly ξ-LIC mechanism $(p,\tilde{x})$, we can find a transfer scheme x such that (p,x) is ε-LIC.*

Proof. Fix $\varepsilon > 0$. Since $e(\cdot)$ admits a Lipschitz selection, there exists $\Lambda > 0$ such that $\frac{\partial \varsigma^i_{k'j}(s^i)}{\partial s^i_{ki}} < \Lambda$ for all $s^i \in S^i$, $k \in \mathcal{K}^i$, $k' \in \mathcal{K}$, $i \in \mathcal{I}$, and $j \neq i$. Recall that $\overline{m} = \max_{i\in\mathcal{I},k\in\mathcal{K},s\in S} \frac{\partial v^i_k(s^i_{ki}, s^{-i}_{ki})}{\partial s^i_{ki}}$. Take $\xi = \frac{\varepsilon}{\overline{m}+\Lambda}$. Take a ξ-robust SCR p and a weakly ξ-LIC mechanism $(p, \tilde{x})$. Construct a transfer scheme x as follows:

$$x^i(s^i, s^{-i}) = -\sum_k p_k(s^i, s^{-i}) v^i_k(s^i, s^{-i}) + \mu^i_{(p,\tilde{x})}(\varsigma^i(s^i)) \quad \forall s^i \in S^i, \forall s^{-i} \in S^{-i}, \forall i \in \mathcal{I}.$$

We are going to show that (p, x) is ε-LIC. Fix i and s^i, t^i. Observe first that for every $k \in L^+(s^i, t^i)$, we have

$$\begin{aligned} &\int p_k(t^i, s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i - \int p_k(\varsigma^i(t^i), s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i \\ \leq &\int_{A_k(t^i,p)\setminus A_k(\varsigma^i(t^i),p)} D^i_k(s^i, t^i, s^{-i}) dG^i \\ \leq &G^i\big(A_k(t^i, p) \setminus A_k(\varsigma^i(t^i), p)\big)\overline{m}(s^i_{ki} - t^i_{ki}) \leq \overline{m}\xi|s^i_{ki} - t^i_{ki}|. \end{aligned}$$

The second inequality follows from the definition of $\overline{m}$; the last inequality follows from the assumption that p is ξ-robust. By an analogous argument, the same conclusion holds for all $k \in L^-(s^i, t^i)$. Thus, for every $k \in \mathcal{K}^i$,

$$\int p_k(\varsigma^i(t^i), s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i \geq \int p_k(t^i, s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i - \overline{m}\xi|s^i_{ki} - t^i_{ki}|. \quad \text{(F40)}$$

By the construction of x and the weak ξ-LIC of $(p, \tilde{x})$, we obtain

$$\begin{aligned} \mu^i_{(p,x)}(s^i) - \mu^i_{(p,x)}(t^i) &= \mu^i_{(p,\tilde{x})}(\varsigma^i(s^i)) - \mu^i_{(p,\tilde{x})}(\varsigma^i(t^i)) \\ &\geq \int \sum_{k\in\mathcal{K}} p_k(\varsigma^i(t^i), s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i - \xi \parallel \varsigma^i(s^i) - \varsigma^i(t^i) \parallel_\infty \\ &= \int \sum_{k\in\mathcal{K}^i} p_k(\varsigma^i(t^i), s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i - \xi \parallel \varsigma^i(s^i) - \varsigma^i(t^i) \parallel_\infty . \end{aligned}$$

The last equality follows from $D^i_k(s^i, t^i, s^{-i}) = 0$ for all $k \in \mathcal{K}^i_0$ and $s^{-i} \in S^{-i}$. Combining this inequality with (F40) yields

$$\begin{aligned} &\mu^i_{(p,x)}(s^i) - \mu^i_{(p,x)}(t^i) \\ \geq &\int \sum_{k\in\mathcal{K}^i} p_k(t^i, s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i - \overline{m}\xi \sum_{k\in\mathcal{K}^i} |s^i_{ki} - t^i_{ki}| - \xi \parallel \varsigma^i(s^i) - \varsigma^i(t^i) \parallel_\infty \\ \geq &\int \sum_{k\in\mathcal{K}^i} p_k(t^i, s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i - \varepsilon \sum_{k\in\mathcal{K}^i} |s^i_{ki} - t^i_{ki}|. \end{aligned}$$

The last inequality follows from $\parallel \varsigma^i(s^i) - \varsigma^i(t^i) \parallel_\infty \leq \Lambda \sum_{k\in\mathcal{K}^i} |s^i_{ki} - t^i_{ki}|$ and the construction of ξ. It follows from the construction of x that

$$\mu^i_{(p,x)}(s^i) \geq \mu^i_{(p,x)}(t^i) + \int \sum_{k\in\mathcal{K}^i} p_k(t^i, s^{-i}) D^i_k(s^i, t^i, s^{-i}) dG^i - \varepsilon \sum_{k\in\mathcal{K}^i} |s^i_{ki} - t^i_{ki}|$$
$$= u^i_{(p,x)}(t^i, s^i) - \varepsilon \sum_{k\in\mathcal{K}^i} |s^i_{ki} - t^i_{ki}|.$$

That is, the mechanism (p, x) is ε-LIC. □

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