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Portfolio Liquidation Games with Self-Exciting Order Flow*

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Abstract

We analyze novel portfolio liquidation games with self-exciting order flow. Both the N-player game and the mean-field game are considered. We assume that players' trading activities have an impact on the dynamics of future market order arrivals thereby generating an additional transient price impact. Given the strategies of her competitors each player solves a mean-field control problem. We characterize open-loop Nash equilibria in both games in terms of a novel mean-field FBSDE system with unknown terminal condition. Under a weak interaction condition we prove that the FBSDE systems have unique solutions. Using a novel sufficient maximum principle that does not require convexity of the cost function we finally prove that the solution of the FBSDE systems do indeed provide open-loop Nash equilibria.

AMS Subject Classification: 93E20, 91B70, 60H30.

Keywords: stochastic games, mean-field games, portfolio liquidation, Hawkes process, singular terminal value

1 Introduction

Models of optimal portfolio liquidation under market impact have received substantial consideration in the financial mathematics and the stochastic control literature in recent years. Starting with the work of Almgren and Chriss (2001) existence and uniqueness of optimal liquidation strategies under various forms of market impact, trading restrictions and model uncertainty have been established by many authors including Ankirchner et al. (2014), Bank and Voß (2018), Fruth et al. (2014), Gatheral and Schied (2011), Graewe et al. (2015), Graewe et al. (2018), Horst et al. (2020), Kratz (2014), Kruse and Popier (2016) and Popier and Zhou (2019).

One of the main characteristics of portfolio liquidation models is the terminal state constraint on the portfolio process. The constraint translates into a singular terminal condition on the associated HJB equation or an unknown terminal condition on the associated adjoint equation when applying stochastic maximum principles. In deterministic settings the state constraint is typically no challenge. In stochastic

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settings, however, it causes significant difficulties when proving the existence of solutions to the HJB or adjoint equation and hence in proving the existence and uniqueness of optimal trading strategies.

The majority of the optimal trade execution literature allows for either instantaneous or transient impact. The first approach, initiated by Bertsimas and Lo (1998) and Almgren and Chriss (2001), describes the price impact as a purely temporary effect that depends only on the present trading rate and does not influence future prices. A second approach, initiated by Obizhaeva and Wang (2013), assumes that the price impact is transient with the impact of past trades on current prices decaying over time. For single player models Graewe and Horst (2017) and Horst and Xia (2019) combined instantaneous and transient impacts into a single model. Assuming that the transient price impact follows an ordinary differential equation with random coefficients driven by the large investor's trading rate they showed that the optimal execution strategies could be characterized in terms of the solutions to multi-dimensional backward stochastic differential equations with singular terminal condition.

This paper studies a game theoretic extension of the liquidation model analyzed in Graewe and Horst (2017) and Horst and Xia (2019). Our key conceptual contribution is to allow for an additional feedback of the large investors' trading activities on future market dynamics. There are many reasons why large selling orders may have an impact on future price dynamics. Extensive selling (or buying) may, for instance diminish the pool of counterparties and/or generate herding effects where other market participants start selling (or buying) in anticipation of further price decreases (or increases). Extensive selling may also attract predatory traders that employ front-running strategies. We refer Brunnermeier and Pedersen (2005) and Carlin et al. (2007) for an in-depth analysis of predatory trading.

Specifically, we assume that the market buy and sell order dynamics follow Hawkes processes whose base intensities depend on the large investors' trading activities. Hawkes processes have recently received considerable attention in the financial mathematics literature as a powerful tool to model self-exciting order flow and its impact on stock price volatility; see Bacry et al. (2013, 2015); El Euch et al. (2018); Jaisson and Rosenbaum (2015); Horst and Xu (2019) and references therein. In the context of liquidation models, they have been employed in Alfonsi and Blanc (2016); Amaral and Papanicolaou (2019); Cartea et al. (2018) albeit in very different settings. Alfonsi and Blanc (2016) considered a variant of model in Obizhaeva and Wang (2013), in which the continuous martingale driving the benchmark price was replaced by a given point process involving mutually exciting Hawkes processes. Amaral and Papanicolaou (2019) modeled the benchmark price by the difference of two mutually exciting processes. Cartea et al. (2018) considered a liquidation model in which the investor placed limit orders whose fill rates depended on a mutually exciting "influential" market order flow. In all three models the intensities of the Hawkes processes were *exogenous*; in our model they are endogenously controlled by the large investors. Cayé and Muhle-Karbe (2016) allowed for some form of endogenous feedback of past trades on future transaction costs but did not model this using Hawkes processes. All the aforementioned papers considered single-player models while our focus is on liquidation games.

We use Hawkes processes to introduce an additional transient price impact, which leads to a mean field control problem for each player. Finite player games with deterministic model parameters and transient impact were studied by Luo and Schied (2019); Schied et al. (2017); Schied and Zhang (2019) and Strehle (2017). We allow all impact parameters and cost coefficients to be stochastic. Liquidation games with instantaneous and permanent impact and with and without strict liquidation constraint have been studied in Carlin et al. (2007); Drapeau et al. (2021); Evangelista and Thamsten (2020); Fu and Horst (2020); Voß (2019). Although our mathematical framework would clearly be flexible enough to allow for an additional permanent impact we deliberately choose not to include a permanent impact as it does not alter the mathematical analysis. Instead, we choose to clarify the effects of self-exciting order flow on equilibrium liquidation strategies in a setting with only transient and instantaneous impact.

We consider both the finite player and the corresponding mean-field liquidation game. Mean-field games (MFGs) of optimal liquidation *without* strict liquidation constraint have been studied by many authors before. Cardaliaguet and Lehalle (2018) considered an MFG where each player has a different risk aversion. Casgrain and Jaimungal (2018, 2020) considered games with partial information and different beliefs, respectively. Huang et al. (2019) considered a game between a major agent who is liquidating a large number of shares and many minor agents that trade against the major player. To the best of our knowledge mean-field (type) games *with* liquidation constraint have only been analyzed by Fu et al. (2021) and Fu and Horst (2020) as well as in the recent work by Evangelista and Thamsten (2020).

Our model is very different from the one studied in the said papers. First, with our choice of feedback effect, in the N-player game each player's best response function is given by the solution to a mean-field rather than a standard control problem. Second, in the N-player game, an individual player's optimization problem is not convex. Third, our model shows a much richer equilibrium dynamics. Anticipating their impact on future order arrivals, the players typically trade more aggressively initially and may take short positions in equilibrium. Moreover we *prove* that in a two player game where one player starts with a strictly positive and the second starts with zero initial position, the second player always shorts the asset in equilibrium, that is, there exists a beneficial round-trip for the second player. While a similar effect has been observed before in, e.g. Fu et al. (2021) and Fu and Horst (2020) in our model the players benefit from their impact on future order flow rather than a pure liquidity provision effect. Finally, numerical simulations suggest that cyclically oscillating trading strategies *may* occur in the single player benchmark model if the impact of the player's trading rate on market dynamics is very strong. Cyclic oscillations have been observed in single-player models before by Gatheral et al. (2012) and in multi-player models by e.g. Schied et al. (2017) and Schied and Zhang (2019). It has been argued by e.g. Alfonsi et al. (2012) and Gatheral et al. (2012) that cyclic fluctuations should be viewed as model irregularities and should hence be avoided, at least in single player models. In our model oscillating strategies can indeed be viewed as "model irregularities" as that they seem to occur only if market impact is too strong for our verification ("no statistical arbitrage") argument to hold. Interestingly, we did not find numerical evidence for cyclically oscillating strategies in the N-player game or the MFG.

We apply a stochastic method to solve the liquidation games. The stochastic maximum principle suggests that the equilibrium trading strategies in both the N-player game and the MFG can be characterized in terms of the solutions to coupled mean-field FBSDE systems. The forward components describe the players' optimal portfolio processes and the expected child order flow; hence their initial and in the case of the portfolio processes also terminal conditions are known. The backward components are the adjoint processes; they describe the respective equilibrium trading rates. Due to the liquidation constraint some of the terminal values are unknown.

We analyze both FBSDE systems within a common mathematical framework. Making a standard affine ansatz the system with unknown terminal condition can be replaced by an FBSDE with known initial and terminal condition, yet singular driver. Proving the existence of a small time solution to this FBSDE is not hard. The challenge is to prove the existence of a global solution on the whole time interval. Extending the continuation method for singular FBSDEs established in Fu et al. (2021) to our higher-dimensional system we prove that the FBSDE system does indeed have a unique solution in a certain space under a weak interaction condition that limits the impact of an individual player on the payoff of other players. Weak interaction conditions have been extensively used in the game theory literature before; see, e.g. Horst (2005) and references therein.

Subsequently, we establish a novel verification argument for the N-player game[1] from which we deduce that the solution to the FBSDE system does indeed give the desired Nash equilibrium. Our maximum principle does not require convexity of the cost function as it is usually the case; see e.g. (Pham, 2009, Theorem 6.4.6). In fact, unlike in Evangelista and Thamsten (2020), Fu et al. (2021), and Fu and Horst (2020), in our model the players' optimization problems are not convex and hence standard verification arguments do not apply. Instead, we establish a novel maximum principle that strongly relies on the liquidation constraint. Our idea is to decompose trading costs into a sum of equilibrium plus round-trip costs and then to show that deviations from the equilibrium strategy (which are round-trips) are costly. The decomposition result provides a sufficient condition for our impact model to be viable. Viability of impact models is not trivial. Huberman and Stanzl (2004) were among the first to point out that

[1] The MFG is convex; hence general verification arguments apply.

market impact may lead to statistical arbitrage and price manipulation. They showed that when the price impact of trades is permanent and time independent, only linear impact functions support viable markets. When impact is transient, the issue of viability is considerably more challenging and the literature is rather sparse. Alfonsi et al. (2012), Gatheral (2011) and Gatheral et al. (2012) discussed viability in *deterministic* single-player models for a variety of impact kernels. To the best of our knowledge our verification result is the first that applies to non-convex multi-player models in stochastic environments.[2]

Finally, we prove that under an additional homogeneity assumption on the players' cost function the sequence of Nash equilibria in the N-player game converges in a suitable sense to the unique equilibrium in the MFG as the number of players tends to infinity. This complements the analysis in Fu et al. (2021) where no such convergence result was established.

The benchmark model where all model parameters are deterministic, except the initial portfolios, is much easier to analyze. In this case, the FBSDE system reduces to an ODE system. The systems for the MFG, the single player model and the two-player model can be solved explicitly. The explicit solution is used to illustrate the possible impact of anticipating one's own impact on future order flow by three specific examples.

The remainder of this paper is organized as follows. The liquidation game is introduced in Section 2. Existence and uniqueness of equilibria in both the N-player game and the corresponding MFG is established in Section 3. Convergence of the N-player equilibria to the unique MFG equilibrium is shown in Section 4. Numerical simulations are provided in Section 5.

Notation. We use the following notation and notational conventions. We denote by $\langle \cdot, \cdot \rangle$ the inner product of two vectors. For a matrix $y \in \mathbb{R}^{n\times m}$, denote by $|y| := \left(\sum_{1\le i\le n, 1\le j\le m} |y_{ij}|^2\right)^{1/2}$ the 2-norm of y. For a $\mathbb{R}$-valued essentially bounded stochastic process y, denote by $y_{\min}$ and by $\|y\|$ its lower bound and upper bound, respectively. For a $\mathbb{R}^{n\times m}$-valued essentially bounded stochastic process y, without confusion, we still denote by $\|y\|$ its upper bound in terms of 2-norm, i.e., $\|y\| := (\sum_{i,j} \|y_{ij}\|^2)^{1/2}$.

For a filtration $\mathscr{F}$ we denote by $L^2_{\mathscr{F}}$ the space of all $\mathscr{F}$ progressively measurable processes such that $\|y\|_{L^2} := \left(\mathbb{E}\left[\int_0^T |y_t|^2\,dt\right]\right)^{\frac{1}{2}} < \infty$. We let $\mathbb{S}^2_{\mathscr{F}}$ be the space of all $\mathscr{F}$ progressively measurable processes with continuous trajectories such that $\|y\|_{\mathbb{S}^2} := \left(\mathbb{E}\left[\sup_{0\le t\le T} |y_t|^2\right]\right)^{\frac{1}{2}} < \infty$ and denote by $\mathcal{H}_{a,\mathscr{F}}$ the subspace of $\mathbb{S}^2_{\mathscr{F}}$ such that $\|y\|_a := \left(\mathbb{E}\left[\sup_{0\le t\le T}\left(\frac{|y_t|}{(T-t)^a}\right)^2\right]\right)^{\frac{1}{2}} < \infty$. Finally, $L^{2,-}_{\mathscr{F}}$ denotes the space of all $\mathscr{F}$ progressively measurable processes such that for each $\epsilon > 0$ it holds that $\mathbb{E}\left[\int_0^{T-\epsilon} |y_t|^2\,dt\right] < \infty$, and $\mathbb{S}^{2,-}_{\mathscr{F}}$ denotes the space of all $\mathscr{F}$ progressively measurable processes with continuous trajectories such that $\|y\|_{\mathbb{S}^{2,-}} := \left(\sup_{\epsilon\ge 0}\mathbb{E}\left[\sup_{0\le t\le T-\epsilon} |y_t|^2\right]\right)^{\frac{1}{2}} < \infty$.

Throughout, C denotes a generic constant that may vary from line to line.

2 The liquidation game

In this paper we introduce a novel portfolio liquidation game with self-exciting order flow. Both the N-player game and the corresponding MFG will be considered. Our starting point is the portfolio liquidation model with instantaneous and persistent price impact analyzed in Graewe and Horst (2017). We briefly review this model in the next subsection before extending it by adding an additional feedback term of mean-field type into the dynamics of the benchmark price process. We assume throughout that

[2]We require traders to be risk averse. The benchmark case of a deterministic risk-neutral single player liquidation model can be solved in closed form as shown in Chen et al. (2021). Even in this case, the impact kernel is different from the ones considered in the above mentioned papers.

randomness is described by a multi-dimensional Brownian motion W, unless otherwise stated, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ that satisfies the usual conditions.

2.1 The single player benchmark model

In Graewe and Horst (2017) the authors analyzed a liquidation model in which the investor needs to unwind an initial portfolio of $\mathcal{X}$ shares over a finite time horizon $[0,T]$ using absolutely continuous trading strategies. Assuming a linear-quadratic cost function, the large investor's stochastic control problem is given by

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \{\eta \xi_s^2 + \xi_s Y_s + \lambda_s X_s^2\}\, ds\right] \tag{2.1}$$

subject to the state dynamics

$$\begin{cases} X_t = \mathcal{X} - \displaystyle\int_0^t \xi_s\, ds, \quad t \in [0,T], \\ X_T = 0, \\ Y_t = \displaystyle\int_0^t \{-\rho_s Y_s + \gamma \xi_s\}\, ds, \quad t \in [0,T]. \end{cases} \tag{2.2}$$

Here, η and γ are positive constants while ρ and λ are progressively measurable, non-negative and essentially bounded stochastic processes. The quantity X_t denotes the number of shares the investor needs to sell at time $t \in [0,T]$, while ξ_t denotes the rate at which the stock is traded at that time. The process Y describes the *persistent* price impact. It can be viewed as a shift in the mid quote price caused by past trades where the impact is measured by impact factor γ. Alternatively, it can be viewed as an additional spread caused by the large investor in a block-shaped limit order book market with constant order book depth $1/\gamma > 0$ as in Horst and Naujokat (2014); Obizhaeva and Wang (2013). This results in an execution price process of the form

$$\tilde{S}_t = S_t - \eta \xi_t - Y_t \tag{2.3}$$

where S is a Brownian martingale that describes the dynamics of the unaffected mid-price process. The essentially bounded process ρ describes the rates at which the order book recovers from past trades. The constant $\eta > 0$ describes an additional *instantaneous* impact as in Almgren and Chriss (2001), Ankirchner et al. (2014), Graewe et al. (2015) or Graewe et al. (2018) among many others. The first two terms of the running cost term in (2.1) capture the expected liquidity cost resulting from the instantaneous and the persistent impact, respectively. The third term can be interpreted as a measure of the market risk associated with an open position. It penalizes slow liquidation.

We are now going to introduce an additional feedback effect into the above model that accounts for the possibility of an additional order flow ("child orders") triggered by the large investor's trading activity. To this end, we assume that the market order dynamics follows a Hawkes process with exponential kernel.

Specifically, we assume that market sell and buy orders arrive according to independent Hawkes processes $N^{\pm}$ with respective intensities

$$\zeta_t^{\pm} := \mu_t^{\pm} + \alpha \int_0^t e^{-\beta(t-s)} dN_s^{\pm}$$

where $\mu^{\pm}$ are the *base intensities* and α, β are deterministic coefficients that capture the impact of past orders on future order flow. In the absence of the large trader we set $\mu^{\pm} \equiv \mu$. In this case the base intensities are equal, and the same number of sell and buy orders arrive on average. In particular, the order flow imbalance $N^+ - N^-$ is a martingale.

In the presence of the large trader the base intensities change to $\mu_t^{\pm} = \mu + \xi_t^{\pm}$ where $\xi^{\pm}$ denotes the positive/negative part of the large investor's liquidation strategy; if $\xi_t > 0$ the investor is selling, else the investor is buying. In this case, the expected intensities satisfy

$$\mathbb{E}[\zeta_t^{\pm}] = \mu + \mathbb{E}[\xi_t^{\pm}] + \alpha \int_0^t e^{-\beta(t-s)}\mathbb{E}[\zeta_s^{\pm}]ds.$$

Thus, if we let $\bar{N}_t^{\pm}$ denote the expected number of sell/buy market orders that arrived over the period $[0,t]$, then the expected net sell order flow imbalance $\bar{N}_t := \bar{N}_t^+ - \bar{N}_t^-$ is given by

$$\bar{N}_t = \int_0^t \left(\mathbb{E}[\zeta_s^+] - \mathbb{E}[\zeta_s^-]\right) ds = \int_0^t \mathbb{E}[\xi_s]ds + \alpha \int_0^t e^{-\beta(t-s)}\bar{N}_s ds. \tag{2.4}$$

As a result, the expected order flow imbalance generated by the large trader can be decomposed into the trader's own expected accumulated order flow plus the expected number of (net) sell child orders

$$C_t = \alpha \int_0^t e^{-\beta(t-s)}\bar{N}_s ds. \tag{2.5}$$

In order to account for the possible feedback effect of the large trader's activity on future order flow and hence price dynamics we suggest to add the average child order flow rate dC_t to the dynamics of the transient impact process, assuming that the child order flow contributes to the price impact in exactly the same way as the large trader's order flow.[3]

Differentiating equation (2.5) we see that

$$dC_t = (-(\beta-\alpha)C_t + \alpha(\mathbb{E}\mathcal{X} - \mathbb{E}X_t))\,dt, \quad C_0 = 0. \tag{2.6}$$

The child order flow rate increases linearly in the investor's expected traded volume $\mathbb{E}\mathcal{X} - \mathbb{E}X$. The child order flow is mean-reverting if $\frac{\alpha}{\beta} < 1$. It is well known that the Hawkes process with constant base intensity is stable in the long term and that each order triggers at most one child order on average if $\frac{\alpha}{\beta} < 1$; see Hawkes and Oakes (1974) for details.

Starting from (2.2) but accounting for the additional child order flow in the dynamics of the market impact process Y results in the following mean-field type control problem for our large investor:

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \left\{\eta_s\xi_s^2 + \xi_s Y_s + \lambda_s X_s^2\right\} ds\right] \tag{2.7}$$

subject to the state dynamics

$$\begin{cases} dX_t = -\xi_t\,dt, \quad t\in[0,T],\\ X_0 = \mathcal{X},\ X_T = 0,\\ dY_t = \Big(-\rho_t Y_t + \gamma_t\big(\xi_t - (\beta-\alpha)C_t + \alpha(\mathbb{E}\mathcal{X} - \mathbb{E}X_t)\big)\Big)dt, \quad t\in[0,T],\\ Y_0 = 0,\\ dC_t = (-(\beta-\alpha)C_t + \alpha(\mathbb{E}\mathcal{X} - \mathbb{E}X_t))\,dt, \quad t\in[0,T],\\ C_0 = 0. \end{cases} \tag{2.8}$$

Remark 2.1. i) It would clearly be desirable to consider the *conditional* child order flow, given the large trader's submission rates instead of the unconditional expectation. The unconditional expectation is considered for mathematical convenience as we are unaware of any tractable representation of the conditional child order flow.

ii) At first sight it might also be desirable to consider the child order process directly rather than the rate dC. This, however, would be inconsistent with the way we model the large trader who is supposed to trade in rates, which, of course, should be viewed as a mathematically tractable approximation of discrete trading.

[3] This assumption can be justified by assuming that the market does not or cannot differentiate different origins of order flow.

2.2 Many player models

Let us now consider a game theoretic extension of the above liquidation model with N strategically interacting investors. The trading rate, initial portfolio and portfolio process of player $i \in \{1, ..., N\}$ are denoted by ξ^i, $\mathcal{X}^i$ and X^i, respectively. The corresponding averages over the set of players are denoted by $\bar{\xi}$, $\bar{\mathcal{X}}$ and $\bar{X}$, respectively. We assume that the initial portfolios are (not necessarily independent) square-integrable random variables.

Assuming that both the child order flow and the impact process are driven by the average trading rate results in the following mean-field type optimization problem for player i given the liquidation strategies ξ^j $(j \neq i)$ of all the other players:

$$\operatorname*{ess\,inf}_{\xi^i \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \eta^i_t(\xi^i_s)^2 + \xi^i_s Y^i_s + \lambda^i_s (X^i_s)^2 \, ds\right] \tag{2.9}$$

subject to

$$\begin{cases} dX^i_t = -\xi^i_t \, ds, \quad t \in [0,T], \\ X^i_0 = \mathcal{X}^i, \; X^i_T = 0, \\ dY^i_t = \Big(-\rho^i_t Y^i_t + \gamma^i_t \Big(\bar{\xi}_t - (\beta^i_t - \alpha^i_t) C^i_t + \alpha^i_t (\mathbb{E}[\bar{\mathcal{X}}] - \mathbb{E}[\bar{X}_t]) \Big) \Big) dt, \quad t \in [0,T] \\ Y^i_0 = 0 \\ dC^i_t = \left(-(\beta^i_t - \alpha^i_t) C^i_t + \alpha^i_t (\mathbb{E}[\bar{\mathcal{X}}] - \mathbb{E}[\bar{X}_t]) \right) dt, \quad t \in [0,T] \\ C^i_0 = 0. \end{cases} \tag{2.10}$$

Under the assumption that all the cost coefficients and model parameters are essentially bounded, $\mathcal{F}$-progressively measurable stochastic processes and that the instantaneous impact term and the risk aversion parameters are uniformly bounded away from zero we prove that the N-player liquidation game admits a Nash equilibrium under a weak interaction condition that limits the impact of an individual player on the trading costs of other players. Since each player affects the state dynamics of other players mainly through the impact parameters γ^i our existence of equilibrium result requires these parameters to be small enough and/or the unaffected processes η^i and λ^i to be large enough. Moreover, we require the stability condition $\frac{\alpha^i}{\beta^i} < 1$ so that child order dynamics is mean-reverting.

Remark 2.2. Assuming that all players trade the same stock in the same venue it is natural to assume that the model parameters and cost coefficients are the same across the players' cost functions, except to the initial portfolios and the risk aversion parameters. We are allowing for additional heterogeneity in the players' cost functions and state dynamics as this does not alter the mathematical analysis.

Under the additional assumption that the player's cost functions are homogeneous in sense that

$$\begin{aligned} \eta^i_t = \eta\left(t, \mathcal{X}^i, (W^i_s)_{0\le s\le t}\right), \quad \lambda^i_t = \lambda\left(t, \mathcal{X}^i, (W^i_s)_{0\le s\le t}\right), \quad \rho^i_t = \rho\left(t, \mathcal{X}^i, (W^i_s)_{0\le s\le t}\right), \\ \alpha^i_t = \alpha\left(t, \mathcal{X}^i, (W^i_s)_{0\le s\le t}\right), \quad \beta^i_t = \beta\left(t, \mathcal{X}^i, (W^i_s)_{0\le s\le t}\right), \quad \gamma^i_t = \gamma\left(t, \mathcal{X}^i, (W^i_s)_{0\le s\le t}\right), \end{aligned} \tag{2.11}$$

for independent Brownian motions $W^1, W^2, ...$ and measurable functions $\eta, \lambda, \rho, \alpha, \beta, \gamma$ and

$$\mathcal{X}^1, \mathcal{X}^2, ... \text{ are i.i.d. square integrable and independent of } W^1, W^2, ... \tag{2.12}$$

we also prove that the equilibrium converges (in a sense to be defined) to the unique equilibrium of a corresponding MFG as the number of players tends to infinity.

The MFG is obtained by first replacing the average quantities $\bar{\xi}$ and $\bar{X}$ by deterministic processes μ and ν, respectively and then by solving a representative player's optimization problem subject to an additional fixed point condition. In the MFG randomness is described by a Brownian motion $\overline{W}$ defined on some filtered probability space $(\Omega, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t), \mathbb{P})$ and all processes are $(\overline{\mathcal{F}}_t)$-progressively measurable. The corresponding MFG is then given by

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\overline{\mathcal{F}}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \{\eta_t(\xi_t)^2 + \xi_t Y_t + \lambda_t (X_t)^2\}\, dt\right] \tag{2.13}$$

subject to the state dynamics

$$\begin{cases} dX_t = -\xi_t\, ds, \quad t \in [0,T], \\ X_0 = \mathcal{X}, \;\; X_T = 0, \\ dY_t = \Big(-\rho_t Y_t + \gamma_t \Big(\mu_t - (\beta_t - \alpha_t) C_t + \alpha_t (\mathbb{E}[\mathcal{X}] - \nu_t)\Big)\Big) dt, \quad t \in [0,T] \\ Y_0 = 0 \\ dC_t = \left(-(\beta_t - \alpha_t) C_t + \alpha_t (\mathbb{E}[\mathcal{X}] - \nu_t)\right) dt, \quad t \in [0,T] \\ C_0 = 0. \end{cases} \tag{2.14}$$

and the equilibrium condition

$$\begin{cases} \mathbb{E}[\xi^*_t(\mu,\nu)] = \mu_t, \quad t \in [0,T], \\ \mathbb{E}[X^*_t(\mu,\nu)] = \nu_t, \quad t \in [0,T]. \end{cases} \tag{2.15}$$

Here $\xi^*(\mu,\nu)$ denotes the unique solution to (2.13) given (μ,ν), and $X^*(\mu,\nu)$ is the corresponding portfolio process.

We prove that the MFG admits a unique solution under a weak interaction condition. Under an additional homogeneity assumption we then prove that the sequence of equilibria in the finite player games converges to the mean-field equilibrium if the number of players tends to infinity.

3 Existence of Equilibria

In this section we provide an existence of equilibrium result for both the N-player and the mean-field liquidation games introduced in the previous section. We first characterize the equilibria of both games in terms of solutions to certain mean-field FBSDE systems with singular terminal conditions. Subsequently, we establish the existence of a unique solution to these systems within a common mathematical framework. Finally, we prove a verification argument from which we deduce the solutions to the FBSDEs do indeed provide the desired Nash equilibria.

3.1 Characterization of open-loop equilibria

We start by characterizing Nash equilibria in the N-player liquidation game. The Hamiltonian associated with the mean-field control problem (2.9) and (2.10) is given by

$$\begin{aligned} H^i = & -\sum_{j=1}^N \xi^j P^{i,j} + \sum_{j=1}^N Q^{i,j}\{-\rho^j Y^j + \gamma^j(\bar{\xi} - (\beta^j - \alpha^j) C^j) + \alpha^j \gamma^j (\mathbb{E}[\bar{\mathcal{X}}] - \mathbb{E}[\bar{X}_t])\} \\ & + \sum_{j=1}^N R^{i,j}\{-(\beta^j - \alpha^j) C^j + \alpha^j (\mathbb{E}[\bar{\mathcal{X}}] - \mathbb{E}[\bar{X}_t])\} + \xi^i Y^i + \eta^i (\xi^i)^2 + \lambda^i (X^i)^2. \end{aligned}$$

Using the same arguments as in Fu et al. (2021); Fu and Horst (2020) the stochastic maximum principle suggests that the best response function of player i given her competitors' actions is given by

$$\xi^{*,i} = \frac{P^{i,i} - Y^i - \frac{\gamma^i}{N} Q^{i,i}}{2\eta^i}, \tag{3.1}$$

where the adjoint processes $(P^{i,j}, Q^{i,j}, R^{i,j})$ $(j = 1, ..., N)$ satisfy the stochastic system

$$\begin{cases} -dP_t^{i,j} = \left(2\lambda_t^i X_t^i \delta_{ij} - \frac{1}{N}\mathbb{E}\left[\alpha_t^j \gamma_t^j Q_t^{i,j}\right] - \frac{1}{N}\mathbb{E}\left[\alpha_t^j R_t^{i,j}\right]\right) dt - Z_t^{P^{i,j}}\, dW_t, \\ -dQ_t^{i,j} = \left(\frac{P_t^{i,i} - Y_t^i - \frac{\gamma_t^i}{N} Q_t^{i,i}}{2\eta_t^i}\delta_{ij} - \rho_t^j Q_t^{i,j}\right) dt - Z_t^{Q^{i,j}}\, dW_t, \\ -dR^{i,j} = \left(-\gamma_t^j(\beta_t^j - \alpha_t^j) Q_t^{i,j} - (\beta_t^j - \alpha_t^j) R_t^{i,j}\right) dt - Z_t^{R^{i,j}}\, dW_t \\ \quad Q_T^{i,j} = R_T^{i,j} = 0, \end{cases} \tag{3.2}$$

with a-priori unknown terminal conditions on the processes $P^{i,j}$. We recall that W denotes a Wiener process of arbitrary dimension. It can be seen from the above system that the processes $P^{i,j}$ for $j \neq i$ are not relevant for the equilibrium dynamics and that $Q^{i,j} = R^{i,j} = 0$ for $j \neq i$ by standard BSDE theory. Putting $P^i := P^{i,i}$, $Q^i := Q^{i,i}$, $R^i := R^{i,i}$ and $M^i := P^i - Y^i$ we arrive at the following coupled mean-field forward-backward system[4]: for $i = 1, ..., N$,

$$\begin{cases} dX_t^i = -\frac{M_t^i - \frac{\gamma_t^i}{N} Q_t^i}{2\eta_t^i}\, dt, \\ dY_t^i = \left\{-\rho_t^i Y_t^i + \gamma_t^i \left(\frac{1}{N}\sum_{j=1}^N \frac{M_t^j - \frac{\gamma_t^j}{N} Q_t^j}{2\eta_t^j} - (\beta_t^i - \alpha_t^i) C_t^i + \alpha_t^i(\mathbb{E}[\bar{\mathcal{X}}] - \mathbb{E}[\bar{X}_t])\right)\right\} dt, \\ dC_t^i = \left\{-(\beta_t^i - \alpha_t^i) C_t^i + \alpha_t^i(\mathbb{E}[\bar{\mathcal{X}}] - \mathbb{E}[\bar{X}_t])\right\} dt \\ -dM_t^i = \left\{\left(2\lambda_t^i X_t^i - \frac{1}{N}\mathbb{E}\left[\alpha_t^i \gamma_t^i Q_t^i\right] - \frac{1}{N}\mathbb{E}\left[\alpha_t^i R_t^i\right]\right)\right. \\ \qquad \left. - \rho_t^i Y_t^i + \gamma_t^i \left(\frac{1}{N}\sum_{j=1}^N \frac{M_t^j - \frac{\gamma_t^j}{N} Q_t^j}{2\eta_t^j} - (\beta_t^i - \alpha_t^i) C_t^i + \alpha_t^i(\mathbb{E}[\bar{\mathcal{X}}] - \mathbb{E}[\bar{X}_t])\right)\right\} dt - Z_t^{M^i}\, dW_t, \\ -dQ_t^i = \left(\frac{M_t^i - \frac{\gamma_t^i}{N} Q_t^i}{2\eta_t^i} - \rho_t^i Q_t^i\right) dt - Z_t^{Q^i}\, dW_t, \\ -dR_t^i = \left(-\gamma_t^i(\beta_t^i - \alpha_t^i) Q_t^i - (\beta_t^i - \alpha_t^i) R_t^i\right) dt - Z_t^{R^i}\, dW_t \\ \quad X_0^i = \mathcal{X}^i,\ Y_0^i = C_0^i = 0,\ Q_T^i = R_T^i = X_T^i = 0. \end{cases} \tag{3.3}$$

In terms of

$$\underline{\mathcal{S}}^i = \begin{pmatrix} Y^i \\ C^i \end{pmatrix},\ A^i = \begin{pmatrix} \rho^i & \gamma^i(\beta^i - \alpha^i) \\ 0 & \beta^i - \alpha^i \end{pmatrix},\ B^i = (B^{i,(1)}, B^{i,(2)}) = \begin{pmatrix} \gamma^i & -\alpha^i\gamma^i \\ 0 & -\alpha^i \end{pmatrix},$$

and

$$\underline{\mathcal{R}}^i = \begin{pmatrix} \frac{\alpha^i\gamma^i}{N}\sum_{j=1}^N \mathbb{E}[\mathcal{X}^j] \\ \frac{\alpha^i}{N}\sum_{j=1}^N \mathbb{E}[\mathcal{X}^j] \end{pmatrix},\ \mathcal{P}^i = \begin{pmatrix} Q^i \\ R^i \end{pmatrix},\ \Theta = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \underline{\chi} = \frac{1}{N}\sum_{j=1}^N \begin{pmatrix} \xi^{*,j} \\ \mathbb{E}[X^j] \end{pmatrix},$$

[4]We prove below that the adjoint processes are determined uniquely by the solution to this mean-field FBSDE system.

the above system can be compactly rewritten as

$$
\begin{cases}
dX_t^i = -\dfrac{M_t^i - \frac{1}{N}\left\langle B_t^{i,(1)}, \mathcal{P}_t^i\right\rangle}{2\eta_t^i}\,dt,\\
d\underline{\mathcal{S}}_t^i = \left(-A_t^i\underline{\mathcal{S}}_t^i + B_t^i\underline{\chi}_t + \underline{\mathcal{R}}_t^i\right)dt\\
-dM_t^i = \left(2\lambda_t^i X_t^i + \left\langle \Theta, -A_t^i\underline{\mathcal{S}}_t^i + B_t^i\underline{\chi}_t + \underline{\mathcal{R}}_t^i\right\rangle + \dfrac{1}{N}\mathbb{E}\left[\left\langle B_t^{i,(2)}, \mathcal{P}_t^i\right\rangle\right]\right)dt - Z_t^{M^i}\,dW_t,\\
-d\mathcal{P}_t^i = \left(-(A_t^i)^\top\mathcal{P}_t^i + \Theta\dfrac{M_t^i - \frac{1}{N}\langle B^{i,(1)}, \mathcal{P}_t^i\rangle}{2\eta_t^i}\right)dt - Z_t^{\mathcal{P}^i}\,dW_t\\
X_0^i = \mathcal{X}^i,\ X_T^i = 0,\ \underline{\mathcal{S}}_0^i = (0,0)^\top,\ \mathcal{P}_T^i = (0,0)^\top.
\end{cases}
\tag{3.4}
$$

The Hamiltonian associated with the representative player's optimization problem in the MFG reads

$$
\begin{aligned}
H = {} & \eta\xi^2 + \xi Y + \lambda X^2 - \xi P + Q\{-\rho Y + \gamma(\mu - (\beta-\alpha)C) + \alpha\gamma(\mathbb{E}[\mathcal{X}] - \nu)\}\\
& + R\{-(\beta-\alpha)C + \alpha(\mathbb{E}[\mathcal{X}] - \nu)\},
\end{aligned}
\tag{3.5}
$$

where (P,Q,R) is the adjoint processes to (X,Y,C). Again, the stochastic maximum principle suggests that the optimal strategy is given by

$$
\xi = \frac{P - Y}{2\eta}.
$$

Putting $M := P - Y$ the candidate equilibrium strategy can be obtained in terms of a solution to the FBSDE system

$$
\begin{cases}
dX_t = -\dfrac{M_t}{2\eta_t}\,dt\\
dY_t = \left(-\rho_t Y_t + \gamma_t\left(\mathbb{E}\left[\dfrac{M_t}{2\eta_t}\right] - (\beta_t-\alpha_t)C_t\right) + \alpha_t\gamma_t(\mathbb{E}[\mathcal{X}] - \mathbb{E}[X_t])\right)dt\\
dC_t = (-(\beta_t-\alpha_t)C_t + \alpha_t(\mathbb{E}[\mathcal{X}] - \mathbb{E}[X_t]))\,dt\\
-dM_t = \left(2\lambda_t X_t - \rho_t Y_t + \gamma_t\left(\mathbb{E}\left[\dfrac{M_t}{2\eta_t}\right] - (\beta_t-\alpha_t)C_t\right) + \alpha_t\gamma_t(\mathbb{E}[\mathcal{X}] - \mathbb{E}[X_t])\right)dt - Z_t^M\,d\overline{W}_t\\
-dQ_t = \left(\dfrac{M_t}{2\eta_t} - \rho_t Q_t\right)dt - Z_t^Q\,d\overline{W}_t\\
-dR_t = (-\gamma_t(\beta_t-\alpha_t)Q_t - (\beta_t-\alpha_t)R_t)\,dt - Z_t^R\,d\overline{W}_t\\
X_0 = \mathcal{X},\ Y_0 = C_0 = 0,\ Q_T = R_T = X_T = 0.
\end{cases}
\tag{3.6}
$$

In terms of

$$
\overline{\mathcal{S}} = \begin{pmatrix} Y \\ C \end{pmatrix},\ A = \begin{pmatrix} \rho & \gamma(\beta-\alpha) \\ 0 & \beta-\alpha \end{pmatrix},\ B = (B^{(1)}, B^{(2)}) = \begin{pmatrix} \gamma & -\alpha\gamma \\ 0 & -\alpha \end{pmatrix},
$$

$$
\overline{\mathcal{R}} = \begin{pmatrix} \alpha\gamma\mathbb{E}[\mathcal{X}] \\ \alpha\mathbb{E}[\mathcal{X}] \end{pmatrix},\ \mathcal{P} = \begin{pmatrix} Q \\ R \end{pmatrix},\ \overline{\chi} = \begin{pmatrix} \mathbb{E}[\frac{M}{2\eta}] \\ \mathbb{E}[X] \end{pmatrix},
$$

this system can be compactly rewritten as

$$
\begin{cases}
dX_t = -\dfrac{M_t}{2\eta_t}\,dt\\
d\overline{\mathcal{S}}_t = \left(-A_t\overline{\mathcal{S}}_t + B_t\overline{\chi}_t + \overline{\mathcal{R}}_t\right)dt\\
-dM_t = \left(2\lambda_t X_t + \langle \Theta, -A_t\overline{\mathcal{S}}_t + B_t\overline{\chi}_t + \overline{\mathcal{R}}_t\rangle\right)dt - Z_t^M\,d\overline{W}_t\\
-d\mathcal{P}_t = \left(-A_t^\top\mathcal{P}_t + \Theta\dfrac{M_t}{2\eta_t}\right)dt - Z_t^{\mathcal{P}}\,d\overline{W}_t\\
X_0 = \mathcal{X},\ X_T = 0,\ \overline{\mathcal{S}}_0 = (0,0)^\top,\ \mathcal{P}_T = (0,0)^\top.
\end{cases}
\tag{3.7}
$$

3.2 The mean field FBSDE

This section provides a unified approach for solving a class of linear mean-field FBSDE systems that contains the systems (3.4) and (3.7) as special cases. Specifically, we consider the FBSDE system

$$\begin{cases} dX_t^i = -\dfrac{M_t^i - \frac{1}{N}\left\langle \widehat{B}_t^{i,(1)}, \mathcal{P}_t^i \right\rangle}{2\eta_t^i}\, dt, \\ d\mathcal{S}_t^i = \left(-A_t^i \mathcal{S}_t^i + K_t^i \chi_t + \mathcal{R}_t^i\right) dt, \\ -dM_t^i = \left(2\lambda_t^i X_t^i + \dfrac{1}{N}\mathbb{E}\left[\left\langle \widehat{B}_t^{i,(2)}, \mathcal{P}_t^i \right\rangle\right] + \left\langle \Theta, -A_t^i \mathcal{S}_t^i + K_t^i \chi_t + \mathcal{R}_t^i \right\rangle\right) dt - Z_t^{M^i}\, dW_t, \\ -d\mathcal{P}_t^i = \left(-(A_t^i)^\top \mathcal{P}_t^i + \Theta \dfrac{M_t^i - \frac{1}{N}\left\langle \widehat{B}_t^{i,(1)}, \mathcal{P}_t^i \right\rangle}{2\eta_t^i}\right) dt - Z_t^{\mathcal{P}^i}\, dW_t, \\ X_0^i = \mathcal{X}^i,\ X_T^i = 0,\ \mathcal{S}_0^i = (0,0)^\top,\ \mathcal{P}_T^i = (0,0)^\top, \end{cases} \tag{3.8}$$

for $i = 1, ..., N$ where $K^i = (K^{i,(1)}, K^{i,(2)}, K^{i,(3)})$ is an $\mathbb{R}^{2\times 3}$-valued stochastic process,

$$\xi^j = \frac{M^j - \frac{1}{N}\left\langle \widehat{B}^{j,(1)}, \mathcal{P}^j \right\rangle}{2\eta^j},$$

and

$$\chi = (\overline{\xi}, \mathbb{E}[\overline{X}], \mathbb{E}[\overline{\xi}])^\top = \left(\frac{1}{N}\sum_{j=1}^N \xi^j, \mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N X^j\right], \frac{1}{N}\sum_{j=1}^N \mathbb{E}[\xi^j]\right)^\top.$$

Remark 3.1. Let $0_{2\times 1}$ be the 2×1 zero matrix. If $K^i = (B^{i,(1)}, B^{i,(2)}, 0_{2\times 1})$, $\widehat{B}^{i,(1)} = B^{i,(1)}$, $\widehat{B}^{i,(2)} = B^{i,(2)}$ and $\mathcal{R}^i = \underline{\mathcal{R}}^i$, then the system (3.8) reduces to (3.4). If $N = 1$, $\widehat{B}^{1,(1)} = \widehat{B}^{1,(2)} = 0$, $K^1 = (0_{2\times 1}, B^{(2)}, B^{(1)})$ and $\mathcal{R}^1 = \overline{\mathcal{R}}$, then it reduces to (3.7).

In order to solve the above system we make the following assumptions.

Assumption 3.2. (i) The processes $A^i, \widehat{B}^i, K^i$ are progressively measurable w.r.t. the natural filtration generated by W and uniformly bounded:

$$\begin{aligned} \|A\| &:= \sup_i \|A^i\| < \infty, \\ \|\widehat{B}^{(1)}\| &:= \sup_i \|\widehat{B}^{i,(1)}\| < \infty,\ \|\widehat{B}^{(2)}\| := \sup_i \|\widehat{B}^{i,(2)}\| < \infty, \\ \|K^{(1)}\| &:= \sup_i \|K^{i,(1)}\| < \infty,\ \|K^{(2)}\| := \sup_i \|K^{i,(2)}\| < \infty,\ \|K^{(3)}\| := \sup_i \|K^{i,(3)}\| < \infty. \end{aligned}$$

(ii) There exists constants $\widehat{\rho} > 0$ and $\widetilde{\rho} > 0$ such that for any $\mathbb{R}^2$-valued process y and $i = 1, \cdots, N$,

$$\mathbb{E}\left[y_t^\top A_t^i y_t\right] \geq \widehat{\rho}\mathbb{E}\left[y_t^\top y_t\right], \quad \mathbb{E}\left[y_t^\top (A_t^i)^\top y_t + y_t^\top \frac{\Theta\langle \widehat{B}_t^{i,(1)}, y_t\rangle}{2N\eta_t^i}\right] \geq \widetilde{\rho}\mathbb{E}\left[y_t^\top y_t\right]. \tag{3.9}$$

(iii) The processes λ^i and η^i are progressively measurable, essentially bounded and there exist constants $\theta_0, \theta_1, \theta_2, \theta_3 > 0$ such that $\lambda_{\min} := \inf_i \lambda^i_{\min}$ and $\eta_{\min} := \inf_i \eta^i_{\min}$ satisfy

$$\begin{cases} 2\lambda_{\min} - \dfrac{\theta_0 + \theta_1 + \theta_2}{2} - \left(1 + \dfrac{1}{\theta_3}\right) \|K^{(2)}\|^2 \left(\dfrac{\|A\|^2}{2\theta_1 \widehat{\rho}^2} + \dfrac{1}{2\theta_2}\right) > 0, \\ 2\eta_{\min} - \dfrac{\|\widehat{B}^{(1)}\|}{N\widetilde{\rho}} - \dfrac{\|\widehat{B}^{(2)}\|^2}{2N^2\widetilde{\rho}^2\theta_0} - \left(1 + \dfrac{\|\widehat{B}^{(1)}\|}{2N\eta_{\min}\widetilde{\rho}}\right)^2 (1+\theta_3)(\|K^{(1)}\| + \|K^{(3)}\|)^2 \left(\dfrac{\|A\|^2}{2\theta_1 \widehat{\rho}^2} + \dfrac{1}{2\theta_2}\right) > 0. \end{cases} \tag{3.10}$$

(iv) The random variables $\mathcal{X}^i$ are square integrable for each $i = 1, \cdots, N$, which are independent of the Brownian motions.

The first assumption is standard. The second assumption essentially means that $A^i + (A^i)^\top$ is uniformly positive definite. The third condition is similar to conditions made in Fu et al. (2021) and Fu and Horst (2020). It states that the impact of other players on an individual player's best response function is weak enough. Specifically, it requires either the cost functions to be dominated by the terms $\eta_t^i(\xi_t^i)^2$ and $\lambda_t^i(X_t^i)^2$ that are unaffected by the choices of other players (large $\lambda_{\min}$ and large $\eta_{\min}$), or the impact of other players on an individual player's cost function and state dynamics to be weak enough.

Remark 3.3. If the number of players is large enough and the processes $\alpha^i, \beta^i, \gamma^i, \rho^i$ are identical across players and constant (cf. Remark 2.2), then condition (3.9) reduces to $4\rho > \gamma^2(\beta-\alpha), \beta > \alpha$ and we can define $\widehat{\rho}$ and $\widetilde{\rho}$ by the minimum eigenvalue of the matrix $\frac{A+A^\top}{2}$, i.e.

$$\widehat{\rho} = \widetilde{\rho} := \frac{\rho+\beta-\alpha-\sqrt{(\rho+\beta-\alpha)^2 - 4\rho(\beta-\alpha)+\gamma^2(\beta-\alpha)^2}}{2}.$$

Moreover, for any choice of $\theta_0, \theta_1, \theta_2$ and θ_3 we can choose λ^i and η^i large enough for (3.10) to be satisfied. We emphasize that both risk aversion and resilience are required to satisfy (3.10). It should be clear that (3.9) and (3.10) are very strong and by no means necessary conditions.

We are now ready to state and prove our main result of this section. It states that our general FBSDE system (3.8) admits a unique solution in a suitable space if Assumption 3.2 is satisfied. The proof is based on an extension of the continuation method introduced in Fu et al. (2021).

Theorem 3.4. *Under Assumption 3.2, there exists a unique solution*

$$(X^i, \mathcal{S}^i, M^i, \mathcal{P}^i, Z^{M^i}, Z^{\mathcal{P}^i}) \in \mathcal{H}_{a,\mathcal{F}} \times \mathbb{S}^2_{\mathcal{F}} \times L^2_{\mathcal{F}} \times \mathcal{H}_{\iota,\mathcal{F}} \times L^{2,-}_{\mathcal{F}} \times L^2_{\mathcal{F}}$$

to the FBSDE system (3.8) *for some positive constants* $a < 1,\ \iota < 1/2$.

Proof. Let $p \in [0,1],\ f^j \in L^2_{\mathcal{F}},\ g^j \in \mathcal{H}_{a,\mathcal{F}}$ for each $j = 1, \cdots, N$, where a is to be determined later. We apply the method of continuation to the following FBSDE indexed by $(p, f^j, g^j)_{j=1,\cdots,N}$:

$$\begin{cases} d\widetilde{X}_t^i = -\dfrac{\widetilde{M}_t^i - \frac{1}{N}\left\langle \widehat{B}_t^{i,(1)}, \widetilde{\mathcal{P}}_t^i \right\rangle}{2\eta_t^i}\, dt, \\ d\widetilde{\mathcal{S}}_t^i = \left(-A_t^i \widetilde{\mathcal{S}}_t^i + K_t^i \widetilde{\chi}_t + \mathcal{R}_t^i\right) dt, \\ -d\widetilde{M}_t^i = \left(2\lambda_t^i \widetilde{X}_t^i + \dfrac{1}{N}\mathbb{E}\left[\left\langle \widehat{B}_t^{i,(2)}, \widetilde{\mathcal{P}}_t^i \right\rangle\right] + \left\langle \Theta, -A_t^i \widetilde{\mathcal{S}}_t^i + K_t^i \widetilde{\chi}_t + \mathcal{R}_t^i \right\rangle\right) dt - Z_t^{\widetilde{M}^i}\, dW_t, \\ -d\widetilde{\mathcal{P}}_t^i = \left(-(A_t^i)^\top \widetilde{\mathcal{P}}_t^i + \Theta \dfrac{p\widetilde{M}_t^i - \frac{1}{N}\left\langle \widehat{B}_t^{i,(1)}, \widetilde{\mathcal{P}}_t^i \right\rangle}{2\eta_t^i} + \Theta f_t^i\right) dt - Z_t^{\widetilde{\mathcal{P}}^i}\, dW_t, \\ \widetilde{X}_0^i = x^i,\ \widetilde{X}_T^i = 0,\ \widetilde{\mathcal{S}}_0^i = (0,0)^\top,\ \widetilde{\mathcal{P}}_T^i = (0,0)^\top, \end{cases} \tag{3.11}$$

where for $j = 1, \cdots, N$,

$$\begin{cases} \widetilde{\xi}^j := \dfrac{p\widetilde{M}^j - \frac{1}{N}\left\langle \widehat{B}^{j,(1)}, \widetilde{\mathcal{P}}^j \right\rangle}{2\eta^j} + f^j \\ \widetilde{\chi} := \dfrac{1}{N}\displaystyle\sum_{j=1}^N \left(\widetilde{\xi}^j, \mathbb{E}[p\widetilde{X}^j + g^j], \mathbb{E}[\widetilde{\xi}^j]\right)^\top. \end{cases}$$

We now make the ansatz

$$\widetilde{M}^i = \mathscr{A}^i \widetilde{X}^i + \mathscr{B}^i.$$

Integration by parts suggests that

$$\begin{cases} -d\mathscr{A}_t^i = \left(2\lambda_t^i - \dfrac{(\mathscr{A}_t^i)^2}{2\eta_t^i}\right) dt - Z_t^{\mathscr{A}^i}\, dW_t, \\ \lim\limits_{t \nearrow T} \mathscr{A}_t^i = +\infty \end{cases} \tag{3.12}$$

and that $\mathscr{B}^i$ satisfies the BSDE

$$
\begin{aligned}
-d\mathscr{B}^i_t = &\left(-\frac{\mathscr{A}^i_t\mathscr{B}^i_t}{2\eta^i_t} + \frac{\mathscr{A}^i_t}{2N\eta^i_t}\left\langle \widehat{B}^{i,(1)}_t, \widetilde{\mathcal{P}}^i_t\right\rangle + \frac{1}{N}\mathbb{E}\left[\left\langle \widehat{B}^{i,(2)}_t, \widetilde{\mathcal{P}}^i_t\right\rangle\right] + \left\langle \Theta, -A^i_t\widetilde{\mathcal{S}}^i_t + K^i_t\widetilde{\chi}_t + \mathcal{R}^i_t\right\rangle\right) dt \\
&- Z^{\mathscr{B}^i}_t\, dW_t
\end{aligned} \tag{3.13}
$$

on $[0,T)$. It has been shown in Ankirchner et al. (2014) and Graewe et al. (2018) that (3.12) admits a unique solution $(\mathscr{A}^i, Z^{\mathscr{A}^i}) \in \mathcal{H}_{-1,\mathcal{F}} \times L^2_{\mathcal{F}}$ and that

$$
\exp\left(-\int_r^s \frac{\mathscr{A}^i_u}{2\eta^i_u}\,du\right) \le \left(\frac{T-s}{T-r}\right)^b, \qquad \text{where } b := \min_i \frac{\eta^i_{\min}}{\|\eta^i\|} \in (0,1]. \tag{3.14}
$$

The existence of a unique solution to (3.13) will be shown in Step 1 below.

We now proceed in two steps. In Step 1 we prove that (3.11) admits a unique solution when $p = 0$. In Step 2 we show that once (3.11) admits a unique solution for some $p \ge 0$ and for any $(f^j, g^j)_{j=1,\cdots,N}$, then the same holds if p is replaced by $p + \sigma$ for every $\sigma \le \sigma_0$ where σ_0 is a strictly positive constant that is independent of p. By iterating p we can then solve (3.11) for $p = 1$. It reduces to (3.8) by letting $f^j = g^j = 0$ for all $j = 1, \cdots, N$.

Step 1. In this step, we prove that the system (3.11) is uniquely solvable in $\mathcal{H}_{a,\mathcal{F}} \times \mathbb{S}^2_{\mathcal{F}} \times L^2_{\mathcal{F}} \times \mathcal{H}_{\iota,\mathcal{F}} \times L^{2,-}_{\mathcal{F}} \times L^2_{\mathcal{F}}$ for some positive constants $a < b$, $\iota < 1/2$ when $p = 0$.

To this end, we first consider the mean-field BSDE for $(\widetilde{\mathcal{P}}^i, Z^{\widetilde{\mathcal{P}}^i})$. This BSDE has a Lipschitz continuous driver and so it has a unique solution in the space $\mathbb{S}^2_{\mathcal{F}} \times L^2_{\mathcal{F}}$; see e.g. (Buckdahn et al., 2009, Theorem 3.1). Taking conditional expectations on both sides yields

$$
\widetilde{\mathcal{P}}^i_t = \mathbb{E}\left[\int_t^T -(A^i_s)^\top \widetilde{\mathcal{P}}^i_s - \Theta\frac{\left\langle \widehat{B}^{i,(1)}_s, \widetilde{\mathcal{P}}^i_s\right\rangle}{2N\eta^i_s} + \Theta f^i_s\,ds \,\middle|\, \mathcal{F}_t\right],
$$

which implies that

$$
\frac{|\widetilde{\mathcal{P}}^i_t|}{(T-t)^\iota} \le \left(\|A\| + \frac{\|\widehat{B}^{(1)}\|}{2N\eta_{\min}}\right)\frac{1}{(T-t)^\iota}\mathbb{E}\left[\int_t^T |\widetilde{\mathcal{P}}^i_s|\,ds \,\middle|\, \mathcal{F}_t\right] + \frac{1}{(T-t)^\iota}\mathbb{E}\left[\int_t^T |f^i_s|\,ds \,\middle|\, \mathcal{F}_t\right].
$$

Next, we take $\mathbb{E}[\sup_{0\le t\le T}(\cdot)^2]$ on both sides of the above inequality. By Hölder's inequality, Doob's maximal inequality and $\iota < 1/2$

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left(\frac{1}{(T-t)^\iota}\mathbb{E}\left[\int_t^T |f^i_s|\,ds \,\middle|\, \mathcal{F}_t\right]\right)^2\right] \\
\le\ &\mathbb{E}\left[\sup_{0\le t\le T}\left(\mathbb{E}\left[\int_0^T |f^i_s|^{\frac{1}{1-\iota}}\,ds \,\middle|\, \mathcal{F}_t\right]\right)^{2(1-\iota)}\right] \le \left(\frac{2-2\iota}{1-2\iota}\right)^{2(1-\iota)} T^{1-2\iota}\mathbb{E}\left[\int_0^T |f^i_s|^2\,ds\right].
\end{aligned}
$$

Similarly, we have that

$$
\mathbb{E}\left[\sup_{0\le t\le T}\left(\frac{1}{(T-t)^\iota}\mathbb{E}\left[\int_t^T |\widetilde{\mathcal{P}}^i_s|\,ds \,\middle|\, \mathcal{F}_t\right]\right)^2\right] \le \left(\frac{2-2\iota}{1-2\iota}\right)^{2(1-\iota)} T^{1-2\iota}\mathbb{E}\left[\int_0^T |\widetilde{\mathcal{P}}^i_s|^2\,ds\right].
$$

Therefore, we conclude that

$$
\mathbb{E}\left[\sup_{0\le t\le T}\left(\frac{|\widetilde{\mathcal{P}}^i_t|}{(T-t)^\iota}\right)^2\right] \le C\left(\|\widetilde{\mathcal{P}}^i\|^2_{\mathbb{S}^2} + \|f^i\|^2_{L^2}\right),
$$

which implies that $\widetilde{\mathcal{P}}^i \in \mathcal{H}_{\iota,\mathcal{F}}$. Next, we consider the process $\widetilde{\mathcal{S}}^i$. Since it solves a linear ODE we get that

$$\mathbb{E}\left[\sup_{0\le t\le T}|\widetilde{\mathcal{S}}^i_t|^2\right] \le C\left(\|\mathcal{R}^i\|^2_{L^2} + \sum_{i=1}^N \|f^i\|^2_{L^2} + \sum_{i=1}^N \|g^i\|^2_a\right).$$

As a result, $\widetilde{\mathcal{S}}^i \in \mathbb{S}^2_{\mathcal{F}}$. Next, we set, for $t \in [0,T)$

$$\begin{aligned}\mathscr{B}^i_t := &\mathbb{E}\left[\int_t^T e^{-\int_t^s \frac{\mathscr{A}^i_r}{2\eta^i_r}\,dr}\left(\frac{\mathscr{A}^i_s}{2N\eta^i_s}\left\langle \widehat{B}^{i,(1)}_s, \widetilde{\mathcal{P}}^i_s\right\rangle + \frac{1}{N}\mathbb{E}\left[\left\langle \widehat{B}^{i,(2)}_s, \widetilde{\mathcal{P}}^i_s\right\rangle\right]\right.\right.\\ &\left.\left.+\left\langle \Theta, -A^i_s\widetilde{\mathcal{S}}^i_s + K^i_s\widetilde{\chi}_t + \mathcal{R}^i_s\right\rangle\right) ds\middle|\mathcal{F}_t\right].\end{aligned}$$

The estimate (3.14) along with Doob's maximal inequality yields a constant $C>0$ s.t. for any $\epsilon > 0$,

$$\mathbb{E}\left[\sup_{0\le t\le T-\epsilon}\left|\mathscr{B}^i_t\right|^2\right] \le C\left(\frac{1}{N}\sum_{j=1}^N \|\widetilde{\mathcal{P}}^j\|^2_\iota + \|\widetilde{\mathcal{S}}^i\|^2_{\mathbb{S}^2} + \|\mathcal{R}^i\|^2_{L^2} + \sum_{i=1}^N \|f^i\|^2_{L^2} + \sum_{i=1}^N \|g^i\|^2_a\right). \tag{3.15}$$

Thus, $\mathscr{B}^i$ belongs to $\mathbb{S}^{2,-}_{\mathcal{F}}$ and so the martingale representation theorem yields a unique process $Z^{\mathscr{B}^i} \in L^{2,-}_{\mathcal{F}}$ such that the pair $(\mathscr{B}^i, Z^{\mathscr{B}^i})$ satisfies the BSDE (3.13).

We now analyze the process $\widetilde{X}^i$. Taking the ansatz $\widetilde{M}^i = \mathscr{A}^i\widetilde{X}^i + \mathscr{B}^i$ into the SDE of $\widetilde{X}^i$ yields

$$\widetilde{X}^i_t = \mathcal{X}^i e^{-\int_0^t \frac{\mathscr{A}^i_r}{2\eta^i_r}\,dr} - \int_0^t e^{-\int_s^t \frac{\mathscr{A}^i_r}{2\eta^i_r}\,dr}\frac{\mathscr{B}^i_s - \frac{1}{N}\left\langle \widehat{B}^{i,(1)}_s, \widetilde{\mathcal{P}}^i_s\right\rangle}{2\eta^i_s}\,ds.$$

Since $a < b \le 1$, it follows from (3.14) that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{\widetilde{X}^i_t}{(T-t)^a}\right|^2\right] &\le C\left(\|\mathcal{X}^i\|_{L^2} + \mathbb{E}\left[\int_0^T\left|\frac{\mathscr{B}^i_s}{(T-s)^a}\right|^2 ds\right] + \mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{\widetilde{\mathcal{P}}^i_t}{(T-t)^\iota}\right|^2\right]\right)\\ &= C\left(\|\mathcal{X}^i\|_{L^2} + \lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^{T-\epsilon}\left|\frac{\mathscr{B}^i_s}{(T-s)^a}\right|^2 ds\right] + \|\widetilde{\mathcal{P}}^i\|^2_\iota\right)\\ &\le C\left(\|\mathcal{X}^i\|_{L^2} + \lim_{\epsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T-\epsilon}\left|\mathscr{B}^i_t\right|^2\right] + \|\widetilde{\mathcal{P}}^i\|^2_\iota\right).\end{aligned}$$

In view of the estimate (3.15) this shows that $\widetilde{X}^i \in \mathcal{H}_{a,\mathcal{F}}$.

It remains to analyze the process $\widetilde{M}^i$. Using the equality $\widetilde{M}^i = \mathscr{A}^i\widetilde{X}^i + \mathscr{B}^i$ and (3.15) again, we see that for each $0 \le \tau < T$

$$\mathbb{E}\left[\sup_{0\le t\le \tau}\left|\widetilde{M}^i_t\right|^2\right] \le \frac{C}{(T-\tau)^{2(1-a)}}\|\widetilde{X}^i\|^2_a + \mathbb{E}\left[\sup_{0\le t\le\tau}\left|\mathscr{B}^i_t\right|^2\right]. \tag{3.16}$$

Moreover, for any $\epsilon > 0$, integration by parts implies that

$$\begin{aligned}&\widetilde{X}^i_{T-\epsilon}\widetilde{M}^i_{T-\epsilon} - \widetilde{X}^i_0\widetilde{M}^i_0\\ =&\int_0^{T-\epsilon}\widetilde{X}^i_t d\widetilde{M}^i_t + \int_0^{T-\epsilon}\widetilde{M}^i_t d\widetilde{X}^i_t\\ =&-\int_0^{T-\epsilon}\widetilde{X}^i_t\left(2\lambda^i_t X^i_t + \frac{1}{N}\mathbb{E}\left[\left\langle \widehat{B}^{i,(2)}_t, \widetilde{\mathcal{P}}^i_t\right\rangle\right] + \left\langle \Theta, -A^i_t\widetilde{\mathcal{S}}^i_t + K^i_t\widetilde{\chi}_t + \mathcal{R}^i_t\right\rangle\right) dt\\ &-\int_0^{T-\epsilon}\widetilde{M}^i_t\frac{\widetilde{M}^i_t - \frac{1}{N}\left\langle \widehat{B}^{i,(1)}_t, \widetilde{\mathcal{P}}^i_t\right\rangle}{2\eta^i_t}\,dt + \text{martingale part}.\end{aligned} \tag{3.17}$$

Since

$$\widetilde{X}^i_{T-\epsilon}\widetilde{M}^i_{T-\epsilon} = \mathscr{A}^i_{T-\epsilon}(\widetilde{X}^i_{T-\epsilon})^2 + \widetilde{X}^i_{T-\epsilon}\mathscr{B}^i_{T-\epsilon} \geq \widetilde{X}^i_{T-\epsilon}\mathscr{B}^i_{T-\epsilon},$$

by taking expectations on both sides and using (3.16) we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\epsilon} 2\lambda^i_t(\widetilde{X}^i_t)^2 + \frac{(\widetilde{M}^i_t)^2}{2\eta^i_t}\,dt\right] \\
&\leq \epsilon'\mathbb{E}\left[\int_0^{T-\epsilon}(\widetilde{M}^i_t)^2\,dt\right] + C(\epsilon')\left(\|\widetilde{X}^i\|^2_a + \|\mathscr{B}^i\|^2_{\mathbb{S}^{2,-}} + \|\widetilde{\mathcal{S}}^i\|^2_{\mathbb{S}^2} + \|\widetilde{\mathcal{P}}^i\|^2_\iota + \|\mathcal{R}^i\|^2_{L^2} + \sum_{i=1}^N \|f^i\|^2_{L^2} + \sum_{i=1}^N \|g^i\|^2_a\right).
\end{aligned}$$

Letting $\epsilon' < \frac{1}{2\|\eta\|}$ and then taking $\epsilon \to 0$, we conclude that $\widetilde{M}^i \in L^2_{\mathcal{F}}$. The martingale representation theorem yields a unique $Z^{\widetilde{M}^i} \in L^{2,-}_{\mathcal{F}}$.

Step 2. We now prove that if (3.11) with parameter p admits a solution in $\mathcal{H}_{a,\mathcal{F}} \times \mathbb{S}^2_{\mathcal{F}} \times L^2_{\mathcal{F}} \times \mathcal{H}_{\iota,\mathcal{F}} \times L^{2,-}_{\mathcal{F}} \times L^2_{\mathcal{F}}$ for any $f^i \in L^2_{\mathcal{F}}, g^i \in \mathcal{H}_{a,\mathcal{F}}$, then there exists a strictly positive constant σ_0 that is independent of p and f^i, g^i such that the same result holds for $p+\sigma$ whenever $\sigma \in [0,\sigma_0]$.

For any $(X^i, M^i) \in \mathcal{H}_{a,\mathcal{F}} \times L^2_{\mathcal{F}}$, it holds that

$$f^i(M) := \sigma\frac{M^i}{2\eta^i} + f^i \in L^2_{\mathcal{F}}, \quad g^i(X) := \sigma X^i + g^i \in \mathcal{H}_{a,\mathcal{F}}.$$

Hence by assumption there exists a unique solution $(\widetilde{X}^i, \widetilde{\mathcal{S}}^i, \widetilde{M}^i, \widetilde{\mathcal{P}}^i, Z^{\widetilde{M}^i}, Z^{\widetilde{\mathcal{P}}^i})$ in $\mathcal{H}_{a,\mathcal{F}} \times \mathbb{S}^2_{\mathcal{F}} \times L^2_{\mathcal{F}} \times \mathcal{H}_{\iota,\mathcal{F}} \times L^{2,-}_{\mathcal{F}} \times L^2_{\mathcal{F}}$ to the FBSDE system (3.11) with $f^i = f^i(M)$ and $g^i = g^i(X)$. It is now sufficient to show that the mapping

$$\Phi : \left((X^j)_{j=1,\cdots,N}, (M^j)_{j=1,\cdots,N}\right) \mapsto \left((\widetilde{X}^j)_{j=1,\cdots,N}, (\widetilde{M}^j)_{j=1,\cdots,N}\right).$$

is a contraction under Assumption 3.2. To this end, we denote for any two stochastic processes H and H' their difference by $\delta H := H - H'$ and use again the representation $\widetilde{M}^i = \mathscr{A}^i\widetilde{X}^i + \mathscr{B}^i$.

Integration by parts implies for any $\epsilon > 0$ that (3.17) holds with $\widetilde{X}^i$ replaced by $\delta\widetilde{X}^i$ and without non-homogenous term. Using the fact that $\delta\widetilde{X}^i_{T-\epsilon}\delta\widetilde{M}^i_{T-\epsilon} \geq \delta\widetilde{X}^i_{T-\epsilon}\delta\mathscr{B}^i_{T-\epsilon}$, we have that

$$\begin{aligned}
&\int_0^{T-\epsilon} 2\lambda^i_t(\delta\widetilde{X}^i_t)^2 + \frac{(\delta\widetilde{M}^i_t)^2}{2\eta^i_t}\,dt \\
&\leq -\delta\widetilde{X}^i_{T-\epsilon}\delta\mathscr{B}^i_{T-\epsilon} + \int_0^{T-\epsilon}\delta\widetilde{M}^i_t\frac{\left\langle\widehat{B}^{i,(1)}_t, \delta\widetilde{\mathcal{P}}^i_t\right\rangle}{2N\eta^i_t}\,dt + \text{martingale part} \\
&\quad - \int_0^{T-\epsilon}\delta\widetilde{X}^i_t\left(\frac{1}{N}\mathbb{E}\left[\left\langle\widehat{B}^{i,(2)}_t, \delta\widetilde{\mathcal{P}}^i_t\right\rangle\right] + \left\langle\Theta, -A^i_t\delta\widetilde{\mathcal{S}}^i_t + K^i_t\delta\chi_t\right\rangle\right)dt.
\end{aligned}$$

Taking expectations on both sides and then letting $\epsilon \to 0$, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T 2\lambda^i_t(\delta\widetilde{X}^i_t)^2 + \frac{(\delta\widetilde{M}^i_t)^2}{2\eta^i_t}\,dt\right] \\
&\leq -\mathbb{E}\left[\int_0^T\delta\widetilde{X}^i_t\left(\frac{1}{N}\mathbb{E}\left[\left\langle\widehat{B}^{i,(2)}_t, \delta\widetilde{\mathcal{P}}^i_t\right\rangle\right] + \left\langle\Theta, -A^i_t\delta\widetilde{\mathcal{S}}^i_t + K^i_t\delta\chi_t\right\rangle\right)dt\right] + \mathbb{E}\left[\int_0^T\delta\widetilde{M}^i_t\frac{\left\langle\widehat{B}^{i,(1)}_t, \delta\widetilde{\mathcal{P}}^i_t\right\rangle}{2N\eta^i_t}\,dt\right].
\end{aligned}$$

Young's inequality and the inequality $|\langle x,y\rangle| \le |x||y|$ for any two vectors x,y imply that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T 2\lambda_t^i(\delta\widetilde{X}_t^i)^2 + \frac{(\delta\widetilde{M}_t^i)^2}{2\eta_t^i}\,dt\right] \\
\le&\frac{\theta_0}{2}\mathbb{E}\left[\int_0^T(\delta\widetilde{X}_t^i)^2\,dt\right] + \frac{\|\widehat{B}^{(2)}\|^2}{2N^2\theta_0}\mathbb{E}\left[\int_0^T|\delta\widetilde{\mathcal{P}}_t^i|^2\,dt\right] \\
&+\frac{\theta_1}{2}\mathbb{E}\left[\int_0^T(\delta\widetilde{X}_t^i)^2\,dt\right] + \frac{\|A\|^2}{2\theta_1}\mathbb{E}\left[\int_0^T|\delta\widetilde{\mathcal{S}}_t^i|^2\,dt\right] \\
&+\frac{\theta_2}{2}\mathbb{E}\left[\int_0^T(\delta\widetilde{X}_t^i)^2\,dt\right] + \frac{1}{2\theta_2}\mathbb{E}\left[\int_0^T\left|K_t^i\delta\widetilde{\chi}_t\right|^2\,dt\right] \\
&+\frac{\theta}{2}\mathbb{E}\left[\int_0^T\left(\frac{\delta\widetilde{M}_t^i}{2\eta_t^i}\right)^2\,dt\right] + \frac{\|\widehat{B}^{(1)}\|^2}{2N^2\theta}\mathbb{E}\left[\int_0^T|\delta\widetilde{\mathcal{P}}_t^i|^2\,dt\right].
\end{aligned}
\tag{3.18}
$$

Applying Itô's formula for $|\delta\widetilde{P}_t^i|^2$, we have that

$$
\begin{aligned}
-|\delta\widetilde{\mathcal{P}}_t^i|^2 =& -2\int_t^T(\delta\widetilde{\mathcal{P}}_s^i)^\top\left(-(A_t^i)^\top\delta\widetilde{\mathcal{P}}_t^i + \Theta\frac{p\delta\widetilde{M}_t^i + \sigma\delta M_t^i - \frac{1}{N}\left\langle\widehat{B}_t^{i,(1)},\delta\widetilde{\mathcal{P}}_t^i\right\rangle}{2\eta_t^i}\right)\,ds \\
&+\int_t^T|\delta Z_s^{\widetilde{\mathcal{P}}^i}|^2\,ds + 2\int_t^T(\delta\widetilde{\mathcal{P}}_s^i)^\top\delta Z_s^{\widetilde{\mathcal{P}}^i}\,dW_s.
\end{aligned}
$$

Recalling the condition (3.9) and using Young's inequality $\langle x,y\rangle \le \frac{\widetilde{\rho}}{2}|x|^2 + \frac{1}{2\widetilde{\rho}}|y|^2$, we obtain that

$$
\mathbb{E}\left[\int_0^T|\delta\widetilde{\mathcal{P}}_t^i|^2\,dt\right] \le \frac{1}{\widetilde{\rho}^2}\mathbb{E}\left[\int_0^T\left(\frac{p\delta\widetilde{M}_t^i + \sigma\delta M_t^i}{2\eta_t^i}\right)^2\,dt\right].
\tag{3.19}
$$

Using similar arguments on $|\delta\widetilde{\mathcal{S}}_t^i|^2$, we get that

$$
|\delta\widetilde{\mathcal{S}}_t^i|^2 = 2\int_0^t(\delta\widetilde{\mathcal{S}}_s^i)^\top\left(-A_t^i\delta\widetilde{\mathcal{S}}_t^i + K_t^i\delta\widetilde{\chi}_t\right)\,ds,
$$

and

$$
\mathbb{E}\left[\int_0^T|\delta\widetilde{\mathcal{S}}_t^i|^2\,dt\right] \le \frac{1}{\rho^2}\mathbb{E}\left[\int_0^T\left|K_t^i\delta\widetilde{\chi}_t\right|^2\,dt\right].
\tag{3.20}
$$

Recalling the definition of $\widetilde{\chi}$ and $\widetilde{\xi}^j$, Remark 3.1, and using Young's inequality again, we have that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T\left|K_t^i\delta\widetilde{\chi}_t\right|^2\,dt\right] \\
\le&(1+\theta_3)(\|K^{(1)}\|+\|K^{(3)}\|)^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T(\delta\widetilde{\xi}_t^j)^2\,dt\right] + \left(1+\frac{1}{\theta_3}\right)\|K^{(2)}\|^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T(p\delta\widetilde{X}_t^j+\sigma\delta X_t^j)^2\,dt\right] \\
\le&(1+\theta_3)(\|K^{(1)}\|+\|K^{(3)}\|)^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T(1+\varepsilon)\left(\frac{p\delta\widetilde{M}_t^j+\sigma\delta M_t^j}{2\eta_t^j}\right)^2 + \left(1+\frac{1}{\varepsilon}\right)\left(\frac{\frac{1}{N}\left\langle\widehat{B}_t^{j,(1)},\delta\widetilde{\mathcal{P}}_t^j\right\rangle}{2\eta_t^j}\right)^2\,dt\right] \\
&+\left(1+\frac{1}{\theta_3}\right)\|K^{(2)}\|^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T\left(p\delta\widetilde{X}_t^j+\sigma\delta X_t^j\right)^2\,dt\right].
\end{aligned}
$$

Letting $\varepsilon := \frac{\|\widehat{B}^{(1)}\|}{2N\eta_{\min}\widetilde{\rho}}$, from the above estimate and (3.19) we have that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|K_t^i\delta\widetilde{\chi}_t\right|^2 dt\right] \\
\le&(1+\theta_3)(\|K^{(1)}\|+\|K^{(3)}\|)^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T\left(1+\varepsilon+\left(1+\frac{1}{\varepsilon}\right)\frac{\|\widehat{B}^{(1)}\|^2}{4N^2\eta_{\min}^2\widetilde{\rho}^2}\right)\left(\frac{p\delta\widetilde{M}_t^j+\sigma\delta M_t^j}{2\eta_t^j}\right)^2 dt\right] \\
&+\left(1+\frac{1}{\theta_3}\right)\|K^{(2)}\|^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T\left(p\delta\widetilde{X}_t^j+\sigma\delta X_t^j\right)^2 dt\right] \\
=&(1+\theta_3)(\|K^{(1)}\|+\|K^{(3)}\|)^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T\left(1+\frac{\|\widehat{B}^{(1)}\|}{2N\eta_{\min}\widetilde{\rho}}\right)^2\left(\frac{p\delta\widetilde{M}_t^j+\sigma\delta M_t^j}{2\eta_t^j}\right)^2 dt\right] \\
&+\left(1+\frac{1}{\theta_3}\right)\|K^{(2)}\|^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T\left(p\delta\widetilde{X}_t^j+\sigma\delta X_t^j\right)^2 dt\right].
\end{aligned} \tag{3.21}$$

Recalling the inequality (3.18), collecting the estimates (3.19)-(3.21) and taking sum from 1 to N on both sides we get

$$\begin{aligned}
&\left(2\lambda_{\min}-\frac{\theta_0+\theta_1+\theta_2}{2}\right)\sum_{i=1}^N\mathbb{E}\left[\int_0^T(\delta\widetilde{X}_t^i)^2\,dt\right]+\left(2\eta_{\min}-\frac{\theta}{2}\right)\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\frac{\delta\widetilde{M}_t^i}{2\eta_t^i}\right)^2 dt\right] \\
\le&\left[\frac{1}{N^2\widetilde{\rho}^2}\left(\frac{\|\widehat{B}^{(2)}\|^2}{2\theta_0}+\frac{\|\widehat{B}^{(1)}\|^2}{2\theta}\right)\right. \\
&\left.+(1+\theta_3)(\|K^{(1)}\|+\|K^{(3)}\|)^2\left(\frac{\|A\|^2}{2\theta_1\widetilde{\rho}^2}+\frac{1}{2\theta_2}\right)\left(1+\frac{\|\widehat{B}^{(1)}\|}{2N\eta_{\min}\widetilde{\rho}}\right)^2\right]\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\frac{p\delta\widetilde{M}_t^i+\sigma\delta M_t^i}{2\eta_t^i}\right)^2 dt\right] \\
&+\left(1+\frac{1}{\theta_3}\right)\|K^{(2)}\|^2\left(\frac{\|A\|^2}{2\theta_1\widetilde{\rho}^2}+\frac{1}{2\theta_2}\right)\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(p\delta\widetilde{X}_t^i+\sigma\delta X_t^i\right)^2 dt\right] \\
\le&(1+\varepsilon)\left[\frac{1}{N^2\widetilde{\rho}^2}\left(\frac{\|\widehat{B}^{(2)}\|^2}{2\theta_0}+\frac{\|\widehat{B}^{(1)}\|^2}{2\theta}\right)\right. \\
&\left.+(1+\theta_3)(\|K^{(1)}\|+\|K^{(3)}\|)^2\left(\frac{\|A\|^2}{2\theta_1\widetilde{\rho}^2}+\frac{1}{2\theta_2}\right)\left(1+\frac{\|\widehat{B}^{(1)}\|}{2N\eta_{\min}\widetilde{\rho}}\right)^2\right]\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\frac{\delta\widetilde{M}_t^i}{2\eta_t^i}\right)^2 dt\right] \\
&+(1+\varepsilon)\left(1+\frac{1}{\theta_3}\right)\|K^{(2)}\|^2\left(\frac{\|A\|^2}{2\theta_1\widetilde{\rho}^2}+\frac{1}{2\theta_2}\right)\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\delta\widetilde{X}_t^i\right)^2 dt\right] \\
&+C\left(1+\frac{1}{\varepsilon}\right)\sigma\left(\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\delta M_t^i\right)^2 dt\right]+\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\delta X_t^i\right)^2 dt\right]\right).
\end{aligned}$$

Thus, choosing $\theta=\frac{\|\widehat{B}^{(1)}\|}{N\widetilde{\rho}}$ and choosing ε small enough, the assumption (3.10) yields

$$\begin{aligned}
&\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\delta\widetilde{M}_t^i\right)^2 dt\right]+\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\delta\widetilde{X}^i\right)^2 dt\right] \\
&\le C\sigma\left(\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\delta M_t^i\right)^2 dt\right]+\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\delta X_t^i\right)^2 dt\right]\right).
\end{aligned}$$

Furthermore, going back to the dynamics of $\widetilde{X}^i$ and using $\widetilde{M}^i = \mathscr{A}^i\widetilde{X}^i + \mathscr{B}^i$, we have that

$$\sum_{i=1}^N \mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{\delta\widetilde{X}^i}{(T-t)^a}\right|^2\right] \le C\sigma\left(\sum_{i=1}^N \mathbb{E}\left[\int_0^T \left(\delta M_t^i\right)^2\,dt\right] + \sum_{i=1}^N \mathbb{E}\left[\int_0^T \left(\delta X_t^i\right)^2\,dt\right]\right).$$

Hence, when σ is small enough, the mapping Φ is a contraction. Iterating p finitely many times until $p=1$ and letting $f^i = g^i = 0$, we obtain the desired result. □

3.3 Verification

Having established the existence of a unique solution to the respective FBSDEs, the candidate optimal strategies are well defined. In this section we provide a verification result that shows that the candidate strategy (3.1) does indeed define a Nash equilibrium of the N-player game (2.9)-(2.10). Our analysis is based on a novel sufficient stochastic maximum principle that does not require convexity of the cost function as it is usually the case; see e.g. (Pham, 2009, Theorem 6.4.6). Instead, our argument strongly relies on the liquidation constraint $X_T^i = 0$. The following is the main result of this section.

Theorem 3.5. *Let* $(X^i, \mathcal{S}^i, M^i, \mathcal{P}^i, Z^{M^i}, Z^{\mathcal{P}^i}) \in \mathcal{H}_{a,\mathcal{F}} \times S^2_{\mathcal{F}} \times L^2_{\mathcal{F}} \times \mathcal{H}_{\iota,\mathcal{F}} \times L^{2,-}_{\mathcal{F}} \times L^2_{\mathcal{F}}$ $(i=1,\cdots,N)$ *be the unique solution to the FBSDE system* (3.4). *Under Assumption 3.2 with the inequalities in (iii) replaced by the following slightly different condition*

$$\begin{cases} \lambda_{\min} - \dfrac{\theta_1+\theta_2}{2} - \dfrac{\|B^{(2)}\|^2}{N^2}\left(1+\dfrac{1}{\theta_3}\right)\left(\dfrac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \dfrac{1}{2\theta_2}\right) > 0, \\ \eta_{\min} - (1+\theta_3)\dfrac{\|B^{(1)}\|^2}{N^2}\left(\dfrac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \dfrac{1}{2\theta_2}\right) > 0, \end{cases} \tag{3.22}$$

the processes $\xi^* = (\xi^{*,1},\cdots,\xi^{*,N})$ *forms an open-loop Nash equilibrium of the* N*-player game* (2.9)-(2.10)*, where*

$$\xi^{*,i} = \frac{M^i - \frac{1}{N}\left\langle B^{i,(1)}, \mathcal{P}^i\right\rangle}{2\eta^i}.$$

Remark 3.6. (1) If Assumption 3.2 (iii) holds, then the condition (3.22) holds for all $N \ge 2$.

(2) The impact process Y is exogenous in the optimization problem of the MFG. Thus, the convexity requirement for the standard sufficient maximum principle holds. We omit the proof of the verification result, which is standard.

In what follows we denote by $(X^i, \underline{\mathcal{S}}^i)$ the states corresponding to the strategy profile $(\xi^i, (\xi^{*,j})_{j\ne i})$ and by $(X^{*,i}, \underline{\mathcal{S}}^{*,i})$ the states corresponding to the strategy profile $(\xi^{*,i}, (\xi^{*,j})_{j\ne i})$. Moreover, we put

$$\underline{\chi} := \frac{1}{N}(\xi^i, \mathbb{E}[X^i])^\top + \frac{1}{N}\sum_{j\ne i}(\xi^{*,j}, \mathbb{E}[X^{*,j}])^\top \quad \text{and} \quad \underline{\chi}^* := \frac{1}{N}\sum_{j=1}^N(\xi^{*,j}, \mathbb{E}[X^{*,j}])^\top.$$

Then it holds that

$$\begin{cases} dX_t^i = -\,\xi_t^i\,dt \\ d\underline{\mathcal{S}}_t^i = \left(-A_t^i\underline{\mathcal{S}}_t^i + B_t^i\underline{\chi}_t + \underline{\mathcal{R}}_t^i\right)\,dt \end{cases} \quad \text{and} \quad \begin{cases} dX_t^{*,i} = -\,\xi_t^{*,i}\,dt \\ d\underline{\mathcal{S}}_t^{*,i} = \left(-A_t^i\underline{\mathcal{S}}_t^{*,i} + B_t^i\underline{\chi}_t^* + \underline{\mathcal{R}}_t^i\right)\,dt. \end{cases}$$

The admissibility of the candidate ξ^* has already been established; in particular, $X_T^{*,i} = 0$ for each $i = 1,\dots,N$ because $X^{*,i} \in \mathcal{H}_{a,\mathcal{F}}$. It remains to prove that

$$J^i(\xi^{*,i}, \xi^{*,-i}) \le J^i(\xi^i, \xi^{*,-i})$$

for each $1 \leq i \leq N$ and any admissible control ξ^i. To this end, we prove that the cost $J^i(\xi^i, \xi^{*,-i})$ can be decomposed into the equilibrium cost plus the cost of a round-trip strategy as

$$\begin{aligned} J^i(\xi^i, \xi^{*,-i}) = J^i(\xi^{*,i}, \xi^{*,-i}) + \mathbb{E}\Bigg[\int_0^T \eta_t^i \left(\xi_t^i - \xi_t^{*,i}\right)^2 + \lambda_t^i \left(X_t^i - X_t^{*,i}\right)^2 \\ + \left(X_t^i - X_t^{*,i}\right) \left\langle \Theta, -A_t^i(\underline{\mathcal{S}}_t^i - \underline{\mathcal{S}}_t^{*,i}) + B_t^i(\underline{\chi}_t - \underline{\chi}_t^*)\right\rangle dt\Bigg] \end{aligned} \tag{3.23}$$

and that the additional cost is non-negative under Assumption (3.22). In order to prove the decomposition (3.23) we proceed in various steps. In a first step, we establish an alternative representation of the cost function.

Lemma 3.7. *The cost associated with the strategy* $(\xi^i, \xi^{*,-i})$ *can be rewritten as*

$$J^i(\xi^i, \xi^{*,-i}) = \mathbb{E}\left[\int_0^T X_t^i \left\langle \Theta, -A_t^i \underline{\mathcal{S}}_t^i + B_t^i \underline{\chi}_t + \underline{\mathcal{R}}_t^i\right\rangle + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2 \, dt\right].$$

Proof. Using integration by parts and $X_T^i = 0$, $\underline{\mathcal{S}}_0^i = 0$, we have that

$$0 = \mathbb{E}\left[X_T^i \left\langle \Theta, \underline{\mathcal{S}}_T^i\right\rangle - X_0^i \left\langle \Theta, \underline{\mathcal{S}}_0^i\right\rangle\right] = \mathbb{E}\left[\int_0^T X_t^i \left\langle \Theta, -A_t^i \underline{\mathcal{S}}_t^i + B_t^i \underline{\chi}_t + \underline{\mathcal{R}}_t^i\right\rangle - \xi_t^i \left\langle \Theta, \underline{\mathcal{S}}_t^i\right\rangle dt\right]. \tag{3.24}$$

As a result,

$$\begin{aligned} J^i(\xi^i, \xi^{*,-i}) &= \mathbb{E}\left[\int_0^T \xi_t^i \left\langle \Theta, \underline{\mathcal{S}}_t^i\right\rangle + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2 \, dt\right] \\ &= \mathbb{E}\left[\int_0^T X_t^i \left\langle \Theta, -A_t^i \underline{\mathcal{S}}_t^i + B_t^i \underline{\chi}_t + \underline{\mathcal{R}}_t^i\right\rangle + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2 \, dt\right]. \end{aligned}$$

□

In view of Lemma 3.7, it holds

$$\begin{aligned} & J^i(\xi^i, \xi^{*,-i}) - J^i(\xi^{*,i}, \xi^{*,-i}) \\ &= \mathbb{E}\left[\int_0^T X_t^i \left\langle \Theta, -A_t^i \underline{\mathcal{S}}_t^i + B_t^i \underline{\chi}_t + \underline{\mathcal{R}}_t^i\right\rangle + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2 \, dt\right] \\ &\quad - \mathbb{E}\left[\int_0^T X_t^{*,i} \left\langle \Theta, -A_t^i \underline{\mathcal{S}}_t^{*,i} + B_t^i \underline{\chi}_t^* + \underline{\mathcal{R}}_t^i\right\rangle + \eta_t^i (\xi_t^{*,i})^2 + \lambda_t^i (X_t^{*,i})^2 \, dt\right] \\ &=: \mathbb{I}. \end{aligned} \tag{3.25}$$

It remains to bring the term on the right-hand side in equation (3.25) into the form (3.23). For this, let

$$\begin{aligned} \mathbb{III} :=& \mathbb{E}\left[\int_0^T \left(2\eta_t^i \xi_t^{*,i} + \frac{1}{N}\left\langle B_t^{i,(1)}, \mathcal{P}_t^i\right\rangle\right)\left(\xi_t^i - \xi_t^{*,i}\right) dt\right] \\ &+ \mathbb{E}\left[\int_0^T \left(X_t^i - X_t^{*,i}\right)\left(2\lambda_t^i X_t^{*,i} + \frac{1}{N}\mathbb{E}\left[\left\langle B_t^{i,(2)}, \mathcal{P}_t^i\right\rangle\right] + \left\langle \Theta, -A_t^i \underline{\mathcal{S}}_t^{*,i} + B_t^i \underline{\chi}_t^* + \underline{\mathcal{R}}_t^i\right\rangle\right) dt\right]. \end{aligned} \tag{3.26}$$

Heuristically, this term equals $\mathbb{E}\left[\int_0^T \left((X_t^i - X_t^{*,i})dM_t^i - M_t^i(dX_t^i - dX_t^{*,i})\right)\right]$. In view of the liquidation constraint, using an integration by parts argument, we expect that $\mathbb{III} = 0$ in which case it remains to bring the difference $\mathbb{I} - \mathbb{III}$ into the form (3.23).

Lemma 3.8. *The representation* (3.23) *holds true.*

Proof. We proceed in two steps. In a first step, we prove that $\mathbb{III} = 0$. Indeed, integration by parts on $[0, T-\epsilon]$ yields that

$$
\begin{aligned}
&\mathbb{E}\left[M^i_{T-\epsilon}\left(X^i_{T-\epsilon} - X^{*,i}_{T-\epsilon}\right) - M^i_0\left(X^i_0 - X^{*,i}_0\right)\right] \\
=& -\mathbb{E}\left[\int_0^{T-\epsilon} M^i_t\left(\xi^i_t - \xi^{*,i}_t\right) dt\right] \\
&-\mathbb{E}\left[\int_0^{T-\epsilon}\left(X^i_t - X^{*,i}_t\right)\left(2\lambda^i_t X^{*,i}_t + \frac{1}{N}\mathbb{E}\left[\left\langle B^{i,(2)}_t, \mathcal{P}^i_t\right\rangle\right] + \left\langle \Theta, -A^i_t \underline{\mathcal{S}}^{*,i}_t + B^i_t \underline{\chi}^*_t + \underline{\mathcal{R}}^i_t\right\rangle\right) dt\right].
\end{aligned}
$$

Letting $\epsilon \to 0$, a similar argument as in the proof of (Fu et al., 2021, Proposition 2.14) yields that

$$
\lim_{\epsilon\to 0}\mathbb{E}\left[M^i_{T-\epsilon}(X^i_{T-\epsilon} - X^{*,i}_{T-\epsilon})\right] = 0.
$$

Thus, dominated convergence implies

$$
\begin{aligned}
&-\mathbb{E}\left[\int_0^T M^i_t\left(\xi^i_t - \xi^{*,i}_t\right) dt\right] \\
=&\mathbb{E}\left[\int_0^T\left(X^i_t - X^{*,i}_t\right)\left(2\lambda^i_t X^{*,i}_t + \frac{1}{N}\mathbb{E}\left[\left\langle B^{i,(2)}_t, \mathcal{P}^i_t\right\rangle\right] + \left\langle \Theta, -A^i_t \underline{\mathcal{S}}^{*,i}_t + B^i_t \underline{\chi}^*_t + \underline{\mathcal{R}}^i_t\right\rangle\right) dt\right].
\end{aligned}
$$

Putting the preceding equation into (3.26) implies that

$$
\mathbb{III} = \mathbb{E}\left[\int_0^T\left(2\eta^i_t\xi^{*,i}_t + \frac{1}{N}\left\langle B^{i,(1)}_t, \mathcal{P}^i_t\right\rangle - M^i_t\right)\left(\xi^i_t - \xi^{*,i}_t\right) dt\right] = 0.
$$

Using integration by parts again yields that

$$
\begin{aligned}
0 =&\mathbb{E}\left[\left\langle \mathcal{P}^i_T, \underline{\mathcal{S}}^i_T - \underline{\mathcal{S}}^{*,i}_T\right\rangle - \left\langle \mathcal{P}^i_0, \underline{\mathcal{S}}^i_0 - \underline{\mathcal{S}}^{*,i}_0\right\rangle\right] \\
=&\mathbb{E}\left[\int_0^T \left\langle (A^i_t)^\top \mathcal{P}^i_t - \Theta\xi^{*,i}_t, \underline{\mathcal{S}}^i_t - \underline{\mathcal{S}}^{*,i}_t\right\rangle + \left\langle \mathcal{P}^i_t, -A^i_t(\underline{\mathcal{S}}^i_t - \underline{\mathcal{S}}^{*,i}_t) + B^i_t(\underline{\chi}_t - \underline{\chi}^*_t)\right\rangle dt\right] \\
=&\mathbb{E}\left[\int_0^T -\xi^{*,i}_t\left\langle \Theta, \underline{\mathcal{S}}^i_t - \underline{\mathcal{S}}^{*,i}_t\right\rangle + \left\langle \mathcal{P}^i_t, B^i_t(\underline{\chi}_t - \underline{\chi}^*_t)\right\rangle dt\right] \\
=&\mathbb{E}\left[\int_0^T -X^{*,i}_t\left\langle \Theta, -A^i_t(\underline{\mathcal{S}}^i_t - \underline{\mathcal{S}}^{*,i}_t) + B^i_t(\underline{\chi}_t - \underline{\chi}^*_t)\right\rangle dt\right] + \mathbb{E}\left[X^{*,i}_T\left\langle \Theta, \underline{\mathcal{S}}^i_T - \underline{\mathcal{S}}^{*,i}_T\right\rangle - X^{*,i}_0\left\langle \Theta, \underline{\mathcal{S}}^i_0 - \underline{\mathcal{S}}^{*,i}_0\right\rangle\right] \\
&+\mathbb{E}\left[\int_0^T \left\langle \mathcal{P}^i_t, B^i_t(\underline{\chi}_t - \underline{\chi}^*_t)\right\rangle dt\right] \\
=&\mathbb{E}\left[\int_0^T -X^{*,i}_t\left\langle \Theta, -A^i_t(\underline{\mathcal{S}}^i_t - \underline{\mathcal{S}}^{*,i}_t) + B^i_t(\underline{\chi}_t - \underline{\chi}^*_t)\right\rangle dt\right] + \mathbb{E}\left[\int_0^T \left\langle \mathcal{P}^i_t, B^i_t(\underline{\chi}_t - \underline{\chi}^*_t)\right\rangle dt\right],
\end{aligned}
$$

where in the fourth equality we use the liquidation constraint $X^{*,i}_T = 0$. Using that $\mathbb{E}[\mathbb{E}[x]y] = \mathbb{E}[x]\mathbb{E}[y] = \mathbb{E}[x\mathbb{E}[y]]$ for any random variables x and y, the second term in the above sum can be rewritten as

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \left\langle \mathcal{P}^i_t, B^i_t(\underline{\chi}_t - \underline{\chi}^*_t)\right\rangle dt\right] \\
=&\frac{1}{N}\mathbb{E}\left[\int_0^T \left\langle \mathcal{P}^i_t, B^{i,(1)}_t\right\rangle\left(\xi^i_t - \xi^{*,i}_t\right) dt\right] + \frac{1}{N}\mathbb{E}\left[\int_0^T \left\langle \mathcal{P}^i_t, B^{i,(2)}_t\right\rangle \mathbb{E}\left[X^i_t - X^{*,i}_t\right] dt\right] \\
=&\frac{1}{N}\mathbb{E}\left[\int_0^T \left\langle \mathcal{P}^i_t, B^{i,(1)}_t\right\rangle\left(\xi^i_t - \xi^{*,i}_t\right) dt\right] + \frac{1}{N}\mathbb{E}\left[\int_0^T \mathbb{E}\left[\left\langle \mathcal{P}^i_t, B^{i,(2)}_t\right\rangle\right]\left(X^i_t - X^{*,i}_t\right) dt\right].
\end{aligned}
$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T X_t^{*,i}\left\langle \Theta, -A_t^i(\underline{\mathcal{S}}_t^i - \underline{\mathcal{S}}_t^{*,i}) + B_t^i(\underline{\chi}_t - \underline{\chi}_t^*)\right\rangle dt\right] \\
=&\frac{1}{N}\mathbb{E}\left[\int_0^T \left\langle \mathcal{P}_t^i, B_t^{i,(1)}\right\rangle \left(\xi_t^i - \xi_t^{*,i}\right) dt\right] + \frac{1}{N}\mathbb{E}\left[\int_0^T \mathbb{E}\left[\left\langle \mathcal{P}_t^i, B_t^{i,(2)}\right\rangle\right]\left(X_t^i - X_t^{*,i}\right) dt\right].
\end{aligned} \tag{3.27}$$

Note that

$$\begin{aligned}
\mathbb{I} - \mathbb{III} =&\mathbb{E}\Bigg[\int_0^T \eta_t^i \left(\xi_t^i - \xi_t^{*,i}\right)^2 + \lambda_t^i\left(X_t^i - X_t^{*,i}\right)^2 + X_t^i\left\langle \Theta, -A_t^i(\underline{\mathcal{S}}_t^i - \underline{\mathcal{S}}_t^{*,i}) + B_t^i(\underline{\chi}_t - \underline{\chi}_t^*)\right\rangle \\
&- \frac{1}{N}\left\langle B_t^{i,(1)}, \mathcal{P}_t^i\right\rangle\left(\xi_t^i - \xi_t^{*,i}\right) - \frac{1}{N}\mathbb{E}\left[\left\langle \mathcal{P}_t^i, B_t^{i,(2)}\right\rangle\right]\left(X_t^i - X_t^{*,i}\right) dt\Bigg].
\end{aligned} \tag{3.28}$$

Plugging (3.27) into (3.28), we get the desired representation. □

We are now ready to finish the proof of the verification result.

Proof of Theorem 3.5. Using the constants appearing in (3.18), we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left(X_t^i - X_t^{*,i}\right)\left\langle \Theta, -A_t^i(\underline{\mathcal{S}}_t^i - \underline{\mathcal{S}}_t^{*,i}) + B_t^i(\underline{\chi}_t - \underline{\chi}_t^*)\right\rangle dt\right] \\
\le& \frac{\theta_1+\theta_2}{2}\mathbb{E}\left[\int_0^T |X_t^i - X_t^{*,i}|^2\right] + \frac{\|A\|^2}{2\theta_1}\mathbb{E}\left[\int_0^T |\underline{\mathcal{S}}_t^i - \underline{\mathcal{S}}_t^{*,i}|^2\, dt\right] + \frac{1}{2\theta_2}\mathbb{E}\left[\int_0^T |B_t^i(\underline{\chi}_t - \underline{\chi}_t^*)|^2\, dt\right].
\end{aligned}$$

The dynamics $\underline{\mathcal{S}}_t^i - \underline{\mathcal{S}}_t^{*,i} = \int_0^t \left(-A_s^i(\underline{\mathcal{S}}_s^i - \underline{\mathcal{S}}_s^{*,i}) + B_s^i(\underline{\chi}_s - \underline{\chi}_s^*)\right) ds$ and the estimate leading to (3.20) imply

$$\mathbb{E}\left[\int_0^T |\underline{\mathcal{S}}_t^i - \underline{\mathcal{S}}_t^{*,i}|^2\, dt\right] \le \frac{1}{\widehat{\rho}^2}\mathbb{E}\left[\int_0^T |B_t^i(\underline{\chi}_t - \underline{\chi}_t^*)|^2\, dt\right].$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left(X_t^i - X_t^{*,i}\right)\left\langle \Theta, -A_t^i(\underline{\mathcal{S}}_t^i - \underline{\mathcal{S}}_t^{*,i}) + B_t^i(\underline{\chi}_t - \underline{\chi}_t^*)\right\rangle dt\right] \\
\le& \frac{\theta_1+\theta_2}{2}\mathbb{E}\left[\int_0^T |X_t^i - X_t^{*,i}|^2\right] + \left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \frac{1}{2\theta_2}\right)\mathbb{E}\left[\int_0^T |B_t^i(\underline{\chi}_t - \underline{\chi}_t^*)|^2\, dt\right] \\
\le& \left(\frac{\theta_1+\theta_2}{2} + \frac{\|B^{(2)}\|^2}{N^2}\left(1 + \frac{1}{\theta_3}\right)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \frac{1}{2\theta_2}\right)\right)\mathbb{E}\left[\int_0^T |X_t^i - X_t^{*,i}|^2\, dt\right] \\
&+ (1+\theta_3)\frac{\|B^{(1)}\|^2}{N^2}\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \frac{1}{2\theta_2}\right)\mathbb{E}\left[\int_0^T |\xi_t^i - \xi_t^{*,i}|^2\, dt\right].
\end{aligned}$$

Due to the decomposition (3.23) and Assumption (3.22), we have

$$\begin{aligned}
&J(\xi, \xi^{*,-i}) - J(\xi^{*,i}, \xi^{*,-i}) \\
\ge& \left(\lambda_{\min} - \frac{\theta_1+\theta_2}{2} - \frac{\|B^{(2)}\|^2}{N^2}\left(1 + \frac{1}{\theta_3}\right)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \frac{1}{2\theta_2}\right)\right)\mathbb{E}\left[\int_0^T \left(X_t^i - X_t^{*,i}\right)^2 dt\right] \\
&+ \left(\eta_{\min} - (1+\theta_3)\frac{\|B^{(1)}\|^2}{N^2}\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \frac{1}{2\theta_2}\right)\right)\mathbb{E}\left[\int_0^T \left(\xi_t^i - \xi_t^{*,i}\right)^2 dt\right] \\
\ge& 0.
\end{aligned}$$

□

Our verification Theorem 3.5 excludes the existence of beneficial round trips for the single player model.

Corollary 3.9. *Under assumptions of Theorem 3.5, there is no beneficial round trip when $N = 1$.*

Proof. Let the initial position $x = 0$ and $\xi^* \equiv 0$, which implies

$$X^* = \underline{\mathcal{S}}^* = \underline{\chi}^* = \underline{\mathcal{R}} = J(\xi^*) \equiv 0. \tag{3.29}$$

Moreover, let ξ be a round trip, that is, $\int_0^T \xi_t \, dt = 0$. In view of Lemma 3.7 and (3.29), the decomposition (3.23) holds with $N = 1$ and the proof of Theorem 3.5 yields $J(\xi) > 0 = J(\xi^*)$. Thus, ξ is not beneficial. □

Remark 3.10. Note that the exclusion of beneficial round trips cannot be obtained from the FBSDE representation of the optimal strategy; this is because our optimization is non-convex and thus uniqueness result cannot be guaranteed. For the two-player game, the next corollary shows that our FBSDE characterization provides the existence of a beneficial round trip.

Corollary 3.11. *If $\mathcal{X}^1 > 0$ and $\mathcal{X}^2 = 0$, then $X^{*,2} \neq 0$. That is, $X^{*,2}$ or $\xi^{*,2}$ is a beneficial round trip.*

Proof. We prove the corollary by contradiction. If $X^{*,2} \equiv 0$, which is equivalent to $\xi^{*,2} \equiv 0$, then $J^2(\xi^{*,2}, \xi^{*,1}) = 0$. In view of the representation (3.23) and Lemma 3.7,

$$\mathbb{E}\left[\int_0^T X_t^2 \langle \Theta, -d\underline{\mathcal{S}}_t^{*,2} \rangle \right] = 0, \quad \text{for any } X^2.$$

This implies that $\mathbb{E}\left[\int_0^T X_t^2 \, dY_t^{*,2}\right] = 0$ for all X^2. Thus, $Y^{*,2} \equiv 0$. From the dynamics of $Y^{*,2}$ we get that

$$\frac{\xi^{*,1}}{2} - (\beta^2 - \alpha^2) C^{*,2} + \frac{\alpha^2}{2} \mathbb{E}\left[\mathcal{X}^1 - X^{*,1}\right] = 0. \tag{3.30}$$

Since $C^{*,2}$ can be expressed in terms of the function $\mathbb{E}\left[\mathcal{X}^1 - X^{*,1}\right]$ and the constants α^2 and β^2 we conclude that (3.30) is an equation for $\mathcal{X}^1 - X^{*,1}$ that only depends on the constants α^2 and β^2. This contradicts the FBSDE characterization (3.4) that states that $X^{*,1}$ and hence $\mathcal{X}^1 - X^{*,1}$ only depends on the coefficients indexed by 1 if $\mathcal{X}^2 = X^{*,2} = \xi^{*,2} = 0$. □

3.4 Approximation by penalization

It has been shown in various settings that the optimal trading strategies in models in which open positions are increasingly penalized converge to optimal trading strategies in models where full liquidation is required; see, e.g. Evangelista and Thamsten (2020); Fu et al. (2021); Horst and Xia (2019) for details. If the strict liquidation constraint is replaced by a penalization $n(X_T^i)^2$ of open positions at the terminal time, the FBSDE system (3.8) changes to

$$\begin{cases} dX_t^i = -\dfrac{M_t^i - \frac{1}{N}\langle \widehat{B}_t^{i,(1)}, \mathcal{P}_t^i \rangle}{2\eta_t^i} \, dt, \\ d\mathcal{S}_t^i = \left(-A_t^i \mathcal{S}_t^i + K_t^i \chi_t + \mathcal{R}_t^i\right) dt, \\ -dM_t^i = \left(2\lambda_t^i X_t^i + \dfrac{1}{N}\mathbb{E}\left[\left\langle \widehat{B}_t^{i,(2)}, \mathcal{P}_t^i \right\rangle\right] + \left\langle \Theta, -A_t^i \mathcal{S}_t^i + K_t^i \chi_t + \mathcal{R}_t^i \right\rangle\right) dt - Z_t^{M^i} \, dW_t, \\ -d\mathcal{P}_t^i = \left(-(A_t^i)^\top \mathcal{P}_t^i + \Theta \dfrac{M_t^i - \frac{1}{N}\langle \widehat{B}_t^{i,(1)}, \mathcal{P}_t^i \rangle}{2\eta_t^i}\right) dt - Z_t^{\mathcal{P}^i} \, dW_t, \\ X_0^i = \mathcal{X}^i, \ \mathcal{S}_0^i = (0,0)^\top, \ M_T^i = 2nX_T^i - \mathcal{S}_T^{i,(1)}, \ \mathcal{P}_T^i = (0,0)^\top, \end{cases} \tag{3.31}$$

where $\mathcal{S}^{i,(1)}$ is the first component of $\mathcal{S}^i$. The same arguments as in the proof of (Fu et al., 2021, Lemma 4.5) show that

$$\mathbb{E}\left[\int_0^T |\mathcal{P}_t^{i,n} - \mathcal{P}_t^i|^2\,dt\right] + \mathbb{E}\left[\int_0^T |M_t^{i,n} - M_t^i|^2\,dt\right] + \mathbb{E}\left[\int_0^T |\mathcal{S}_t^{i,n} - \mathcal{S}_t^i|^2\,dt\right] \to 0.$$

From this, we immediately obtain that the model with liquidation constraint can be approximated by a sequence of models with increasing penalization. Specifically, using the same arguments as in the proof of (Fu et al., 2021, Theorem 4) it is not difficult to prove the following approximation result.

Proposition 3.12. *Let* $(X^i, \mathcal{S}^i, M^i, \mathcal{P}^i, Z^{M^i}, Z^{\mathcal{P}^i})$ *and* $(X^{i,n}, \mathcal{S}^{i,n}, M^{i,n}, \mathcal{P}^{i,n}, Z^{M^{i,n}}, Z^{\mathcal{P}^{i,n}})$ *be the solutions of* (3.8) *and* (3.31)*, respectively. Then,*

$$\mathbb{E}\left[\sup_{0\le t\le T} |X_t^{i,n} - X_t^i|^2\right] + \mathbb{E}\left[\sup_{0\le t\le T} |\mathcal{S}_t^{i,n} - \mathcal{S}_t^i|^2\right] \to 0 \quad \text{as } n \to \infty.$$

4 From many player games to mean-field games

In this section we prove the convergence of the Nash equilibria in the N-player game to the Nash equilibrium of the corresponding MFG under the homogeneity conditions (2.11) and (2.12). This is achieved by establishing the convergence of the solutions to the FBSDE system (3.4) to the solution to the corresponding mean-field FBSDE (3.7) as $N \to \infty$. More precisely, let

$$\left(\overline{X}^i, \overline{\mathcal{S}}^i, \overline{M}^i, \overline{\mathcal{P}}^i, Z^{\overline{M}^i}, Z^{\overline{\mathcal{P}}^i}\right) \in \mathcal{H}_{a,\mathcal{F}} \times \mathbb{S}^2_{\mathcal{F}} \times L^2_{\mathcal{F}} \times \mathcal{H}_{\iota,\mathcal{F}} \times L^{2,-}_{\mathcal{F}} \times L^2_{\mathcal{F}}$$

be the unique solution to the mean-field FBSDE (3.7) with $\overline{W} = W^i$, $\mathcal{X} = \mathcal{X}^i$, $\lambda = \lambda^i$, $\eta = \eta^i$, $\rho = \rho^i$, $\alpha = \alpha^i$, $\beta = \beta^i$ and $\gamma = \gamma^i$. Using the Yamada-Watanabe result for mean-field FBSDE established in (Fu et al., 2021, Lemma 3.2), there exists a measurable function Σ independent of i such that

$$(\overline{X}_t^i, \overline{\mathcal{S}}_t^i, \overline{M}_t^i, \overline{\mathcal{P}}_t^i) = \Sigma(t, \mathcal{X}^i, W^i_{\cdot\wedge t}). \tag{4.1}$$

In particular, the mean field equilibrium state and control satisfy

$$\nu_t = \mathbb{E}[\overline{X}_t^i] \qquad \text{and} \qquad \mu_t = \mathbb{E}\left[\frac{\overline{M}_t^i}{2\eta_t^i}\right].$$

Lemma 4.1. *It holds that*

$$\mathbb{E}\left[\int_0^T \left(\frac{1}{N}\sum_{j=1}^N \frac{\overline{M}_t^j}{2\eta_t^j} - \mu_t\right)^2 dt\right] \xrightarrow{N\to\infty} 0, \tag{4.2}$$

and

$$\mathbb{E}\left[\sup_{0\le t\le T} \left(\frac{1}{N}\sum_{j=1}^N \overline{X}_t^j - \nu_t\right)^2 dt\right] \xrightarrow{N\to\infty} 0. \tag{4.3}$$

Proof. By (2.11), (2.12) and (4.1), $\frac{\overline{M}_t^k}{2\eta_t^k}$ and $\frac{\overline{M}_t^j}{2\eta_t^j}$ are independent and identically distributed for $k \neq j$. By Theorem 3.4, there exists a constant C independent of i such that

$$\mathbb{E}\left[\int_0^T \left(\frac{\overline{M}_t^i}{2\eta_t^i}\right)^2 dt\right] \le C. \tag{4.4}$$

Moreover, it follows that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \left(\frac{1}{N}\sum_{j=1}^N \frac{\overline{M}_t^j}{2\eta_t^j} - \mu_t\right)^2 dt\right] \\
=&\frac{1}{N^2}\mathbb{E}\left[\int_0^T \sum_{k\neq j}\left(\frac{\overline{M}_t^k}{2\eta_t^k} - \mu_t\right)\left(\frac{\overline{M}_t^j}{2\eta_t^j} - \mu_t\right) dt\right] + \frac{1}{N^2}\mathbb{E}\left[\int_0^T \sum_{k=1}^N \left(\frac{\overline{M}_t^k}{2\eta_t^k} - \mu_t\right)^2 dt\right] \\
\leq&\frac{4C}{N} \xrightarrow{N\to\infty} 0.
\end{aligned}
$$

By considering the dynamics of $\overline{X}^i$, the convergence (4.3) follows. $\square$

Let $(X^i, \underline{\mathcal{S}}^i, M^i, \mathcal{P}^i, Z^{M^i}, Z^{\mathcal{P}^i})$ be the unique solution of (3.4) and

$$
\left(\delta X^i, \delta\mathcal{S}^i, \delta M^i, \delta\mathcal{P}^i, \delta Z^{M^i}, \delta Z^{\mathcal{P}^i}\right) := \left(X^i - \overline{X}^i, \underline{\mathcal{S}}^i - \overline{\mathcal{S}}^i, M^i - \overline{M}^i, \mathcal{P}^i - \overline{\mathcal{P}}^i, Z^{M^i} - Z^{\overline{M}^i}e_i, Z^{\mathcal{P}^i} - Z^{\overline{\mathcal{P}}^i}e_i\right),
$$

where e_i denotes the ith unit vector in $\mathbb{R}^N$. The FBSDE

$$
\begin{cases}
d\delta X_t^i = -\dfrac{\delta M_t^i - \frac{1}{N}\left\langle B_t^{i,(1)}, \mathcal{P}_t^i\right\rangle}{2\eta_t^i}\, dt, \\
d\delta\mathcal{S}_t^i = \left(-A_t^i\delta\mathcal{S}_t^i + B_t^i\delta\chi_t + \delta\mathcal{R}_t^i\right) dt, \\
-d\delta M_t^i = \left(2\lambda_t^i\delta X_t^i + \dfrac{1}{N}\mathbb{E}\left[\left\langle B_t^{i,(2)}, \mathcal{P}_t^i\right\rangle\right] + \left\langle\Theta, -A_t^i\delta\mathcal{S}_t^i + B_t^i\delta\chi_t + \delta\mathcal{R}_t^i\right\rangle\right) dt - \delta Z_t^{M^i}\, dW_t, \\
-d\delta\mathcal{P}_t^i = \left(-(A_t^i)^\top\delta\mathcal{P}_t^i + \Theta\dfrac{\delta M_t^i - \frac{1}{N}\left\langle B_t^{i,(1)}, \mathcal{P}_t^i\right\rangle}{2\eta_t^i}\right) dt - \delta Z_t^{\mathcal{P}^i}\, dW_t, \\
\delta X_0^i = 0,\ \delta X_T^i = 0,\ \delta\mathcal{S}_0^i = (0,0)^\top,\ \delta\mathcal{P}_T^i = (0,0)^\top,
\end{cases}
\tag{4.5}
$$

where

$$
\delta\chi = \left(\frac{1}{N}\sum_{j=1}^N \frac{M^j - \frac{1}{N}\left\langle B^{j,(1)}, \mathcal{P}^j\right\rangle}{2\eta^j} - \mathbb{E}\left[\frac{\overline{M}^i}{2\eta^i}\right], \frac{1}{N}\sum_{j=1}^N \mathbb{E}[X^j] - \mathbb{E}[\overline{X}^i]\right)^\top
$$

and

$$
\delta\mathcal{R}^i = \left(\frac{\alpha^i\gamma^i}{N}\sum_{j=1}^N \mathbb{E}[\mathcal{X}^j] - \alpha^i\gamma^i\mathbb{E}[\mathcal{X}^i], \frac{\alpha^i}{N}\sum_{j=1}^N \mathbb{E}[\mathcal{X}^j] - \alpha^i\mathbb{E}[\mathcal{X}^i]\right)^\top \equiv (0,0)^\top
$$

has a unique solution. This allows us to establish the convergence of the Nash equilibria of the N-player game to the mean field solution as $N\to\infty$.

Theorem 4.2. *Let* (3.9) *and* (3.10) *hold for all N large enough. The following convergence holds*

$$
\mathbb{E}\left[\int_0^T |\delta M_t^i|^2\, dt\right] + \mathbb{E}\left[\sup_{0\leq t\leq T} |\delta X_t^i|^2\, dt\right] \xrightarrow{N\to\infty} 0.
$$

As a result, the optimal strategy of player i in the N-player game converges to the one in MFG, i.e.,

$$
\mathbb{E}\left[\int_0^T |\xi_t^{*,i,N} - \overline{\xi}_t^{*,i}|^2\, dt\right] \to 0,
$$

where $\xi^{*,i,N} := \frac{M^i - \frac{1}{N}\left\langle B^{i,(1)}, \mathcal{P}^i\right\rangle}{2\eta^i}$ *and* $\overline{\xi}^{*,i} := \frac{\overline{M}^i}{2\eta^i}$.

Proof. Using $M_t^j = \delta M_t^j + \overline{M}_t^j$ and $X_t^j = \delta X_t^j + \overline{X}_t^j$ we have that

$$\begin{aligned}
\delta\chi = &\left(\frac{1}{N}\sum_{j=1}^{N}\frac{\delta M^j}{2\eta^j} + \frac{1}{N}\sum_{j=1}^{N}\frac{\overline{M}^j}{2\eta^j} - \mathbb{E}\left[\frac{\overline{M}^i}{2\eta^j}\right],\ \frac{1}{N}\sum_{j=1}^{N}\mathbb{E}[\delta X^j] + \frac{1}{N}\sum_{j=1}^{N}\mathbb{E}[\overline{X}^j] - \mathbb{E}\left[\overline{X}^i\right]\right)^\top \\
&+ \left(-\frac{1}{N^2}\sum_{j=1}^{N}\frac{\langle B^{j,(1)}, \mathcal{P}^j\rangle}{2\eta^j},\ 0\right)^\top \\
= &\left(\frac{1}{N}\sum_{j=1}^{N}\frac{\delta M^j}{2\eta^j} + \frac{1}{N}\sum_{j=1}^{N}\frac{\overline{M}^j}{2\eta^j} - \mathbb{E}\left[\frac{\overline{M}^i}{2\eta^j}\right],\ \frac{1}{N}\sum_{j=1}^{N}\mathbb{E}[\delta X^j]\right)^\top \\
&+ \left(-\frac{1}{N^2}\sum_{j=1}^{N}\frac{\langle B^{j,(1)}, \mathcal{P}^j\rangle}{2\eta^j},\ 0\right)^\top.
\end{aligned} \tag{4.6}$$

In view of (3.19) and (3.20), we have that

$$\mathbb{E}\left[\int_0^T |\mathcal{P}_t^i|^2\,dt\right] \le \frac{1}{\widehat{\rho}^2}\mathbb{E}\left[\int_0^T \left(\frac{M_t^i}{2\eta_t^i}\right)^2 dt\right] = \frac{1}{\widehat{\rho}^2}\mathbb{E}\left[\int_0^T \left(\frac{\delta M_t^i + \overline{M}_t^i}{2\eta_t^i}\right)^2 dt\right]. \tag{4.7}$$

and that

$$\mathbb{E}\left[\int_0^T |\delta\mathcal{S}_t^i|^2\,dt\right] \le \frac{1}{\widehat{\rho}^2}\mathbb{E}\left[\int_0^T |B_t^i\delta\chi_t|^2\right]. \tag{4.8}$$

Taking (4.6) into (4.5), following the proof of Theorem 3.4 and using (4.7) and (4.8), we obtain

$$\begin{aligned}
&\left(2\lambda_{\min} - \frac{\theta_0+\theta_1+\theta_2}{2}\right)\mathbb{E}\left[\int_0^T (\delta X_t^i)^2\,dt\right] + \left(2\eta_{\min} - \frac{\theta}{2}\right)\mathbb{E}\left[\int_0^T \left(\frac{\delta M_t^i}{2\eta_t^i}\right)^2 dt\right] \\
\le &\left(\frac{\|B^{(2)}\|^2}{\theta_0} + \frac{\|B^{(1)}\|^2}{\theta}\right)\frac{1}{2N^2}\mathbb{E}\left[\int_0^T |\mathcal{P}_t^i|^2\,dt\right] + \left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \frac{1}{2\theta_2}\right)\mathbb{E}\left[\int_0^T |B^i\delta\chi_t|^2\,dt\right] \\
\le &\left(\frac{\|B^{(2)}\|^2}{\theta_0} + \frac{\|B^{(1)}\|^2}{\theta}\right)\frac{1}{2N^2}\mathbb{E}\left[\int_0^T |\mathcal{P}_t^i|^2\,dt\right] \\
&+ (1+\theta_3)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \frac{1}{2\theta_2}\right)\|B^{(1)}\|^2\mathbb{E}\left[\int_0^T \left(\frac{1}{N}\sum_{j=1}^{N}\frac{\delta M^j}{2\eta^j}\right.\right. \\
&\left.\left.+\frac{1}{N}\sum_{j=1}^{N}\frac{\overline{M}^j}{2\eta^j} - \mathbb{E}\left[\frac{\overline{M}^i}{2\eta^j}\right] - \frac{1}{N^2}\sum_{j=1}^{N}\frac{\langle B^{j,(1)}, \mathcal{P}^j\rangle}{2\eta^j}\right)^2 dt\right] \\
&+ \left(1+\frac{1}{\theta_3}\right)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \frac{1}{2\theta_2}\right)\|B^{(2)}\|^2\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}\left[\int_0^T |\delta X_t^j|^2\,dt\right] \\
\le &\left(\frac{\|B^{(2)}\|^2}{\theta_0} + \frac{\|B^{(1)}\|^2}{\theta}\right)\frac{1}{2N^2}\mathbb{E}\left[\int_0^T |\mathcal{P}_t^i|^2\,dt\right] \\
&+ (1+\epsilon)^2(1+\theta_3)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \frac{1}{2\theta_2}\right)\|B^{(1)}\|^2\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}\left[\int_0^T \left(\frac{\delta M^j}{2\eta^j}\right)^2 dt\right] \\
&+ (1+\epsilon)\left(1+\frac{1}{\epsilon}\right)(1+\theta_3)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2} + \frac{1}{2\theta_2}\right)\|B^{(1)}\|^2\mathbb{E}\left[\int_0^T \left(\frac{1}{N}\sum_{j=1}^{N}\frac{\overline{M}^j}{2\eta^j} - \mathbb{E}\left[\frac{\overline{M}^i}{2\eta^j}\right]\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+\left(1+\frac{1}{\epsilon}\right)(1+\theta_3)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2}+\frac{1}{2\theta_2}\right)\frac{\|B^{(1)}\|^4}{4\eta_{\min}^2N^2}\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T|\mathcal{P}_t^j|^2\,dt\right]\\
&+\left(1+\frac{1}{\theta_3}\right)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2}+\frac{1}{2\theta_2}\right)\|B^{(2)}\|^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T|\delta X_t^j|^2\,dt\right]\\
\leq{}&(1+\epsilon)\left(\frac{\|B^{(2)}\|^2}{\theta_0}+\frac{\|B^{(1)}\|^2}{\theta}\right)\frac{1}{2N^2\widehat{\rho}^2}\mathbb{E}\left[\int_0^T\left(\frac{\delta M_t^i}{2\eta_t^i}\right)^2dt\right]\\
&+\left(1+\frac{1}{\epsilon}\right)\left(\frac{\|B^{(2)}\|^2}{\theta_0}+\frac{\|B^{(1)}\|^2}{\theta}\right)\frac{1}{2N^2\widehat{\rho}^2}\mathbb{E}\left[\int_0^T\left(\frac{\overline{M}_t^i}{2\eta_t^i}\right)^2dt\right]\\
&+(1+\epsilon)^2(1+\theta_3)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2}+\frac{1}{2\theta_2}\right)\|B^{(1)}\|^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T\left(\frac{\delta M^j}{2\eta^j}\right)^2dt\right]\\
&+(1+\epsilon)\left(1+\frac{1}{\epsilon}\right)(1+\theta_3)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2}+\frac{1}{2\theta_2}\right)\|B^{(1)}\|^2\mathbb{E}\left[\int_0^T\left(\frac{1}{N}\sum_{j=1}^N\frac{\overline{M}^j}{2\eta^j}-\mathbb{E}\left[\frac{\overline{M}^i}{2\eta^j}\right]\right)^2\right]\\
&+(1+\epsilon)\left(1+\frac{1}{\epsilon}\right)(1+\theta_3)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2}+\frac{1}{2\theta_2}\right)\frac{\|B^{(1)}\|^4}{4\eta_{\min}^2N^2\widehat{\rho}^2}\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T\left(\frac{\delta M_t^j}{2\eta_t^j}\right)^2dt\right]\\
&+\left(1+\frac{1}{\epsilon}\right)^2(1+\theta_3)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2}+\frac{1}{2\theta_2}\right)\frac{\|B^{(1)}\|^4}{4\eta_{\min}^2N^2\widehat{\rho}^2}\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T\left(\frac{\overline{M}_t^j}{2\eta_t^j}\right)^2dt\right]\\
&+\left(1+\frac{1}{\theta_3}\right)\left(\frac{\|A\|^2}{2\theta_1\widehat{\rho}^2}+\frac{1}{2\theta_2}\right)\|B^{(2)}\|^2\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T|\delta X_t^j|^2\,dt\right].
\end{aligned}
$$

Letting $\theta=\frac{\|B^{(1)}\|}{N\widehat{\rho}}$, ϵ be small enough, N be large enough, taking average and upper limit on both sides, we obtain by (3.10)

$$
\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\frac{\delta M_t^i}{2\eta_t^i}\right)^2dt\right]+\limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\delta X_t^i\right)^2dt\right]\\
\leq{}&C\limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\int_0^T\left(\frac{1}{N}\sum_{j=1}^N\frac{\overline{M}_t^j}{2\eta_t^j}-\mathbb{E}\left[\frac{\overline{M}_t^i}{2\eta_t^i}\right]\right)^2dt\right]\\
&+\limsup_{N\to\infty}O\left(\frac{1}{N}\right)\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\int_0^T\left(\frac{\overline{M}_t^j}{2\eta_t^j}\right)^2dt\right]\\
={}&0.
\end{aligned}
$$

Going back to the inequality for $\mathbb{E}\left[\int_0^T\left(\frac{\delta M_t^i}{2\eta_t^i}\right)^2dt\right]$ and $\mathbb{E}\left[\int_0^T\left(\delta X_t^i\right)^2dt\right]$, we have

$$
\mathbb{E}\left[\int_0^T\left(\frac{\delta M_t^i}{2\eta_t^i}\right)^2dt\right]+\mathbb{E}\left[\int_0^T\left(\delta X_t^i\right)^2dt\right]\xrightarrow{N\to\infty}0.
$$

Furthermore,

$$
\mathbb{E}\left[\sup_{0\leq t\leq T}|\delta X_t^i|^2\,dt\right]\leq C\mathbb{E}\left[\int_0^T\left(\delta M_t^i\right)^2dt\right]+\frac{C}{N}\mathbb{E}\left[\int_0^T\left(\delta M_s^i+\overline{M}_s^i\right)^2ds\right]\xrightarrow{N\to\infty}0.
$$

□

5 Deterministic benchmark models

In this section we consider three deterministic benchmark examples. In Section 5.1 we consider a MFG where all model parameters except the initial portfolios are deterministic. In Section 5.2 and Section 5.3 we consider a single player model and a two player model, respectively, where all model parameters including initial portfolios are deterministic. In all deterministic models, our FBSDE systems reduce to ODE systems. Existence and uniqueness of solutions to these systems can easily be established; in particular, Assumption 3.2 is not required. For simplicity we also replace the strict liquidation constraint by a penalization $n(X^i_T)^2$ of open positions at the terminal time. This considerably simplifies our numerical analysis; see Section 3.4. We display the solutions to the corresponding ODE system for various choices of model parameters.

5.1 The mean-field game

If all model parameters except the initial positions are deterministic constants, then the stochastic integral terms drop out of the FBSDE system (3.6). Taking expectations on both sides in (3.6) and putting

$$\mathbb{F} := (\mathbb{E}[X], \mathbb{E}[Y], \mathbb{E}[C])^\top \quad \text{and} \quad \mathbb{B} := (\mathbb{E}[P], \mathbb{E}[Q], \mathbb{E}[R])^\top,$$

we obtain that

$$\begin{cases} \mathbb{F}' = \varphi_{00}\mathbb{F} + \varphi_{01}\mathbb{B} + F^0, \\ \mathbb{B}' = \varphi_{10}\mathbb{F} + \varphi_{11}\mathbb{B}, \\ \mathbb{F}_0 = \begin{pmatrix} \mathbb{E}[\mathcal{X}] \\ 0 \\ 0 \end{pmatrix}, \ \mathbb{B}_T = \begin{pmatrix} 2n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbb{F}_T, \end{cases} \tag{5.1}$$

where

$$\varphi_{00} = \begin{pmatrix} 0 & \frac{1}{2\eta} & 0 \\ -\alpha\gamma & -\rho - \frac{\gamma}{2\eta} & -\gamma(\beta-\alpha) \\ -\alpha & 0 & -(\beta-\alpha) \end{pmatrix}, \quad \varphi_{01} = \begin{pmatrix} -\frac{1}{2\eta} & 0 & 0 \\ \frac{\gamma}{2\eta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^0 = \begin{pmatrix} 0 \\ \gamma\alpha\mathbb{E}[\mathcal{X}] \\ \alpha\mathbb{E}[\mathcal{X}] \end{pmatrix},$$

and

$$\varphi_{10} = \begin{pmatrix} -2\lambda & 0 & 0 \\ 0 & \frac{1}{2\eta} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varphi_{11} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2\eta} & \rho & 0 \\ 0 & \gamma(\beta-\alpha) & (\beta-\alpha) \end{pmatrix}.$$

Making the ansatz $\mathbb{B} = D\mathbb{F} + D^0$ yields the following ODE system for D and D^0:

$$\begin{cases} D' = -D\varphi_{01}D - D\varphi_{00} + \varphi_{11}D + \varphi_{10}, & D_T = \begin{pmatrix} 2n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ (D^0)' = (\varphi_{11} - D\varphi_{10})D^0 - DF^0, & D^0_T = (0,0,0)^\top. \end{cases} \tag{5.2}$$

Let $\Phi(T,t) = e^{\mathscr{P}(T-t)}$ be the fundamental solution to (5.1), where

$$\mathscr{P} = \begin{pmatrix} \varphi_{00} & \varphi_{01} \\ \varphi_{10} & \varphi_{11} \end{pmatrix}.$$

From (5.1) one has

$$
\begin{aligned}
0 &= (D_T, -I_{3\times3}) \begin{pmatrix} \mathbb{F}_T \\ \mathbb{B}_T \end{pmatrix} \\
&= (D_T, -I_{3\times3})\Phi(T,t) \begin{pmatrix} \mathbb{F}_t \\ \mathbb{B}_t \end{pmatrix} + (D_T, -I_{3\times3}) \int_t^T \Phi(T,s)\, ds \begin{pmatrix} F^0 \\ O_{3\times1} \end{pmatrix} \\
&= (D_T, -I_{3\times3})\Phi(T,t) \begin{pmatrix} I_{3\times3} \\ O_{3\times3} \end{pmatrix} \mathbb{F}_t + (D_T, -I_{3\times3})\Phi(T,t) \begin{pmatrix} O_{3\times3} \\ I_{3\times3} \end{pmatrix} \mathbb{B}_t \\
&\quad + (D_T, -I_{3\times3}) \int_t^T \Phi(T,s)\, ds \begin{pmatrix} F^0 \\ O_{3\times1} \end{pmatrix},
\end{aligned}
$$

where $I_{3\times3}, O_{3\times3}$ and $O_{3\times1}$ are the 3×3 identity matrix and $3\times3, 3\times1$ zero matrices, respectively. If $(D_T, -I_{3\times3})\Phi(T,t) \begin{pmatrix} O_{3\times3} \\ I_{3\times3} \end{pmatrix}$ is invertible, which will be the case in our simulations, a direct calculation shows that the unique solution to (5.2) is given by

$$
D_t = -\left[(D_T, -I_{3\times3})\Phi(T,t) \begin{pmatrix} O_{3\times3} \\ I_{3\times3} \end{pmatrix}\right]^{-1} (D_T, -I_{3\times3})\Phi(T,t) \begin{pmatrix} I_{3\times3} \\ O_{3\times3} \end{pmatrix} \tag{5.3}
$$

and

$$
D_t^0 = -\left[(D_T, -I_{3\times3})\Phi(T,t) \begin{pmatrix} O_{3\times3} \\ I_{3\times3} \end{pmatrix}\right]^{-1} (D_T, -I_{3\times3}) \int_t^T \Phi(T,s)\, ds \begin{pmatrix} F^0 \\ O_{3\times1} \end{pmatrix}. \tag{5.4}
$$

Having derived an explicit solution for the expected equilibrium portfolio process allows us to derive an explicit solution for the equilibrium portfolio process itself. It is not difficult to see that

$$
\begin{cases}
(X_t - \mathbb{E}[X_t])' = -\dfrac{P_t - \mathbb{E}[P_t]}{2\eta} \\
-(P_t - \mathbb{E}[P_t])' = 2\lambda(X_t - \mathbb{E}[X_t]) \\
X_0 - \mathbb{E}[X_0] = \mathcal{X} - \mathbb{E}[\mathcal{X}] \\
P_T - \mathbb{E}[P_T] = 2n(X_T - \mathbb{E}[X_T])
\end{cases}
$$

which is approximated by

$$
\begin{cases}
(X_t - \mathbb{E}[X_t])' = -\dfrac{P_t - \mathbb{E}[P_t]}{2\eta} \\
-(P_t - \mathbb{E}[P_t])' = 2\lambda(X_t - \mathbb{E}[X_t]) \\
X_0 - \mathbb{E}[X_0] = \mathcal{X} - \mathbb{E}[\mathcal{X}] \\
X_T - \mathbb{E}[X_T] = 0.
\end{cases}
$$

Making the ansatz $P - \mathbb{E}[P] = A(X - \mathbb{E}[X])$ yields

$$
A' = \frac{A^2}{2\eta} - 2\lambda, \quad A_T = \infty,
$$

or equivalently,

$$
A_t = 2\sqrt{\eta\lambda} \coth\left(\sqrt{\frac{\lambda}{\eta}}(T-t)\right).
$$

Thus, we get that

$$
X_t - \mathbb{E}[X_t] = (\mathcal{X} - \mathbb{E}[\mathcal{X}])e^{-\int_0^t \frac{A_s}{2\eta}\, ds} = (\mathcal{X} - \mathbb{E}[\mathcal{X}]) \frac{\sinh\left(\sqrt{\frac{\lambda}{\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\lambda}{\eta}}T\right)}
$$

and hence the optimal position approximately equals

$$X_t \approx \mathbb{E}[X_t] + (\mathcal{X} - \mathbb{E}[\mathcal{X}]) \frac{\sinh\left(\sqrt{\frac{\lambda}{\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\lambda}{\eta}}T\right)}.$$

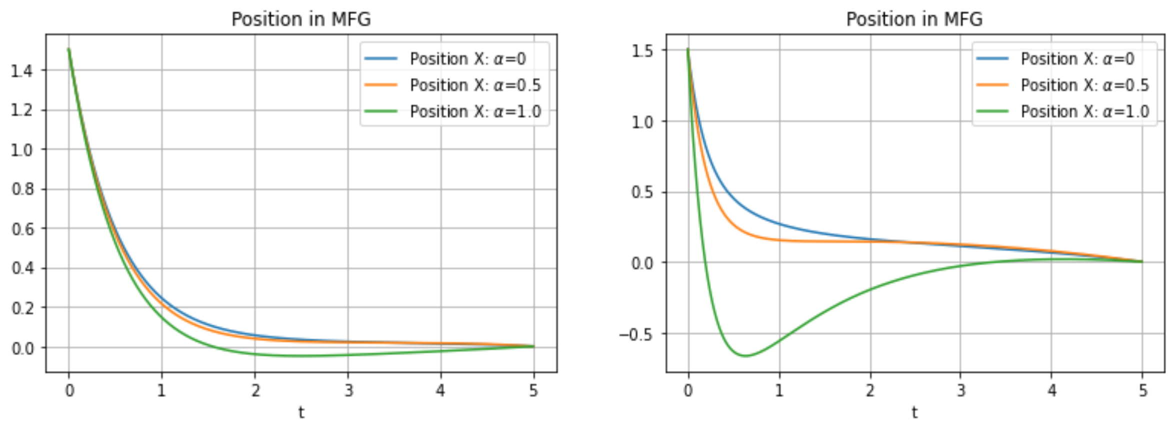

Figure 1: Dependence of equilibrium portfolio process on the market impact parameter α, $\gamma = 0.1$(left) and $\gamma = 1$(right). Other parameters are chosen as $\eta = 0.1$, $\rho = 0.2$, $\lambda = 0.3$, $\beta = 1.1$, $x = 1.5$, $\mathbb{E}[\mathcal{X}] = 1$ and $T = 5$.

The MFG is convex; hence no additional verification arguments are required. Figure 1 displays the equilibrium portfolio processes in an MFG for varying degrees of child order flow and transient market impact. We can see from both pictures that short positions do not occur in equilibrium if the impact as measured by the quantities α and γ is small. For near critical values of α it is optimal for the representative player to unwind his position before the terminal time, and then to take a negative position that he closes at the end of the trading period. This effect increases significantly in the impact parameter γ. The result is intuitive; the larger α and γ, the stronger the representative player benefits from the inertia in market order flow when closing a short position.

5.2 Single player model

When $N = 1$ and all model parameters are deterministic constants, then our mean-field FBSDE reduces to a forward-backward ODE system, and can be rewritten as

$$\begin{cases} \mathbb{F}' = \psi_{00}\mathbb{F} + \psi_{01}\mathbb{B} + F^0, \\ \mathbb{B}' = \psi_{10}\mathbb{F} + \psi_{11}\mathbb{B}, \\ \mathbb{F}_0 = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \ \mathbb{B}_T = \begin{pmatrix} 2n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbb{F}_T, \end{cases} \tag{5.5}$$

where

$$\mathbb{F} = (X, Y, C) \quad \text{and} \quad \mathbb{B} = (P, Q, R),$$

and

$$\psi_{00} = \begin{pmatrix} 0 & \frac{1}{2\eta} & 0 \\ -\alpha\gamma & -\rho - \frac{\gamma}{2\eta} & -\gamma(\beta-\alpha) \\ -\alpha & 0 & -(\beta-\alpha) \end{pmatrix}, \quad \psi_{01} = \begin{pmatrix} -\frac{1}{2\eta} & \frac{\gamma}{2\eta} & 0 \\ \frac{\gamma}{2\eta} & -\frac{\gamma^2}{2\eta} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^0 = \begin{pmatrix} 0 \\ \gamma\alpha x \\ \alpha x \end{pmatrix},$$

$$\psi_{10} = \begin{pmatrix} -2\lambda & 0 & 0 \\ 0 & \frac{1}{2\eta} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \psi_{11} = \begin{pmatrix} 0 & \alpha\gamma & \alpha \\ -\frac{1}{2\eta} & \rho + \frac{\gamma}{2\eta} & 0 \\ 0 & \gamma(\beta-\alpha) & (\beta-\alpha) \end{pmatrix}.$$

Making again a linear ansatz $\mathbb{B} = \mathscr{D}\mathbb{F} + \mathscr{D}^0$, yields

$$\begin{cases} \mathscr{D}' = -\mathscr{D}\psi_{01}\mathscr{D} - \mathscr{D}\psi_{00} + \psi_{11}\mathscr{D} + \psi_{10}, & \mathscr{D}_T = \begin{pmatrix} 2n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ (\mathscr{D}^0)' = (\psi_{11} - \mathscr{D}\psi_{01})\mathscr{D}^0 - \mathscr{D}F^0, & \mathscr{D}^0_T = (0,0,0)^\top, \end{cases} \tag{5.6}$$

and the same argument as in the previous section shows that the unique solution to the above ODE system is given by

$$\mathscr{D}_t = -\left[(\mathscr{D}_T, -I_{3\times 3})\Psi(T,t)\begin{pmatrix} O_{3\times 3} \\ I_{3\times 3} \end{pmatrix}\right]^{-1} (\mathscr{D}_T, -I_{3\times 3})\Psi(T,t)\begin{pmatrix} I_{3\times 3} \\ O_{3\times 3} \end{pmatrix}$$

and

$$\mathscr{D}^0_t = -\left[(\mathscr{D}_T, -I_{3\times 3})\Psi(T,t)\begin{pmatrix} O_{3\times 3} \\ I_{3\times 3} \end{pmatrix}\right]^{-1} (\mathscr{D}_T, -I_{3\times 3}) \int_t^T \Psi(T,s)\, ds \begin{pmatrix} F^0 \\ O_{3\times 1} \end{pmatrix},$$

where $\Psi(T,t) = e^{\mathscr{G}(T-t)}$ and

$$\mathscr{G} = \begin{pmatrix} \psi_{00} & \psi_{01} \\ \psi_{10} & \psi_{11} \end{pmatrix}.$$

Note that $\int_t^T \psi(T,s)\, ds = \mathscr{G}^{-1}(e^{\mathscr{G}(T-t)} - I_{6\times 6})$ as long as $\mathscr{G}$ is invertible. This is indeed the case because $\beta > \alpha$ and so

$$\det(\mathscr{G}) = -\rho\lambda(\beta - \alpha)^2 \left(\frac{\gamma}{\eta^2} + \frac{\rho}{\eta}\right) \neq 0.$$

In particular, the ODE system can be solved explicitly. The solution to the ODE system yields candidate optimal portfolios; portfolios for various choices of model parameters are shown in Figure 2. The left figure shows the portfolio process for various degrees of child order flow when $\gamma = 1$, $\beta = 1.1$ and $\lambda = 0.3$. We see that the initial trading rate increases in α and that it is optimal to oversell for near-critical values of α. The right picture shows the portfolio process for different degrees of transient market impact. For very large values of γ cyclic fluctuations in the optimal portfolio process emerge. Oscillating strategies arise when a trader expects his impact on future order flow to be very strong. Aggressively selling generates additional sell flow at later points in time from which the trader may benefit when switching from selling to buying.

Cyclic oscillations can be viewed as a form of transaction-triggered price manipulation. As already argued by Alfonsi et al. (2012) they are economically undesirable and should, as Gatheral et al. (2012) write (p.456), "be regarded as an additional model irregularity that should be excluded." Our numerical simulations suggest that cyclic oscillations occur only for unreasonably large values of γ that violate the assumptions of our verification theorem. Although the assumptions of our verification theorem are far from being necessary, it seems natural that some bound on the impact of traders' on market dynamics is necessary to exclude arbitrage and/or price manipulation. Interestingly, we found no numerical evidence that cyclic oscillations may occur in game-theoretic settings. This suggests that within our modelling framework strategic interactions may stabilize markets. We leave a more detailed analysis of this question for future research.

5.3 Two player model

If $N = 2$ and all model parameters are deterministic constants, then our mean-field FBSDE reduces to a forward-backward ODE system, and can be rewritten as

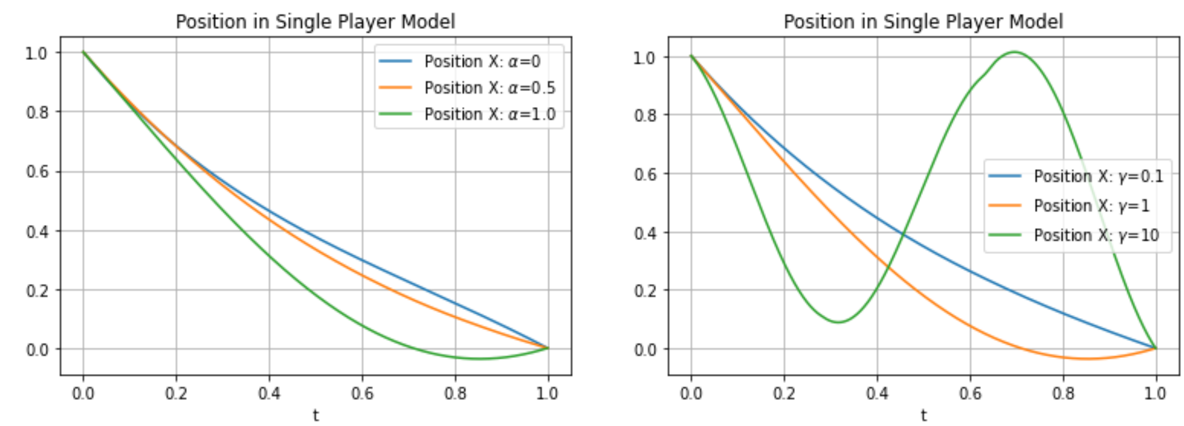

Figure 2: Dependence of optimal portfolio process on the market impact parameters α and γ, $\gamma = 1$(left) and $\alpha = 1$(right). Parameters are chosen as $\eta = 0.1$, $\rho = 0.2$, $\lambda = 0.3$, $\beta = 1.1$, $x = 1$ and $T = 1$.

$$\begin{cases} \mathbb{F}' = \phi_{00}\mathbb{F} + \phi_{01}\mathbb{B} + F^0, \\ \mathbb{B}' = \phi_{10}\mathbb{F} + \phi_{11}\mathbb{B}, \\ \mathbb{F}_0 = \begin{pmatrix} x^1 \\ 0 \\ 0 \\ x^2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbb{B}_T = \begin{pmatrix} 2n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbb{F}_T, \end{cases} \tag{5.7}$$

where

$$\mathbb{F} = (X^{(1)}, Y^{(1)}, C^{(1)}, X^{(2)}, Y^{(2)}, C^{(2)})^\top \quad \text{and} \quad \mathbb{B} = (P^{(1)}, Q^{(1)}, R^{(1)}, P^{(2)}, Q^{(2)}, R^{(2)})^\top$$

and

$$\phi_{00} = \begin{pmatrix} 0 & \frac{1}{2\eta_1} & 0 & 0 & 0 & 0 \\ -\frac{\gamma^1\alpha^1}{2} & -\rho^1 - \frac{\gamma^1}{4\eta^1} & -\gamma^1(\beta^1 - \alpha^1) & -\frac{\gamma^1\alpha^1}{2} & -\frac{\gamma^1}{4\eta^2} & 0 \\ -\frac{\alpha^1}{2} & 0 & -(\beta^1 - \alpha^1) & -\frac{\alpha^1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\eta_2} & 0 \\ -\frac{\gamma^2\alpha^2}{2} & -\frac{\gamma^2}{4\eta^1} & 0 & -\frac{\gamma^2\alpha^2}{2} & -\rho^2 - \frac{\gamma^2}{4\eta^2} & -\gamma^2(\beta^2 - \alpha^2) \\ -\frac{\alpha^2}{2} & 0 & 0 & -\frac{\alpha^2}{2} & 0 & -(\beta^2 - \alpha^2) \end{pmatrix},$$

$$\phi_{01} = \begin{pmatrix} -\frac{1}{2\eta^1} & \frac{\gamma^1}{4\eta^1} & 0 & 0 & 0 & 0 \\ \frac{\gamma^1}{4\eta^1} & -\frac{(\gamma^1)^2}{8\eta^1} & 0 & \frac{\gamma^1}{4\eta^2} & -\frac{\gamma^1\gamma^2}{8\eta^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2\eta^2} & \frac{\gamma^2}{4\eta^2} & 0 \\ \frac{\gamma^2}{4\eta^1} & -\frac{\gamma^1\gamma^2}{8\eta^1} & 0 & \frac{\gamma^2}{4\eta^2} & -\frac{(\gamma^2)^2}{8\eta^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \phi_{10} = \begin{pmatrix} -2\lambda^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2\eta^1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\eta^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\phi_{11} = \begin{pmatrix} 0 & \frac{\alpha^1\gamma^1}{2} & \frac{\alpha^1}{2} & 0 & 0 & 0 \\ -\frac{1}{2\eta^1} & \rho^1 + \frac{\gamma^1}{4\eta^1} & 0 & 0 & 0 & 0 \\ 0 & \gamma^1(\beta^1 - \alpha^1) & (\beta^1 - \alpha^1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha^2\gamma^2}{2} & \frac{\alpha^2}{2} \\ 0 & 0 & 0 & -\frac{1}{2\eta^2} & \rho^2 + \frac{\gamma^2}{4\eta^2} & 0 \\ 0 & 0 & 0 & 0 & \gamma^2(\beta^2 - \alpha^2) & (\beta^2 - \alpha^2) \end{pmatrix},$$

and

$$F^0 = \left(0, \frac{\gamma^1\alpha^1}{2}(x^1+x^2), \frac{\alpha^1}{2}(x^1+x^2), 0, \frac{\gamma^2\alpha^2}{2}(x^1+x^2), \frac{\alpha^2}{2}(x^1+x^2)\right)^\top.$$

Again making the ansatz $\mathbb{B} = \mathcal{D}\mathbb{F} + \mathcal{D}^0$, where

$$\mathcal{D}' = -\mathcal{D}\phi_{01}\mathcal{D} - \mathcal{D}\phi_{00} + \phi_{11}\mathcal{D} + \phi_{10}, \qquad \mathcal{D}_T = \begin{pmatrix} 2n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.8}$$

$$(\mathcal{D}^0)' = (\phi_{11} - \mathcal{D}\phi_{01})\mathcal{D}^0 - \mathcal{D}F^0, \qquad \mathcal{D}^0_T = (0,0,0,0,0,0)^\top$$

the same arguments as in the mean-field case yield the unique solution

$$\mathcal{D}_t = -\left[(\mathcal{D}_T, -I_{6\times 6})\Phi(T,t)\begin{pmatrix} O_{6\times 6} \\ I_{6\times 6}\end{pmatrix}\right]^{-1}(\mathcal{D}_T, -I_{6\times 6})\Phi(T,t)\begin{pmatrix} I_{6\times 6} \\ O_{6\times 6}\end{pmatrix}$$

and

$$\mathcal{D}^0_t = -\left[(\mathcal{D}_T, -I_{6\times 6})\Phi(T,t)\begin{pmatrix} O_{6\times 6} \\ I_{6\times 6}\end{pmatrix}\right]^{-1}(\mathcal{D}_T, -I_{6\times 6})\int_t^T \Phi(T,s)\,ds\begin{pmatrix} F^0 \\ O_{6\times 1}\end{pmatrix},$$

where $\Phi(T,t) = e^{\mathcal{G}(T-t)}$ and

$$\mathcal{G} = \begin{pmatrix} \phi_{00} & \phi_{01} \\ \phi_{10} & \phi_{11}\end{pmatrix}.$$

There is no explicit expression for the integral since $\det(\mathcal{G}) \equiv 0$. Figure 3 shows equilibrium positions in a two player model with different degrees of transient market impact. In both cases, Player 2 benefits from the presence of Player 1; there is a beneficial round-trip for this player in equilibrium. As expected the round-trip is stronger (more convex) for larger degrees of transient impact.

As pointed out above, we did not find numerical evidence for the occurrence of cyclic oscillations in the 2-Player game. As illustrated by Figure 4 some form of oscillation may occur for large values of γ but the oscillations are not as regular as in the single player case and are not cyclic, even if both players are completely identical.

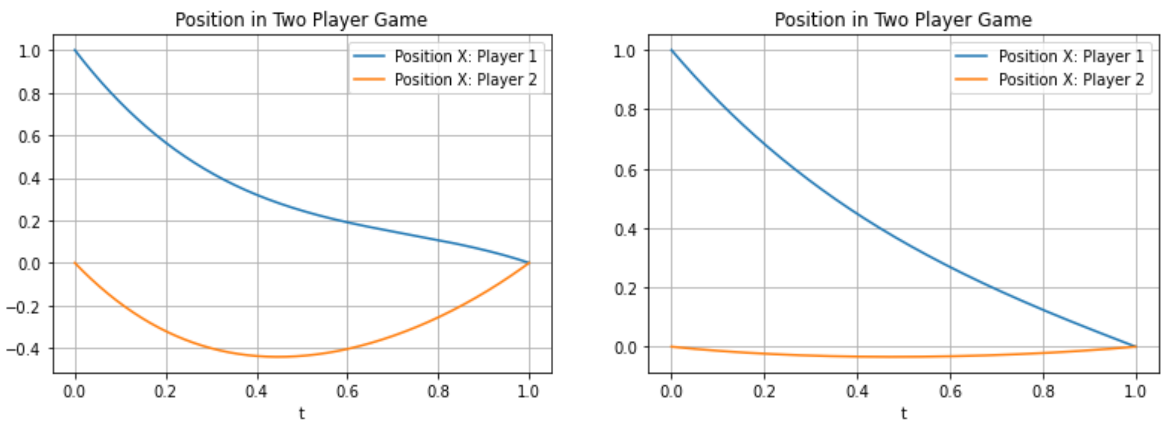

Figure 3: Portfolio process in the two player game under different parameters γ, $\gamma = 1$(left) and $\gamma = 0.1$(right). Other parameters are chosen as $\eta_1 = \eta_2 = 0.1$, $\rho_1 = \rho_2 = 0.2$, $\lambda_1 = \lambda_2 = 0.3$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 1.1$, $x_1 = 1, x_2 = 0$, and $T = 1$.

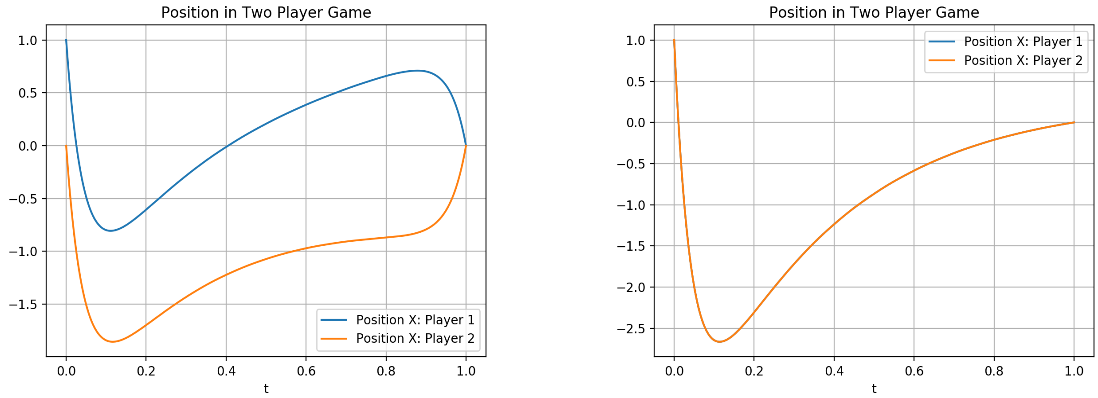

Figure 4: Portfolio process in the two player game for $\gamma = 10$ and different (left) respectively same (right) initial portfolios.

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