Objective Rationality Foundations for (Dynamic) α-MEU

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Objective rationality foundations for (dynamic) α-MEU*

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Abstract

We show how incorporating Gilboa, Maccheroni, Marinacci, and Schmeidler's (2010) notion of objective rationality into the α-MEU model of choice under ambiguity (Hurwicz, 1951) can overcome several challenges faced by the baseline model without objective rationality. The decision-maker (DM) has a *subjectively rational* preference $\succsim^\wedge$, which captures the complete ranking over acts the DM expresses when forced to make a choice; in addition, we endow the DM with a (possibly incomplete) *objectively rational* preference $\succsim^*$, which captures the rankings the DM deems uncontroversial. Under the *objectively founded* α-MEU model, $\succsim^\wedge$ has an α-MEU representation and $\succsim^*$ has a unanimity representation à la Bewley (2002), where both representations feature the same utility index and set of beliefs. While the axiomatic foundations of the baseline α-MEU model are still not fully understood, we provide a simple characterization of its objectively founded counterpart. Moreover, in contrast with the baseline model, the model parameters are uniquely identified. Finally, we provide axiomatic foundations for prior-by-prior Bayesian updating of the objectively founded α-MEU model, while we show that, for the baseline model, standard updating rules can be ill-defined.

Keywords: ambiguity, α-MEU, objective rationality, updating.

1 Introduction

A widely used model of choice under ambiguity is the α-maxmin expected utility (α-MEU) criterion, dating back to Hurwicz (1951). This criterion represents a decision-maker's (DM's) preference $\succsim^\wedge$ over (Anscombe-Aumann) acts f by considering the weighted average of each act's worst-case and best-case expected utility,

$$\alpha \min_{\mu \in P} \mathbb{E}_\mu[u(f)] + (1-\alpha) \max_{\mu \in P} \mathbb{E}_\mu[u(f)], \tag{1}$$

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according to some weight $\alpha \in [0, 1]$, closed and convex set P of beliefs over states, and nonconstant and affine utility u over outcomes. Unlike Gilboa and Schmeidler's (1989) maxmin expected utility criterion (i.e., the special case when $\alpha = 1$), the general α-MEU model does not assume that the DM is uncertainty-averse (Schmeidler, 1989). Instead, in line with various experimental evidence (see the survey by Trautmann and van de Kuilen, 2015), (1) allows the DM to display a mix of ambiguity-averse and ambiguity-seeking tendencies, and the weight α and set of beliefs P are often interpreted as simple parameterizations of the DM's ambiguity attitude and perception of ambiguity, respectively. This has contributed to the model's popularity in applied work, which has employed α-MEU representations in both static and dynamic settings.[1]

Despite its popularity, the foundations of the α-MEU model are still not fully understood. First, there is no known axiomatic characterization of the model in the standard domain of preferences over acts, except for several special cases that restrict the structure of the belief set P (see Section 1.1). Second, the preference $\succsim^\wedge$ does not uniquely identify α and P, complicating the interpretation of these parameters as capturing the DM's ambiguity attitude and perception: Proposition 1 in this paper characterizes the extent of multiplicity. Third, as we show in Example 1, the lack of identification of the model parameters creates the following problem for dynamic extensions of α-MEU: Common belief-updating rules, such as prior-by-prior Bayesian updating of all beliefs in P, are ill-defined at the level of preferences, as different representations of the same ex-ante preference $\succsim^\wedge$ may give rise to different updated preferences.

In this paper, we show how these challenges can be addressed by incorporating the notion of objective rationality (Gilboa, Maccheroni, Marinacci, and Schmeidler, 2010, henceforth, GMMS) into the α-MEU model. We interpret $\succsim^\wedge$ as the DM's *subjectively rational* preference, which captures the complete ranking the DM expresses when forced to choose between any two acts. In addition, we endow the DM with a (possibly incomplete) *objectively rational* preference $\succsim^*$, which models the rankings that appear uncontroversial to the DM. We consider a joint representation of $\succsim^\wedge$ and $\succsim^*$, where for some utility u, set of beliefs P, and weight α:

1. The subjectively rational preference $\succsim^\wedge$ admits an α-MEU representation based on u, P, and α.

2. The objectively rational preference $\succsim^*$ is represented by u and P in the sense of Bewley (2002); that is, act f is deemed uncontroversially better than g if and only if the

[1] In static settings, see for instance Cherbonnier and Gollier (2015); Chen, Katuščák, and Ozdenoren (2007); Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010); Ahn, Choi, Gale, and Kariv (2014); in dynamic settings, see Saghafian (2018); Georgalos (2019); Beissner, Lin, and Riedel (2020).

expected utility under u of f dominates the expected utility of g for every belief in P.

Thus, under this *objectively founded α-MEU* model, the DM employs the α-MEU criterion as a forced-choice completion of Bewley's (2002) unanimity criterion, where both criteria are based on the same set of beliefs P and the same utility u over outcomes.

Turning to the aforementioned challenges, we first show that the objectively founded α-MEU model admits a simple axiomatic characterization (Theorem 1). We impose Bewley's (2002) axioms on the objectively rational preference; that is, $\succsim^*$ satisfies all subjective expected utility axioms, except that completeness is only assumed for the ranking over constant acts. The subjectively rational preference is required to be invariant biseparable (Ghirardato, Maccheroni, and Marinacci, 2004); that is, $\succsim^\wedge$ satisfies all subjective expected utility axioms, except that independence is only imposed for mixtures with constant acts. The final and key axiom, security-potential dominance, disciplines the completion rule from $\succsim^*$ to $\succsim^\wedge$: We require the DM to subjectively prefer act f to act g whenever f is both "more secure" than g and has "more potential" than g, where security and potential are defined in terms of the objective ranking against certain prospects.

Second, in contrast with the baseline model without objective rationality, the parameters α and P in Theorem 1 are uniquely identified. Thus, the interpretation of α and P as the DM's ambiguity attitude and perception is behaviorally founded, making it possible to conduct behavioral comparisons of these parameters (Section 4.2).

Finally, in contrast with Example 1, we show that prior-by-prior Bayesian updating of the objectively founded α-MEU model admits well-defined preference foundations. Suppose the DM's ex-ante subjective and objective preferences have an objectively founded α-MEU representation (u, P, α). Upon learning that the state of the world is contained in some event E, the DM forms conditional subjective and objective preferences $\succsim^\wedge_E$ and $\succsim^*_E$. Theorem 2 characterizes when $\succsim^\wedge_E$ and $\succsim^*_E$ admit an objectively founded α-MEU representation whose utility is u and whose set of beliefs P^E is derived from the unique ex-ante belief set P by prior-by-prior Bayesian updating. The key axioms impose an intertemporal analog of security-potential dominance on the relationship between the ex-ante objective preference and the conditional subjective preference, as well as dynamic consistency on the ex-ante and conditional objective preferences. Under these axioms, the weights α_E in the conditional representation can vary across different events E, allowing for the possibility that the information the DM obtains might affect his ambiguity attitude. We characterize this variation in ambiguity attitudes behaviorally, including the special case when $\alpha_E = \alpha$ is history-independent.

1.1 Related literature

GMMS propose the objective and subjective rationality approach, and characterize when $\succsim^*$ and $\succsim^\wedge$ admit Bewley and maxmin expected utility representations with a common set of beliefs P and utility u. We impose the same axioms as GMMS on $\succsim^*$ and $\succsim^\wedge$ individually, but relax their main axiom, caution, that concerns the relationship between $\succsim^*$ and $\succsim^\wedge$ (see Section 4.1).[2] Several papers extend the results in GMMS in different directions. Kopylov (2009) characterizes when $\succsim^\wedge$ admits an ε-contamination representation. Cerreia-Vioglio (2016) allows $\succsim^\wedge$ to be a general uncertainty-averse preference (Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2011). Faro and Lefort (2019) characterize prior-by-prior Bayesian updating under the Bewley-maxmin model in GMMS.[3] Grant, Rich, and Stecher (2019) use a condition that is equivalent to security-potential dominance (along with weaker assumptions on $\succsim^\wedge$) to characterize a representation in which the subjectively rational model—ordinal Hurwicz expected utility—is more general than α-MEU; they do not characterize α-MEU and do not study belief updating.[4] We note that most aforementioned papers consider subjectively rational models that have well-understood foundations based on the primitive $\succsim^\wedge$ alone, and the focus is on understanding the consistency of the objective and subjective models. In contrast, in the present paper, the subjectively rational model—α-MEU—is not well-understood in isolation, and incorporating objective rationality plays a key role in enabling its axiomatic characterization, identification, and dynamic extension.

Several papers characterize α-MEU representations in terms of the subjectively rational preference $\succsim^\wedge$ alone, but impose specific assumptions on the structure of the belief set P. Kopylov (2003) considers the case in which P consists of beliefs that are derived from a particular class of subjectively risky acts. Ghirardato, Maccheroni, and Marinacci (2004) require P to coincide with the Bewley set of the largest independent subrelation of $\succsim^\wedge$; for finite state spaces, Eichberger, Grant, Kelsey, and Koshevoy (2011) show that this case reduces to maxmin or maxmax expected utility (see Remark 1). Chateauneuf, Eichberger, and Grant (2007) consider a neo-additive capacity model that evaluates each act according to a convex combination of the least favorable prize, most favorable prize, and the expected utility with respect to a fixed probability. Gul and Pesendorfer (2015) study the case in which P is the set of measures that are consistent with some benchmark belief μ over a given sigma-

[2]GMMS also introduce a slight strengthening of caution, termed default to certainty, under which C-independence can be dropped from the assumptions on $\succsim^\wedge$.

[3]See also Bastianello, Faro, and Santos (2020) and Ceron and Vergopoulos (2020) for the connection with dynamic consistency.

[4]Relatedly, Nehring (2009) studies the compatibility of an incomplete preference over events and a complete preference over Savage acts. He considers the case where the latter preference is invariant biseparable, but does not characterize the special case of α-MEU.

algebra of events. Klibanoff, Mukerji, Seo, and Stanca (2020) consider a product state space $S = Y^\infty$ and assume that P consists of i.i.d. distributions. In contrast with these papers, we incorporate objective rationality but allow for general belief sets P. In addition, we highlight the lack of identification of the general α-MEU model and the resulting difficulties for defining belief-updating, and we show how the objectively founded model can address these challenges. To the best of our knowledge, Theorem 2 provides the first axiomatic foundation for a dynamic extension of α-MEU.

Finally, Hill (2019) enriches the standard domain in a different direction, by considering a preference over acts f that map each state s to a *set* of objective lotteries $f(s)$. He characterizes an α-MEU representation $\alpha \min_{\mu \in P} \mathbb{E}_\mu[w(f)] + (1-\alpha) \max_{\mu \in P} \mathbb{E}_\mu[w(f)]$, where the utility $w(f(s)) = \alpha \min_{p \in f(s)} \mathbb{E}_p[u] + (1-\alpha) \max_{p \in f(s)} \mathbb{E}_p[u]$ over sets of lotteries also takes an α-MEU form. Relatedly, Jaffray (1994) and Olszewski (2007) directly consider preferences over sets of objective lotteries and characterize α-MEU representations for such preferences.

2 Model

2.1 Setup

Let Z be a set of prizes and let $\Delta(Z)$ denote the space of probability measures with finite support over Z. We refer to typical elements $p, q \in \Delta(Z)$ as lotteries. Let S be a finite set of states. An *(Anscombe-Aumann) act* is a mapping $f : S \to \Delta(Z)$. Let $\mathcal{F}$ be the space of all acts, with typical elements f, g, h. For any $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, define the mixture $\alpha f + (1-\alpha)g \in \mathcal{F}$ to be the act that in each state $s \in S$ yields lottery $\alpha f(s) + (1-\alpha)g(s) \in \Delta(Z)$. As usual, we identify each lottery $p \in \Delta(Z)$ with the constant act that yields lottery p in all states $s \in S$.

Let $\Delta(S)$ denote the set of all probability measures over S, which we embed in $\mathbb{R}^S$ and endow with the Euclidean topology. We refer to typical elements $\mu, \nu \in \Delta(S)$ as beliefs. Given any act $f \in \mathcal{F}$ and function $u : \Delta(Z) \to \mathbb{R}$, let $u(f)$ denote the element of $\mathbb{R}^S$ defined by $u(f)(s) = u(f(s))$ for all $s \in S$, and let $\mathbb{E}_\mu[u(f)] := \mu \cdot u(f)$. Given any functions $u, v : \Delta(Z) \to \mathbb{R}$, we write $u \approx v$ if u is a positive affine transformation of v.

We follow GMMS in endowing the DM with two binary relations, $\succsim^\wedge$ and $\succsim^*$, over $\mathcal{F}$. Relation $\succsim^\wedge$ is the DM's ***subjectively rational*** (for short, ***subjective***) preference, which models the rankings the DM expresses when forced to choose between any two acts and, as such, is complete. Relation $\succsim^*$ is the DM's ***objectively rational*** (for short, ***objective***) preference, which captures the rankings that appear uncontroversial to the DM and, as such, may be incomplete.[5] As usual, $\succ^\wedge$ and $\sim^\wedge$ (resp., $\succ^*$ and $\sim^*$) denote the asymmetric and symmetric parts of $\succsim^\wedge$ (resp., $\succsim^*$).

2.2 Representation

We are interested in the following joint representation of $\succsim^*$ and $\succsim^\wedge$:

Definition 1. *An **objectively founded** α**-MEU representation** of $(\succsim^\wedge, \succsim^*)$ consists of a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$, a nonempty, closed and convex set of beliefs $P \subseteq \Delta(S)$, and a weight $\alpha \in [0,1]$ such that*

*(i.) (u, P, α) is an α-**MEU representation** of $\succsim^\wedge$; that is, for all $f, g \in \mathcal{F}$,*

$$f \succsim^\wedge g \Leftrightarrow \alpha \min_{\mu \in P} \mathbb{E}_\mu[u(f)] + (1-\alpha) \max_{\mu \in P} \mathbb{E}_\mu[u(f)] \geq \alpha \min_{\mu \in P} \mathbb{E}_\mu[u(g)] + (1-\alpha) \max_{\mu \in P} \mathbb{E}_\mu[u(g)]. \tag{2}$$

*(ii.) (u, P) is a **Bewley representation** of $\succsim^*$; that is, for all $f, g \in \mathcal{F}$,*

$$f \succsim^* g \iff \mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\mu[u(g)] \ \forall \mu \in P. \tag{3}$$

The first condition says that when forced to choose between any two acts, the DM employs the α-MEU criterion based on utility u, set of beliefs P, and weight α. The second condition enriches the basic α-MEU model by requiring this choice procedure to be *objectively founded*, in the sense that the *same* set of beliefs P and utility u also represent the rankings the DM considers uncontroversial: Specifically, the DM deems act f uncontroversially better than act g if and only if the expected utility under u of f dominates the expected utility of g for every belief in P. Thus, the objectively founded α-MEU model captures a DM who uses the α-MEU criterion as a completion of an underlying unanimity criterion à la Bewley (2002).

3 α-MEU without objective rationality

Before proceeding to study objectively founded α-MEU representations, we point to three challenges for the baseline α-MEU model without objective rationality. First, as discussed in the introduction, there is no known axiomatization of general α-MEU representations in terms of the subjective preference $\succsim^\wedge$ alone.

[5] Kopylov (2009) interprets $\succsim^*$ as a "firm" preference that the DM would not want to revise at an interim stage (prior to the resolution of any uncertainty). He notes that $\succsim^*$ could in principle be elicited by charging a small monetary cost for the option to revise the preference at the interim stage.

Second, the belief set P and weight α in representation (2) are not uniquely pinned down by $\succsim^\wedge$, complicating the common interpretation of these parameters as capturing the DM's ambiguity perception and attitude, respectively. The following result characterizes which pairs (P, α) give rise to the same preference, extending a result in Siniscalchi (2006).[6] Given any nonempty, closed and convex sets $P, Q \subseteq \Delta(S)$ and any $\gamma \geq 1$, we call Q the ***γ-expansion*** of P if

$$Q = \gamma P + (1-\gamma)P := \{\gamma\nu + (1-\gamma)\nu' : \nu, \nu' \in P\}. \tag{4}$$

Observe that (4) implies $Q \supseteq P$, with $Q = P$ if $\gamma = 1$.[7]

Proposition 1. *Suppose (u_1, P_1, α_1) and (u_2, P_2, α_2) are α-MEU representations of $\succsim_1^\wedge$ and $\succsim_2^\wedge$, respectively, such that $\alpha_i \neq 1/2$ and P_i is not a singleton for $i = 1, 2$, and $\alpha_1 \leq \alpha_2$.*[8]

Then $\succsim_1^\wedge = \succsim_2^\wedge$ if and only if $u_1 \approx u_2$ and one of the following two statements holds:

(i). $\alpha_1, \alpha_2 > 1/2$ and P_1 is the γ-expansion of P_2 for $\gamma = \dfrac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1}$;

(ii). $\alpha_1, \alpha_2 < 1/2$ and P_2 is the γ-expansion of P_1 for $\gamma = \dfrac{1 - \alpha_1 - \alpha_2}{1 - 2\alpha_2}$.

Proposition 1 shows that while the DM's subjective preference pins down whether the weight α is greater or less than $1/2$, a range of different weights can be used to represent the same preference $\succsim^\wedge$. For each such weight α, the corresponding set of beliefs P is uniquely determined. In case 1, weight α_1 suggests a less extreme attitude towards ambiguity than α_2 (as $\alpha_2 \geq \alpha_1$ is closer to $\{0, 1\}$ than α_1), but the corresponding set of priors P_1 is larger than P_2, suggesting greater perceived ambiguity. In case 2, the opposite relationship obtains.

Third, we highlight that the non-uniqueness of the set of priors P poses a challenge for defining belief-updating under α-MEU. To illustrate, we focus on prior-by-prior Bayesian updating, which has been used in several applications.[9] Consider a DM whose ex-ante preference $\succsim^\wedge$ admits an α-MEU representation (u, P, α). Suppose the DM is informed that

[6] Siniscalchi (2006) (Proposition 6.1) considers the special case when $\succsim^\wedge$ admits a maxmin expected utility representation whose belief set P is bounded away from $\Delta(S)$ and shows that there is a continuum of α-MEU representations of $\succsim^\wedge$ with $\alpha < 1$ and belief sets $Q \supsetneq P$. His proof uses a different (but equivalent) definition of γ-expansion.

[7] Note that while the set $\gamma P + (1-\gamma)P \subseteq \mathbb{R}^S$ need not in general be a subset of the simplex $\Delta(S)$, condition (4) implicitly imposes this as $Q \subseteq \Delta(S)$.

[8] Rogers and Ryan (2012) cover the case where one belief set $P_i = \{\mu\}$ is a singleton ($\succsim_i$ is subjective expected utility): In this case, $\succsim_i^\wedge = \succsim_j^\wedge$ iff $u_i \approx u_j$ and either (i) $P_j = \{\mu\}$ or (ii) $\alpha_j = 1/2$ and P_j is centrally symmetric around μ. Appendix B considers the case where $\alpha_i = 1/2$ and P_i, P_j are not singletons (which implies $\alpha_j = 1/2$). As we show, this case admits a greater multiplicity of belief sets than in Proposition 1.

[9] See the references in Footnote 1. Analogous issues arise for other updating rules, such as maximum likelihood updating.

the true state of nature is contained in some event $E \subseteq S$ and based on this information forms a conditional preference $\succsim_E^\wedge$. Consider deriving $\succsim_E^\wedge$ from $\succsim^\wedge$ by prior-by-prior Bayesian updating of all beliefs in P. That is, assuming that $\mu(E) > 0$ for all $\mu \in P$, let $\succsim_E^\wedge$ be induced by the α-MEU representation (u, P^E, α) whose conditional set of beliefs is

$$P^E := \{\mu^E : \mu \in P\}, \text{ where } \mu^E(F) := \frac{\mu(E \cap F)}{\mu(E)} \quad \forall F \subseteq S. \tag{5}$$

The following example shows that this approach is not well-defined at the level of preferences. Indeed, if the ex-ante preference $\succsim^\wedge$ admits multiple α-MEU representations, prior-by-prior updating can induce a different conditional preference $\succsim_E^\wedge$ depending on which ex-ante representation is used:

Example 1. Suppose $S = \{1, 2, 3\}$. Fix any nonconstant affine utility u, and consider the two α-MEU representations (u, P_i, α_i), where

$$\alpha_1 = \frac{3}{4}, \qquad P_1 = \mathrm{co}\left\{(\frac{5}{6}, \frac{1}{12}, \frac{1}{12}), (\frac{1}{6}, \frac{5}{12}, \frac{5}{12})\right\},$$

$$\alpha_2 = 1, \qquad P_2 = \mathrm{co}\left\{(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\right\}.$$

Let $\gamma = \dfrac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1} = 3/2$, and note that P_1 is the γ-expansion of P_2. Thus, by Proposition 1, the two representations represent the same ex-ante preference $\succsim^\wedge$. Now, consider the event $E = \{1, 2\}$. The prior-by-prior Bayesian updates of P_1 and P_2 are

$$P_1^E = \mathrm{co}\left\{(\frac{10}{11}, \frac{1}{11}, 0), (\frac{2}{7}, \frac{5}{7}, 0)\right\}, \quad P_2^E = \mathrm{co}\left\{(\frac{4}{5}, \frac{1}{5}, 0), (\frac{1}{2}, \frac{1}{2}, 0)\right\}.$$

However, the γ-expansion of P_2^E is $\mathrm{co}\left\{(\frac{19}{20}, \frac{1}{20}, 0), (\frac{7}{20}, \frac{13}{20}, 0)\right\} \neq P_1^E$. Hence, by Proposition 1, the conditional preferences represented by (u, P_1^E, α_1) and (u, P_2^E, α_2) are not the same. ▲

An implication of Example 1 is that Pires's (2002) coherency axiom (dubbed conditional certainty equivalent consistency by Eichberger, Grant, and Kelsey, 2007), which characterizes prior-by-prior updating under maxmin expected utility and several of its extensions, need not hold for prior-by-prior updating under α-MEU. Indeed, this axiom uniquely pins down the conditional preference $\succsim_E^\wedge$, in contrast with Example 1.

4 Objectively founded α-MEU representations

We now show how incorporating objective rationality into the α-MEU model makes it possible to overcome the challenges discussed in the previous section.

4.1 Characterization and uniqueness

This section provides an axiomatic characterization of objectively founded α-MEU representations and shows that the pair $(\succsim^\wedge, \succsim^*)$ uniquely determines P and α. Our characterization imposes the same five axioms as GMMS on $\succsim^\wedge$ and $\succsim^*$ individually, but differs from GMMS in what we assume about the relationship between $\succsim^*$ and $\succsim^\wedge$.

First, we impose two basic rationality conditions, along with a continuity assumption, on both $\succsim^\wedge$ and $\succsim^*$. We state this axiom for a generic binary relation $\succsim$ on $\mathcal{F}$:

Axiom 1 (Basic conditions)**.**

1. *Preorder:* $\succsim$ *is reflexive, transitive, and nondegenerate.*
2. *Monotonicity: If* $f, g \in \mathcal{F}$ *and* $f(s) \succsim g(s)$ *for all* $s \in S$*, then* $f \succsim g$*.*
3. *Mixture continuity: If* $f, g \in \mathcal{F}$*, then the sets* $\{\lambda \in [0,1] : \lambda f + (1-\lambda)g \succsim h\}$ *and* $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda)g\}$ *are closed in* $[0,1]$*.*

The following two axioms are specific to the objective preference $\succsim^*$:

Axiom 2 (C-Completeness)**.** *If* $p, q \in \Delta(Z)$*, then either* $p \succsim^* q$ *or* $q \succsim^* p$*.*

Axiom 3 (Independence)**.** *If* $f, g, h \in \mathcal{F}$ *and* $\alpha \in (0,1]$*, then*

$$f \succsim^* g \Longleftrightarrow \alpha f + (1-\alpha)h \succsim^* \alpha g + (1-\alpha)h.$$

A binary relation on $\mathcal{F}$ satisfying Axioms 1–3 is called a ***Bewley preference***. Such preferences satisfy all subjective expected utility axioms, except that completeness is only imposed on the ranking over constant acts. C-completeness assumes that any difficulties the DM might have in determining an uncontroversial ranking are due to uncertainty, rather than incompleteness of tastes over certain outcomes. As is well-known (Bewley, 2002, GMMS), $\succsim^*$ is a Bewley preference if and only if it admits a Bewley representation (3).

The next two assumptions are specific to the subjective preference $\succsim^\wedge$:

Axiom 4 (Completeness)**.** *If* $f, g \in \mathcal{F}$*, then either* $f \succsim^\wedge g$ *or* $g \succsim^\wedge f$*.*

Axiom 5 (C-Independence)**.** *If $f, g \in \mathcal{F}$, $p \in \Delta(Z)$, and $\alpha \in (0,1]$, then*

$$f \succsim^\wedge g \Longleftrightarrow \alpha f + (1-\alpha)p \succsim^\wedge \alpha g + (1-\alpha)p.$$

A binary relation on $\mathcal{F}$ satisfying Axioms 1, 4, and 5 is called an ***invariant biseparable preference***. Unlike Bewley preferences, such preferences satisfy completeness, but differ from subjective expected utility in that independence is only assumed for mixtures with constant acts. We refer the reader to GMMS for a rationale for imposing C-independence on $\succsim^\wedge$, and to Ghirardato, Maccheroni, and Marinacci (2004), Amarante (2009), and Frick, Iijima, and Le Yaouanq (2019) for representations of invariant biseparable preferences.

Our key axiom disciplines the completion rule from $\succsim^*$ to $\succsim^\wedge$. Consider any $f, g \in \mathcal{F}$. As in Kopylov (2009), we say that f ***is more secure*** than g if for all $p \in \Delta(Z)$,

$$g \succsim^* p \Longrightarrow f \succsim^* p.$$

We say that f ***has more potential*** than g if for all $p \in \Delta(Z)$,

$$p \not\succsim^* g \Longrightarrow p \not\succsim^* f.$$[10]

Axiom 6 (Security-potential dominance)**.** *If $f, g \in \mathcal{F}$ and f is both more secure than g and has more potential than g, then $f \succsim^\wedge g$.*

Axiom 6 captures that in choosing between two uncertain acts f and g, the DM might compare how f and g rank objectively against prospects that are certain. Two dimensions might matter to the DM in comparing an uncertain act f with a constant act p. On the one hand, an ambiguity-averse DM might favor the "security" of certain prospects, and thus seek the assurance that f uncontroversially dominates p. On the other hand, an ambiguity-seeking DM might be drawn to the "potential" of uncertain prospects, and thus be content as long as p does not uncontroversially dominate f. If f is more secure (resp., has more potential) than g, then f performs at least as well as g along the first (resp., second) dimension. Security-potential dominance allows for the possibility that both dimensions are relevant to the DM, reflecting the idea that the α-MEU criterion accommodates a mix of ambiguity-averse and ambiguity-seeking tendencies. Thus, Axiom 6 only requires the DM to choose f over g if f is both more secure and has more potential than g.

Given transitivity of $\succsim^*$, note that if $f \succsim^* g$, then f is more secure than g and has

[10] Kopylov (2009) introduces the notion of more security to define a strengthening of uncertainty aversion he calls cautious independence, and uses this to characterize the ε-contamination model. He uses the notion of more potential to characterize its uncertainty-seeking counterpart.

more potential than g; thus, security-potential dominance implies the following consistency condition imposed by GMMS. This condition (together with Axiom 4) captures that the subjectively rational preference is a completion of the objectively rational preference:

Consistency. *If $f, g \in \mathcal{F}$ and $f \succsim^* g$, then $f \succsim^\wedge g$.*

By contrast, Axiom 6 does not entail the main substantive assumption in GMMS. This assumption requires the DM's completion rule to be cautious, in the sense that unless a general act f is uncontroversially superior to a constant act p, the DM prefers to choose the constant act:

Caution. *If $f \in \mathcal{F}$, $p \in \Delta(Z)$ and $f \not\succsim^* p$, then $p \succsim^\wedge f$.*

While GMMS show that caution and consistency characterize when the invariant biseparable preference $\succsim^\wedge$ is a maxmin expected utility completion of the Bewley preference $\succsim^*$, the following result shows that security-potential dominance characterizes α-MEU completions. Moreover, in contrast with Proposition 1, for the objectively founded α-MEU model, the parameters P and α are uniquely identified.

Theorem 1. *The following are equivalent:*

(i). $\succsim^*$ *is a Bewley preference,* $\succsim^\wedge$ *is an invariant biseparable preference, and the pair* $(\succsim^\wedge, \succsim^*)$ *jointly satisfies security-potential dominance.*

(ii). $(\succsim^\wedge, \succsim^*)$ *admits an objectively founded α-MEU representation* (u, P, α).

Moreover, in this case, u is unique up to positive affine transformation, P is unique, and α is unique if $\succsim^*$ is not complete.

The main observation in the proof is that, given a Bewley representation (u, P) of $\succsim^*$, security-potential dominance ensures that for any f and g,

$$\left[\min_{\mu \in P} \mathbb{E}_\mu[u(f)] \geq \min_{\mu \in P} \mathbb{E}_\mu[u(g)] \text{ and } \max_{\mu \in P} \mathbb{E}_\mu[u(f)] \geq \max_{\mu \in P} \mathbb{E}_\mu[u(g)]\right] \Longrightarrow f \succsim^\wedge g.$$

Based on this, we find a weight α such that (u, P, α) is an α-MEU representation of $\succsim^\wedge$. The uniqueness of u and P follows from the uniqueness of Bewley representations. Given that P is unique, $\succsim^\wedge$ pins down α, unless $P = \{\mu\}$ is a singleton (i.e., $\succsim^*$ is complete). In the latter case, $\succsim^* = \succsim^\wedge$ is the subjective expected utility preference corresponding to belief μ, and α can be chosen arbitrarily.

Remark 1. For any invariant biseparable preference $\succsim^\wedge$, Ghirardato, Maccheroni, and Marinacci (2004) define the ***unambiguous preference*** $\succsim^u$ as the largest independent subrelation of $\succsim^\wedge$; equivalently, $f \succsim^u g$ means that $\lambda f + (1-\lambda)h \succsim^\wedge \lambda g + (1-\lambda)h$ for all $\lambda \in (0,1]$ and $h \in \mathcal{F}$. They show that $\succsim^u$ admits a Bewley representation (u, C) for some set of beliefs C.[11] Under the assumptions of Theorem 1, we have that $f \succeq^* g$ implies $f \succeq^u g$, or equivalently $C \subseteq P$. However, the opposite implication is typically not true. Thus, the unambiguous ranking $f \succsim^u g$ is necessary but not sufficient for the DM to deem f uncontroversially superior to g. As a result, Theorem 1 avoids the existence problem for finite-state α-MEU representations highlighted by Eichberger, Grant, Kelsey, and Koshevoy (2011): While requiring P to be the Bewley set of $\succsim^u$ implies that either $\alpha = 1$ (maxmin), $\alpha = 0$ (maxmax), or P is a singleton (subjective expected utility), Theorem 1 imposes no such restrictions. ▲

Finally, strengthening security-potential dominance as follows characterizes the extreme cases of objectively founded maxmin ($\alpha = 1$) and maxmax ($\alpha = 0$) expected utility:

Axiom 7 (Security dominance)**.** *If $f, g \in \mathcal{F}$ and f is more secure than g, then $f \succsim^\wedge g$.*

Axiom 8 (Potential dominance)**.** *If $f, g \in \mathcal{F}$ and f has more potential than g, then $f \succsim^\wedge g$.*

Corollary 1. *The following are equivalent:*

(i). $\succsim^*$ *is a Bewley preference,* $\succsim^\wedge$ *is an invariant biseparable preference, and the pair* $(\succsim^\wedge, \succsim^*)$ *jointly satisfies security (resp. potential) dominance.*

(ii). $(\succsim^*, \succsim^\wedge)$ *admits an objectively founded α-MEU representation (u, P, α) with $\alpha = 1$ (resp. $\alpha = 0$).*

Corollary 1 provides an alternative to GMMS's foundation for maxmin expected utility completions. In particular, imposing security dominance on the completion rule is equivalent (given Axioms 1–5) to caution and consistency.

4.2 Comparative statics

The unique identification of the parameters α and P in Theorem 1 behaviorally founds their interpretation as ambiguity attitude and perception and motivates conducting comparative statics. Consider two individuals $(\succsim_i^\wedge, \succsim_i^*)_{i=1,2}$ with objectively founded α-MEU representations $(u_i, P_i, \alpha_i)_{i=1,2}$. The belief sets (and utilities) are fully determined by the objective

[11] They also use the derived relation $\succsim^u$ to characterize the special case of α-MEU where the belief set P equals the induced C: Their Proposition 19 shows that $\succsim^\wedge$ admits such an α-MEU representation if and only if it is invariant biseparable and $C^u(f) = C^u(g)$ implies $f \sim^\wedge g$, where $C^u(f) := \{p \in \Delta(Z) : \forall q \in \Delta(Z),\ [q \succsim^u f \implies q \succsim^u p]\ \&\ [f \succsim^u q \implies p \succsim^u q]\}$ is the set of unambiguous certainty equivalents of f.

Bewley preferences $\succsim_i^*$, and the comparative statics of P_i are well-understood.[12] Moreover, when $u_1 \approx u_2$ and $P_1 = P_2$, standard arguments imply that $\alpha_1 \geq \alpha_2$ if and only if individual 1's subjective preference is *more ambiguity averse* than individual 2's (Ghirardato and Marinacci, 2002), in the sense that for all $p \in \Delta(Z)$ and $f \in \mathcal{F}$,

$$p \succsim_2^\wedge f \Longrightarrow p \succsim_1^\wedge f. \tag{6}$$

We now show how, by considering both subjective and objective preferences, one can compare ambiguity attitudes α_i across individuals whose perceived ambiguity P_i need not be the same.

Definition 2. *We call* $(\succsim_1^\wedge, \succsim_1^*)$ ***more security oriented*** *than* $(\succsim_2^\wedge, \succsim_2^*)$ *if the following condition holds: Whenever* $f, g \in \mathcal{F}$ *are such that for all* $p \in \Delta(Z)$, $f \succeq_1^* p \Leftrightarrow g \succeq_2^* p$ *and* $p \succeq_1^* f \Leftrightarrow p \succeq_2^* g$, *then for any* $p \in \Delta(Z)$,

$$p \succsim_2^\wedge g \Longrightarrow p \succsim_1^\wedge f.$$

Suppose that in terms of objective comparisons against constant acts, individual 1 ranks act f the same way as individual 2 ranks act g. Thus, objectively, f has the same level of security and potential for individual 1 as act g has for individual 2. If, subjectively, individual 1 is more inclined to prefer constant acts over f than individual 2 is to prefer constant acts over g, this suggests that individual 1's choices are more driven by security considerations than individual 2's. This is the content of Definition 2. Note that when $\succsim_1^* = \succsim_2^*$, more security orientation implies that $\succsim_1^\wedge$ is more ambiguity averse than $\succsim_2^\wedge$ in the sense of (6).

The following result shows that for a fixed utility u, a higher α corresponds to more security orientation, even across individuals with different belief sets. In Section 4.3, we will use this result to compare updated preferences $\succsim_E^\wedge$ across different events E.

Proposition 2. *Suppose* $(\succsim_i^\wedge, \succsim_i^*)_{i=1,2}$ *admit objectively founded* α*-MEU representations* $(u_i, P_i, \alpha_i)_{i=1,2}$, *where* $u_1 \approx u_2$ *and* P_i *is not a singleton for* $i = 1, 2$. *The following are equivalent:*

(i). $(\succsim_1^\wedge, \succsim_1^*)$ *is more security oriented than* $(\succsim_2^\wedge, \succsim_2^*)$.

(ii). $\alpha_1 \geq \alpha_2$.

[12]In particular, if $u_1 \approx u_2$, then $P_1 \subseteq P_2$ if and only if $\succsim_2^* \subseteq \succsim_1^*$.

4.3 Dynamic extension

Finally, the fact that the objectively founded α-MEU model uniquely determines a set of priors makes it possible to provide preference foundations for a dynamic extension of the model, avoiding the problem highlighted in Example 1. Fix ex-ante subjective and objective preferences $(\succsim^\wedge, \succsim^*)$ that admit an objectively founded α-MEU representation whose unique belief set is P. Let $\mathcal{E}_P$ denote the collection of events $E \subseteq S$ such that $\mu(E) > 0$ for all $\mu \in P$.[13] For any $E \in \mathcal{E}_P$, denote by $(\succsim^\wedge_E, \succsim^*_E)$ the DM's subjective and objective preferences conditional on learning that the true state is in E. In this section, we characterize when $(\succsim^\wedge_E, \succsim^*_E)$ is induced by an objectively founded α-MEU representation whose conditional set of beliefs P^E is the prior-by-prior Bayesian update (5) of the ex-ante set P.

Given any $E \in \mathcal{E}_P$, we impose intertemporal restrictions that relate the conditional subjective and objective preferences $(\succsim^\wedge_E, \succsim^*_E)$ to the ex-ante objective preference $\succsim^*$. For any $f, g \in \mathcal{F}$, let $f_E g$ denote the act such that $f_E g(s) = f(s)$ for all $s \in E$ and $f_E g(s) = g(s)$ for all $s \notin E$. Our intertemporal restriction on objective preferences is the same as in Faro and Lefort (2019):

Axiom 9 (Objective dynamic consistency). *If $f, g \in \mathcal{F}$, then*

$$f_E g \succsim^* g \iff f \succsim^*_E g.$$

Axiom 9 requires that if at the ex-ante stage, the DM deems f uncontroversially better than g when comparing their outcomes in event E, then ex post, upon learning that event E has realized, the DM continues to consider f uncontroversially better than g, and vice versa.

For the relationship between $\succsim^*$ and $\succsim^\wedge_E$, we impose an intertemporal analog of security-potential dominance. Call f ***more secure than*** g ***at*** E if for all $p \in \Delta(Z)$,

$$g_E p \succsim^* p \Longrightarrow f_E p \succsim^* p.$$

Likewise, f ***has more potential than*** g ***at*** E if for all $p \in \Delta(Z)$,

$$p \not\succsim^* g_E p \Longrightarrow p \not\succsim^* f_E p.$$

Axiom 10 (Intertemporal security-potential dominance). *If $f, g \in \mathcal{F}$ and f is both more secure than g at E and has more potential than g at E, then $f \succsim^\wedge_E g$.*

Axiom 10 requires that if at the ex-ante stage, act f offers both more security and more potential than g when only considering their outcomes in event E, then ex post, upon

[13] Note that $E \in \mathcal{E}_P$ if and only if there exist $p, q, r \in \Delta(Z)$ such that $p_E q \succsim^* r$ and $r \succ^* q$.

learning that event E has realized, the DM will choose f over g.

These conditions yield a foundation for the following dynamic extension of the objectively founded α-MEU model:

Theorem 2. *Suppose $(\succsim^\wedge, \succsim^*)$ admits an objectively founded α-MEU representation (u, P, α). Then for any $E \in \mathcal{E}_P$, the following are equivalent:*

(i). $\succsim^\wedge_E$ is an invariant biseparable preference and the pair $(\succsim^*, \succsim^\wedge_E)$ jointly satisfies intertemporal security-potential dominance.

(ii). There exists $\alpha_E \in [0,1]$ such that (u, P^E, α_E) is an α-MEU representation of $\succsim^\wedge_E$.

Moreover, α_E is unique if there exist $f, g \in \mathcal{F}$ such that f and $g_E f$ are not $\succsim^*$-comparable. Finally, $(\succsim^*, \succsim^*_E)$ satisfies objective dynamic consistency if and only if (u, P^E) is a Bewley representation of $\succsim^*_E$.

Thus, intertemporal security-potential dominance (along with the usual assumption that subjective preferences are invariant biseparable) characterizes when the conditional subjective preferences $(\succsim^\wedge_E)_{E \in \mathcal{E}_P}$ admit α-MEU representations with the same utility index u as the ex-ante preferences and with sets of beliefs P^E that are prior-by-prior Bayesian updates of the ex-ante set P. At the same time, as in Ghirardato, Maccheroni, and Marinacci (2008) and Faro and Lefort (2019), objective dynamic consistency characterizes when the conditional objective preferences $(\succsim^*_E)_{E \in \mathcal{E}_P}$ admit Bewley representations (u, P^E). Note that the subjective preferences $\succsim^\wedge$ and $\succsim^\wedge_E$ need not satisfy dynamic consistency.[14]

Note also that the unique weights α_E in the conditional α-MEU representations can vary across different events E. Thus, intertemporal security-potential dominance allows for the possibility that the DM's ambiguity attitude might be affected by the nature of the information he obtains—for example, ambiguity attitudes might differ following "surprising" (low ex-ante likelihood) vs. "unsurprising" events.[15]

Proposition 2 makes it possible to compare how different information affects the DM's ambiguity attitude: In particular, $\alpha_E \geq \alpha_F$ if and only if information E makes the DM more security oriented than information F in the sense of Definition 2. Imposing this condition

[14] Beissner, Lin, and Riedel (2020) consider prior-by-prior updating of a given α-MEU representation and show that, in contrast with maxmin expected utility, imposing Epstein and Schneider's (2003) rectangularity condition on belief sets is not sufficient to ensure dynamic consistency. Siniscalchi (2011) provides a general analysis of dynamic choice without dynamic consistency.

[15] For neo-additive capacities, Eichberger, Grant, and Kelsey (2010) show that the Dempster-Shafer (resp. Optimistic) updating rule renders conditional preferences following *all* events uniformly more ambiguity-averse (resp. ambiguity-seeking). Dillenberger and Rozen (2015) explore history-dependent *risk* attitudes. They focus on the effect of past payoff realizations, as opposed to realized information.

on all events E and F characterizes the history-independent case, where all α_E are equal to the ex-ante weight $\alpha = \alpha_S$:

Corollary 2. *Suppose that $(\succsim_E^{\wedge}, \succsim_E^{*})$ admits an objectively founded α-MEU representation (u, P^E, α_E) for each $E \in \mathcal{E}_P$. Then for each $E, F \in \mathcal{E}_P$ such that $\succsim_E^{*}$ and $\succsim_F^{*}$ are incomplete, the following are equivalent:*

(i). $(\succsim_E^{\wedge}, \succsim_E^{*})$ is more security oriented than $(\succsim_F^{\wedge}, \succsim_F^{*})$.

(ii). $\alpha_E \geq \alpha_F$.

A Proofs

A.1 Preliminaries

Call a functional $I : \mathbb{R}^S \to \mathbb{R}$ *monotonic* if $I(\phi) \geq I(\psi)$ for all $\phi, \psi \in \mathbb{R}^S$ with $\phi \geq \psi$; *normalized* if $I(\underline{a}) = a$ for all $a \in \mathbb{R}$, where $\underline{a} \in \mathbb{R}^S$ denotes the vector with $\underline{a}(s) = a$ for all $s \in S$; *constant-additive* if $I(\phi + \underline{a}) = I(\phi) + a$ for all $\phi \in U$ and $a \in \mathbb{R}$; *positively homogeneous* if $I(a\phi) = aI(\phi)$ for all $\phi \in U$ and $a \in \mathbb{R}_+$; and *constant-linear* if I is constant-additive and positively homogeneous. It is easy to see that any constant-linear functional I on $\mathbb{R}^S$ is normalized.

Throughout this appendix, for any non-empty, closed and convex $P \subseteq \Delta(S)$ and $\phi \in \mathbb{R}^S$, let

$$M_P(\phi) := \max_{\mu \in P} \phi \cdot \mu, \quad m_P(\phi) := \min_{\mu \in P} \phi \cdot \mu.$$

The following lemma, which is used in the proof of Theorem 1, provides a condition under which a functional I can be expressed as a weighted sum of two α-MEU functionals. The proof employs a similar argument as Lemma B.5 in Ghirardato, Maccheroni, and Marinacci (2004), which considers weighted sums of maxmin and maxmax ($\alpha' = 1$, $\alpha'' = 0$).

Lemma A.1. *Fix any $\alpha', \alpha'' \in [0, 1]$ with $\alpha' > \alpha''$ and any non-empty, convex, and compact $P \subseteq \Delta(S)$, and define the functionals $I', I'' : \mathbb{R}^S \to \mathbb{R}$ by*

$$I'(\phi) := \alpha' m_P(\phi) + (1 - \alpha')M_P(\phi), \quad I''(\phi) := \alpha'' m_P(\phi) + (1 - \alpha'')M_P(\phi), \forall \phi \in \mathbb{R}^S.$$

Consider any constant-linear functional $I : \mathbb{R}^S \to \mathbb{R}$ such that for all $\phi, \psi \in \mathbb{R}^S$,

$$[I'(\phi) \geq I'(\psi) \text{ and } I''(\phi) \geq I''(\psi)] \Longrightarrow I(\phi) \geq I(\psi). \tag{7}$$

Then there exists $\beta \in [0,1]$ *such that for each* $\phi \in \mathbb{R}^S$,

$$I(\phi) = \beta I'(\phi) + (1-\beta) I''(\phi). \tag{8}$$

Proof. By (7), there exists some functional $W : \mathbb{R}^2 \to \mathbb{R}$ such that I satisfies $I(\phi) = W(I'(\phi), I''(\phi))$ for all $\phi \in \mathbb{R}^S$. Moreover, (7) and monotonicity of I' and I'' implies that I is monotonic.

Suppose first that $P = \{\mu\}$ is a singleton. Then $I'(\phi) = I''(\phi) = \mu \cdot \phi$ for all ϕ. By (7), I is ordinally equivalent to I' (and hence linear): Indeed, $I'(\phi) \geq I'(\psi)$ implies $I(\phi) \geq I(\psi)$ by assumption; moreover, if $I'(\phi) > I'(\psi)$, then $I(\phi) > I(\psi)$, because if $I(\phi) \leq I(\psi)$, then for $\varepsilon > 0$ small enough, we have $I(\phi) < I(\psi + \underline{\varepsilon})$ and $I'(\phi) > I'(\psi + \underline{\varepsilon})$ by constant-additivity of I and I', contradicting (7). Since I is normalized (as it is constant linear), this implies that $I = I' = I''$. Thus, (8) holds for any $\beta \in [0,1]$.

Suppose next that P is not a singleton. Then there exists $\phi \in \mathbb{R}^S$ such that $M_P(\phi) > m_P(\phi)$, and hence $I'(\phi) < I''(\phi)$. For any such ϕ, there exists a unique $\beta(\phi) \in \mathbb{R}$ such that

$$I(\phi) = \beta(\phi) I'(\phi) + (1-\beta(\phi)) I''(\phi).$$

In particular,

$$\beta(\phi) = \frac{I(\phi) - I''(\phi)}{I'(\phi) - I''(\phi)} = -\frac{I(\phi) - I''(\phi)}{I''(\phi) - I'(\phi)} = -I(\psi),$$

where $\psi := \frac{\phi - \underline{I''(\phi)}}{I''(\phi) - I'(\phi)}$ and the last equality holds since I is constant linear. Note that $I'(\psi) = -1$ and $I''(\psi) = 0$. Thus, $\beta(\phi) = -W(-1,0) := \beta$, which does not depend on ϕ. Hence, for β thus defined, (8) holds for all ϕ with $I'(\phi) < I''(\phi)$. By a continuity argument, (8) holds for all $\phi \in \mathbb{R}^S$.[16]

Finally, we show that $\beta \in [0,1]$. Suppose that $\beta < 0$. Then for any ϕ such that $I'(\phi) < I''(\phi)$, we have $I(\phi) > I''(\phi) = I(\underline{I''(\phi)})$, which contradicts (7), as $I'(\underline{I''(\phi)}) = I''(\phi) > I'(\phi)$ and $I''(\underline{I''(\phi)}) = I''(\phi)$. If $\beta > 1$, we obtain a contradiction in an analogous manner. $\square$

The following lemma shows that for a given α, the sets of priors in the α-MEU functional are uniquely identified:

Lemma A.2. *Fix any* $\alpha \in [0,1]$ *with* $\alpha \neq 1/2$ *and any non-empty, closed and convex* $P_1, P_2 \subseteq \Delta(S)$. *For* $i = 1,2$ *and each* $\phi \in \mathbb{R}^S$, *let* $I_i(\phi) := \alpha m_{P_i}(\phi) + (1-\alpha) M_{P_i}(\phi)$. *If* $I_1(\phi) = I_2(\phi)$ *for all* $\phi \in \mathbb{R}^S$, *then* $P_1 = P_2$.

[16] Specifically, consider any ϕ with $I'(\phi) = I''(\phi)$. Then $\mu \cdot \phi = \mu' \cdot \phi$ for any $\mu, \mu' \in P$. Hence, for any ψ with $I'(\psi) < I''(\psi)$ and any $\lambda \in (0,1)$, we have $I'(\lambda\phi + (1-\lambda)\psi) = \lambda I'(\phi) + (1-\lambda) I'(\psi) < \lambda I''(\phi) + (1-\lambda) I''(\psi) = I''(\lambda\phi + (1-\lambda)\psi)$. Thus, (8) holds for $\lambda\phi + (1-\lambda)\psi$. Since this is true for all $\lambda \in (0,1)$, continuity of I, I', I'' implies that (8) holds for ϕ.

Proof. Take any $\phi \in \mathbb{R}^S$. Then for each $i = 1, 2$,

$$I_i(-\phi) = -\alpha M_{P_i}(\phi) - (1-\alpha) m_{P_i}(\phi).$$

But $I_1(\phi) = I_2(\phi)$ and $I_1(-\phi) = I_2(-\phi)$ implies $M_{P_1}(\phi) = M_{P_2}(\phi)$, because

$$\begin{aligned}
& (1-\alpha) I_1(\phi) + \alpha I_1(-\phi) = (1-\alpha) I_2(\phi) + \alpha I_2(-\phi) \\
\Longleftrightarrow \quad & (1-2\alpha) M_{P_1}(\phi) = (1-2\alpha) M_{P_2}(\phi) \\
\Longleftrightarrow \quad & M_{P_1}(\phi) = M_{P_2}(\phi),
\end{aligned}$$

where the last equivalence uses $\alpha \neq 1/2$. Since this is true for any $\phi \in \mathbb{R}^S$, the support functions of P_1 and P_2 coincide, which implies $P_1 = P_2$. □

The following lemma, which is used in the proof of Proposition 1, characterizes γ-expansions in terms of the relationship between the corresponding support functions.

Lemma A.3. *Consider two non-empty, closed and convex sets $P, Q \subseteq \Delta(S)$, and $\gamma \geq 1$. Then Q is the γ-expansion of P if and only if, for each $\phi \in \mathbb{R}^S$,*

$$M_P(\phi) = \frac{\gamma}{2\gamma-1} M_Q(\phi) + \frac{\gamma-1}{2\gamma-1} m_Q(\phi), \quad m_P(\phi) = \frac{\gamma-1}{2\gamma-1} M_Q(\phi) + \frac{\gamma}{2\gamma-1} m_Q(\phi). \tag{9}$$

Proof. Suppose first that Q is the γ-expansion of P. Then $\mu \in Q$ if and only if $\mu = \gamma\nu + (1-\gamma)\nu'$ for some $\nu, \nu' \in P$. Since $\gamma \geq 1$, this implies that for any $\phi \in \mathbb{R}^S$,

$$M_Q(\phi) = \gamma M_P(\phi) + (1-\gamma) m_P(\phi), \quad m_Q(\phi) = \gamma m_P(\phi) + (1-\gamma) M_P(\phi).$$

Solving this system yields (9).

Conversely, suppose (9) holds for all $\phi \in \mathbb{R}^S$. For any $s \in S$, define $\phi^s \in \mathbb{R}^S$ by $\phi^s(s) = 1$ and $\phi^s(s') = 0$ for each $s' \neq s$. By (9), we have

$$\begin{aligned}
\gamma \min_{\nu \in P} \nu(s) &= \gamma m_P(\phi^s) = \frac{\gamma^2}{2\gamma-1} m_Q(\phi^s) + \frac{\gamma(\gamma-1)}{2\gamma-1} M_Q(\phi^s) \\
&\geq \frac{(\gamma-1)^2}{2\gamma-1} m_Q(\phi^s) + \frac{\gamma(\gamma-1)}{2\gamma-1} M_Q(\phi^s) \quad \text{since } \gamma^2 \geq (\gamma-1)^2 \text{ and } m_Q(\phi^s) \geq 0 \\
&= (\gamma-1) M_P(\phi^s) \\
&= (\gamma-1) \max_{\nu \in P} \nu(s).
\end{aligned}$$

This shows that for any $s \in S$ and $\nu, \nu' \in P$, we have $\gamma\nu(s) + (1-\gamma)\nu'(s) \geq 0$. Thus, $P^\gamma := \gamma P + (1-\gamma) P$ is a subset of $\Delta(S)$. Moreover, P^γ is non-empty (since it contains P), closed, and convex. Hence, for any $\phi \in \mathbb{R}^S$, we have

$$\frac{\gamma}{2\gamma-1}M_{P^\gamma}(\phi)+\frac{\gamma-1}{2\gamma-1}m_{P^\gamma}(\phi)=M_P(\phi)=\frac{\gamma}{2\gamma-1}M_Q(\phi)+\frac{\gamma-1}{2\gamma-1}m_Q(\phi),$$

where the first equality holds by the "only if" direction of the lemma and the second by (9). By Lemma A.2, this implies $Q = P^\gamma$, that is, Q is the γ-expansion of P. □

A.2 Proof of Proposition 1

"If" direction. For each $\phi \in \mathbb{R}^S$ and $i = 1, 2$, let $M_i(\phi) := \max_{\mu \in P_i} \phi \cdot \mu$, $m_i(\phi) := \min_{\mu \in P_i} \phi \cdot \mu$, and let $I_i(\phi) := \alpha_i m_i(\phi) + (1-\alpha_i)M_i(\phi)$. Suppose case 1 in the proposition holds; the argument for case 2 is analogous. Note that since $\alpha_2 \geq \alpha_1 > \frac{1}{2}$, we have $\gamma := \frac{\alpha_1+\alpha_2-1}{2\alpha_1-1} \geq 1$. Since P_1 is the γ-expansion of P_2, Lemma A.3 implies that for any $\phi \in \mathbb{R}^S$,

$$M_2(\phi)=\frac{\gamma}{2\gamma-1}M_1(\phi)+\frac{\gamma-1}{2\gamma-1}m_1(\phi),\ \ m_2(\phi)=\frac{\gamma}{2\gamma-1}m_1(\phi)+\frac{\gamma-1}{2\gamma-1}M_1(\phi).$$

Then, for any $\phi \in \mathbb{R}^S$,

$$\begin{aligned}&\alpha_2 m_2(\phi)+(1-\alpha_2)M_2(\phi)\\ =&\alpha_2[\frac{\gamma}{2\gamma-1}m_1(\phi)+\frac{\gamma-1}{2\gamma-1}M_1(\phi)]+(1-\alpha_2)[\frac{\gamma}{2\gamma-1}M_1(\phi)+\frac{\gamma-1}{2\gamma-1}m_1(\phi)]\\ =&\alpha_1 m_1(\phi)+(1-\alpha_1)M_1(\phi),\end{aligned}$$

as $\alpha_2\gamma/(2\gamma-1)+(1-\alpha_2)(\gamma-1)/(2\gamma-1) = \alpha_1$ by the definition of γ. Thus, given $u_1 \approx u_2$, (u_1, P_1, α_1) and (u_2, P_2, α_2) represent the same preference.

"Only if" direction. Assume that (u_1, P_1, α_1) and (u_2, P_2, α_2) with $\alpha_1 \leq \alpha_2$ represent the same preference, and again let $M_i(\phi) := \max_{\mu \in P_i} \phi \cdot \mu$, $m_i(\phi) := \min_{\mu \in P_i} \phi \cdot \mu$ for $i = 1, 2$ and any $\phi \in \mathbb{R}^S$. Standard arguments imply $u_1 \approx u_2$. Suppose that $\alpha_1 < 1/2$ (the case $\alpha_1 > 1/2$ is analogous). Since P_i is not a singleton for $i = 1, 2$, Corollary 2 in Frick, Iijima, and Le Yaouanq (2019) implies that $\alpha_2 < \frac{1}{2}$.[17]

[17] Given any invariant biseparable preference $\succsim$ and event $E \subseteq S$, define the *matching probability* $m(E) \in [0, 1]$ of E by the indifference condition $x_E y \sim m(E)\delta_x + (1-m(E))\delta_y$, where $x, y \in Z$ are any two outcomes such that $\delta_x \succ \delta_y$. Define the *ambiguity aversion index* of E by $AA(E) := 1 - m(E) - m(E^c)$. Corollary 2 in Frick, Iijima, and Le Yaouanq (2019) shows that if $\succsim$ admits an α-MEU representation (u, P, α) where P is not a singleton, then $\alpha \geq 1/2$ (resp. $\alpha \leq 1/2$) if and only if $AA(E) \geq 0$ (resp. $AA(E) \leq 0$) for all E. Since (u_i, P_i, α_i) represent the same preference $\succsim^\wedge$ and P_i is not a singleton, this implies that $\alpha_1 < 1/2$ if and only if $\alpha_2 < \frac{1}{2}$.

Arguments similar to the proof of Lemma A.2 show that, for any $\phi \in \mathbb{R}^S$,

$$\alpha_1 m_1(\phi) + (1-\alpha_1)M_1(\phi) = \alpha_2 m_2(\phi) + (1-\alpha_2)M_2(\phi)$$

and

$$(1-\alpha_1)m_1(\phi) + \alpha_1 M_1(\phi) = (1-\alpha_2)m_2(\phi) + \alpha_2 M_2(\phi).$$

Solving this system yields

$$M_1(\phi) = \frac{1-\alpha_1-\alpha_2}{1-2\alpha_1}M_2(\phi) + \frac{\alpha_2-\alpha_1}{1-2\alpha_1}m_2(\phi), \quad m_1(\phi) = \frac{1-\alpha_1-\alpha_2}{1-2\alpha_1}m_2(\phi) + \frac{\alpha_2-\alpha_1}{1-2\alpha_1}M_2(\phi)$$

which can be written as

$$M_1(\phi) = \frac{\gamma}{2\gamma-1}M_2(\phi) + \frac{\gamma-1}{2\gamma-1}m_2(\phi), \quad m_1(\phi) = \frac{\gamma}{2\gamma-1}m_2(\phi) + \frac{\gamma-1}{2\gamma-1}M_2(\phi),$$

where $\gamma := \frac{1-\alpha_1-\alpha_2}{1-2\alpha_2}$. By Lemma A.3, this implies that P_2 is the γ-expansion of P_1. $\square$

A.3 Proof of Theorem 1

We show that (i) implies (ii); verifying that (ii) implies (i) is standard. Since $\succsim^*$ is a Bewley preference, it admits a Bewley representation (u, P) (see, e.g., Theorem 1 in GMMS). Moreover, since $\succsim^\wedge$ is invariant biseparable, there exists a nonconstant and affine utility $v : \Delta(Z) \to \mathbb{R}$ and a unique constant-linear and monotonic functional $I : \mathbb{R}^S \to \mathbb{R}$ such that $I \circ u$ represents $\succsim^\wedge$ (see, e.g., Lemma 1 in Ghirardato, Maccheroni, and Marinacci (2004)). Note that for any $p, q \in \Delta(Z)$, security-potential dominance implies the three-way equivalence

$$u(p) \geq u(q) \iff p \text{ is more secure and has more potential than } q \iff v(p) \geq v(q).$$

Hence, $v \approx u$, and we can assume without loss that $v = u$.

Observe that f is more secure than g if and only if $\min_{\mu \in P} \mu \cdot u(f) \geq \min_{\mu \in P} \mu \cdot u(g)$. Likewise, f has more potential than g if and only if $\max_{\mu \in P} \mu \cdot u(f) \geq \max_{\mu \in P} \mu \cdot u(g)$. Hence, since $I \circ u$ represents $\succsim^\wedge$, security-potential dominance (and constant linearity of I) implies that for any $\phi, \psi \in \mathbb{R}^S$ with $\min_{\mu \in P} \mu \cdot \phi \geq \min_{\mu \in P} \mu \cdot \psi$ and $\max_{\mu \in P} \mu \cdot \phi \geq \max_{\mu \in P} \mu \cdot \psi$, we have $I(\phi) \geq I(\psi)$. Thus, Lemma A.1 yields some $\alpha \in [0, 1]$ such that

$$I(\phi) = \alpha \min_{\mu \in P} \mu \cdot \phi + (1-\alpha) \max_{\mu \in P} \mu \cdot \phi, \forall \phi \in \mathbb{R}^S.$$

This shows that (u, P, α) is an objectively founded α-MEU representation of $(\succsim^*, \succsim^\wedge)$.

For the moreover part, cardinal uniqueness of u and uniqueness of P follows from the uniqueness properties of Bewley representations (e.g., Theorem 1 in GMMS). Finally, whenever $\succsim^*$ is incomplete, then P is not a singleton. Thus, there exists $\phi \in \mathbb{R}^S$ with $\min_{\mu\in P} \mu\cdot\phi < \max_{\mu\in P} \mu \cdot \phi$. For any $\alpha' \neq \alpha$, this implies $\alpha' \min_{\mu\in P} \mu \cdot \phi + (1-\alpha') \max_{\mu\in P} \mu \cdot \phi \neq I(\phi)$. Hence, α is unique by uniqueness of I. □

A.4 Proof of Corollary 1

We show that (i) implies (ii); verifying the other direction is standard. Consider the case in which security dominance holds (the argument when potential dominance holds is analogous). By Theorem 1, $(\succsim^*, \succsim^\wedge)$ admits some objectively founded α-MEU representation (u, P, α). If P is a singleton, the representation does not depend on the value of α, and we can set $\alpha = 1$. If P is not a singleton, then there exist $f \in \mathcal{F}$ and $p \in \Delta(Z)$ such that $\min_{\mu\in P} \mu \cdot u(f) = u(p) < \max_{\mu\in P} \mu \cdot u(f)$. Thus, p is more secure than f, and hence by security dominance $p \succsim^\wedge f$. By the representation, this means

$$u(p) \geq \alpha \min_{\mu\in P} \mu \cdot u(f) + (1-\alpha) \max_{\mu\in P} \mu \cdot u(f),$$

which is only possible if $\alpha = 1$. □

A.5 Proof of Proposition 2

Observe first that there exist $\phi, \psi \in [-1, 1]^S$ such that $m_{P_1}(\phi) = m_{P_2}(\psi) < M_{P_1}(\phi) = M_{P_2}(\psi)$. Indeed, given that P_1 and P_2 are not singletons, there exist $\phi', \psi' \in \mathbb{R}^S$ such that $m_{P_1}(\phi') < M_{P_1}(\phi')$ and $m_{P_2}(\psi') < M_{P_2}(\psi')$. By setting $\phi := \varepsilon \frac{\phi' - m_{P_1}(\phi')}{M_{P_1}(\phi') - m_{P_1}(\phi')}$ and $\psi := \varepsilon \frac{\psi' - m_{P_2}(\psi')}{M_{P_2}(\psi') - m_{P_2}(\psi')}$ for a sufficiently small $\varepsilon > 0$, we obtain $\phi, \psi \in [-1, 1]^S$ and $m_{P_1}(\phi) = m_{P_2}(\psi) = 0 < \varepsilon = M_{P_1}(\phi) = M_{P_2}(\psi)$.

Since $u_1 \approx u_2$, we can assume without loss that $u_1 = u_2 =: u$ and (up to performing a suitable positive affine transformation) that $[-1, 1] \subseteq u(Z)$. Given this, consider any $f, g \in \mathcal{F}$, and observe that the equivalences $f \succeq_1^* p \Leftrightarrow g \succeq_2^* p$ and $p \succeq_1^* f \Leftrightarrow p \succeq_2^* g$ hold for all constant acts p, if and only if, $\min_{\mu\in P_1} \mu \cdot u(f) = \min_{\mu\in P_2} \mu \cdot u(g)$ and $\max_{\mu\in P_1} \mu \cdot u(f) = \max_{\mu\in P_2} \mu \cdot u(g)$. The equivalence of (i) and (ii) then follows from the fact that $\min_{\mu\in P_1} \mu \cdot u(f) = \min_{\mu\in P_2} \mu \cdot u(g) > \max_{\mu\in P_1} \mu \cdot u(f) = \max_{\mu\in P_2} \mu \cdot u(g)$ for some $f, g \in \mathcal{F}$, which follows from the observation in the previous paragraph. □

A.6 Proof of Theorem 2

Suppose $(\succsim^*, \succsim^\wedge)$ admits an objectively founded α-MEU representation (u, P, α). We will show that (i) implies (ii); verifying the other direction is standard. We will invoke the following lemma:

Lemma A.4. *Fix any $E \in \mathcal{E}_\mathcal{P}$ and $f, g \in \mathcal{F}$. Then f is more secure than g at E if and only if $\min_{\mu \in P^E} \mu \cdot u(f) \geq \min_{\mu \in P^E} \mu \cdot u(g)$, and f has more potential than g at E if and only if $\max_{\mu \in P^E} \mu \cdot u(f) \geq \max_{\mu \in P^E} \mu \cdot u(g)$.*

Proof. For any $\phi \in \mathbb{R}^S$ and $a \in \mathbb{R}$ define $\phi_E a \in \mathbb{R}^S$ by $\phi_E a(s) = \phi(s)$ if $s \in E$ and $\phi_E \psi(s) = a$ if $s \notin E$. Observe that

$$
\begin{aligned}
& f \text{ is more secure than } g \text{ at } E \\
\iff \quad & \left[\min_{\mu \in P} \mu \cdot (u(g)_E u(p)) \geq u(p) \implies \min_{\mu \in P} \mu \cdot (u(f)_E u(p)) \geq u(p)\right] \ \forall p \in \Delta(Z) \\
\iff \quad & \left[\min_{\mu \in P^E} \mu \cdot u(g) \geq u(p) \implies \min_{\mu \in P^E} \mu \cdot u(f) \geq u(p)\right] \ \forall p \in \Delta(Z) \\
\iff \quad & \min_{\mu \in P^E} \mu \cdot u(f) \geq \min_{\mu \in P^E} \mu \cdot u(g),
\end{aligned}
$$

where the second equivalence uses the fact that for any $\phi \in \mathbb{R}^S$ and $a \in \mathbb{R}$, we have $\min_{\mu \in P^E} \mu \cdot \phi \geq a$ if and only if $\min_{\mu \in P} \mu \cdot (\phi_E a) \geq a$. The argument when f has more potential than g is analogous. $\square$

Suppose $\succsim_E^\wedge$ is an invariant biseparable preference and the pair $(\succsim^*, \succsim_E^\wedge)$ jointly satisfies intertemporal security-potential dominance. Given Lemma A.4, the same argument as in Theorem 1 implies that there exists $\alpha_E \in [0, 1]$ such that (u, P^E, α_E) represents $\succsim_E^\wedge$. Moreover, α_E is unique if P^E is not a singleton. The latter holds if there exist $f, g \in \mathcal{F}$ and $\mu, \nu \in P$ such that $\mu^E \cdot (u(f) - u(g)) \geq 0$ and $\nu^E \cdot (u(f) - u(g)) < 0$, that is, if f and $g_E f$ are not $\succsim^*$-comparable.

Finally, the fact that $(\succsim^*, \succsim_E^*)$ satisfies objective dynamic consistency if and only if (u, P^E) is a Bewley representation of $\succsim_E^*$ follows from the same arguments as Theorem 1 in Ghirardato, Maccheroni, and Marinacci (2008) (see also Theorem 1 in Faro and Lefort, 2019). $\square$

A.7 Proof of Corollary 2

Immediate from Proposition 2. $\square$

B Additional results

The following result complements the identification result in Proposition 1 by covering the remaining case where $\alpha_i = 1/2$ for some i. Given two subsets A and B of $\Delta(S)$, we write $A - B := \{a - b : a \in A, b \in B\}$.

Proposition B.1. *Suppose $(u_1, P_1, 1/2)$ and (u_2, P_2, α_2) are α-MEU representations of $\succsim_1^\wedge$ and $\succsim_2^\wedge$, respectively, where P_i is not a singleton for $i = 1, 2$. Then $\succsim_1^\wedge = \succsim_2^\wedge$ if and only if $u_1 \approx u_2$, $\alpha_2 = 1/2$, and $P_1 - P_2 = P_2 - P_1$.*

Proof. The condition $u_1 \approx u_2$ is standard, and we can thus assume $u_1 = u_2 = u$ without loss of generality. Moreover, since $\alpha_1 = 1/2$ and the P_i are not singletons, Corollary 2 in Frick, Iijima, and Le Yaouanq (2019) (see Footnote 17) implies that if $\succsim_1^\wedge = \succsim_2^\wedge$, then $\alpha_2 = 1/2$.

Given that $\alpha_1 = \alpha_2 = 1/2$, the condition $\succsim_1^\wedge = \succsim_2^\wedge$ is equivalent to

$$\frac{1}{2} \max_{\mu \in P_1} \mathbb{E}_\mu[u(f)] + \frac{1}{2} \min_{\mu \in P_1} \mathbb{E}_\mu[u(f)] = \frac{1}{2} \max_{\mu \in P_2} \mathbb{E}_\mu[u(f)] + \frac{1}{2} \min_{\mu \in P_2} \mathbb{E}_\mu[u(f)]$$

for all acts f. Re-arranging yields

$$\max_{(\mu_1, \mu_2) \in P_1 \times P_2} \mathbb{E}_{\mu_1}[u(f)] - \mathbb{E}_{\mu_2}[u(f)] = \max_{(\mu_1, \mu_2) \in P_1 \times P_2} \mathbb{E}_{\mu_2}[u(f)] - \mathbb{E}_{\mu_1}[u(f)],$$

that is,

$$\max_{\mu \in P_1 - P_2} \mathbb{E}_\mu[u(f)] = \max_{\mu \in P_2 - P_1} \mathbb{E}_\mu[u(f)].$$

Since $P_1 - P_2$ and $P_2 - P_1$ are both closed and convex subsets of $\mathbb{R}^S$, the above property is true for all f if and only if $P_1 - P_2 = P_2 - P_1$. $\square$

The multiplicity of belief sets allowed by Proposition B.1 is greater than in Proposition 1. Indeed, $P_1 - P_2 = P_2 - P_1$ is satisfied if P_2 is the γ-expansion of P_1 (or vice versa) for some $\gamma \geq 1$, irrespective of the value of γ. However, in contrast with Proposition 1, the opposite implication is not true, as $P_1 - P_2 = P_2 - P_1$ can hold even if P_1 and P_2 are not nested. The following example illustrates this: Consider $|S| = 3$, take ε with $0 < \varepsilon < 1/3$, and define

$$P_1 := \{\mu \in \Delta(S) : \mu_1 = \frac{1}{3}, |\mu_2 - \frac{1}{3}| \leq \varepsilon\} \text{ and } P_2 := \{\mu \in \Delta(S) : |\mu_1 - \frac{1}{3}| \leq \varepsilon, \mu_2 = \frac{1}{3}\}.$$

The sets P_1 and P_2 are not nested but satisfy

$$P_1 - P_2 = P_2 - P_1 = \{(\nu_1, \nu_2, -\nu_1 - \nu_2) : |\nu_1| \leq \varepsilon, |\nu_2| \leq \varepsilon\}.$$

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