Speculative Trade and Market Newcomers

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Speculative Trade and Market Newcomers*,†

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Arguing that in the real world *relatively optimistic* inexperienced investors are prey for *relatively pessimistic* veteran traders, we formalize this intuitive conjecture as a proven proposition in a simple model. This agreement to disagree leads to a perpetual bubble, in which more experienced, but less optimistic, investors keep selling overpriced assets to less experienced traders. As in a fraction of the uniform-experience literature, lack of short-selling makes room for the success of such bubble schemes. This previous literature did not allow for persistent effects of experience on beliefs and, instead, relied on more direct assumptions of belief heterogeneity. Although we map experience into beliefs in a specific way, the intuition behind the perpetual bubble involves the above-mentioned disagreement patterns, not belief formation itself.

Keywords: speculative trade; price bubble; experience; optimism; belief heterogeneity; non-Bayesian learning; short-selling

JEL Codes: D8; D9; G1; G4

1. Introduction

Young or immigrant-turned investors are inexperienced compared to incumbent traders. On the one hand, the veteran investors may simply overwhelm the novices and act as price-stabilizers, in the sense of abating deviations of prices from fundamentals, or even

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burst bubbles. On the other hand, the experienced investors may systematically take advantage of newcomers and be themselves willing to pay a premium for an opportunity to resell the asset to a novice above the fundamental value. We present an intuitive sufficient condition for a perpetual bubble of this type and argue that our results extend beyond our deliberately simple, tractable model. They suggest that one of real-world manifestations of such a bubble is experience cohorts' agreement to disagree such that at least occasionally the most experienced are not the most optimistic about fundamentals. The more frequently the most experienced are relatively pessimistic as all traders learn about the true dividend-generating process of some asset, the more bubbly we expect this asset's prices to be.

Our grounds for such public disagreement about fundamentals in terms of the primitives are short-sales constraints that permit newcomers' prior beliefs to differ from the incumbents' already updated beliefs in equilibrium. If one insists on modeling rather than arbitrarily assuming differences in beliefs prior to trade, we do so in terms of possibly coarse and nonstatistical learning about the past and probabilistic learning while trading. Similar differentiation between learning about the past and from experience, but for different purposes, appeared in theoretical models of *Schraeder* (2015), *Collin-Dufresne et al.* (2016), and *Ehling et al.* (2017). To support our claim of generality, we proceed to giving an intuition independent of this specific way of obtaining belief heterogeneity (à la *Harrison and Kreps*, 1978; *Morris*, 1996; *Scheinkman and Xiong*, 2003; *Werner*, 2018).

For that, we first need to observe that asset prices will be at least as high as the (subjective) fundamental valuations of the most optimistic traders. As potential asset-holders, they should be able to anticipate, that, should the real economy even perform worse than they hope, at least at the end of their lives they might get a premium: By assumption, there is a chance of them being eventually relatively pessimistic, and thus having opportunities to sell assets above their updated fundamental valuations to more optimistic traders. These anticipations make today's the most optimistic willing to pay above their current fundamental valuations. The result is a perpetual bubble in the sense that always many asset prices (those affected) are strictly greater than the most optimistic fundamental valuations (maxima over all traders) of these assets.

Ultimately, the question whether the most experienced express more pessimistic views about fundamentals is empirical, but supporting empirical evidence of persistent age differences in investors' expectations exists. When stock prices were reaching two-year lows during the last week of March 2001, investors above age 60 were more likely to consider the stocks to be overvalued (39% vs. 25%) according to *Dreman et al.* (2001). They also compare this downturn period's survey measures of investors' expectations with those from a period of rapid rise in stock prices in 1998 and do not find significant differences. On a different note, in our model the most experienced are permanently the most pessimistic for a nonnegligible set of parameter values pertaining to traders' beliefs, not just relatively pessimistic, and not just occasionally.

Admittedly, perpetual bubbles fueled by repeated trade between veteran investors and novices, as under our condition, are only one aspect of the interaction between different experience cohorts in financial markets. We think that such perpetual bubbles may coexist with a significant degree of price stabilization and bubble bursting attributable to actions of relatively experienced investors, as in a fraction of the previous literature. For instance, experience as a source of pessimism helped explain the dotcom bubble burst—a record number of IPO lockups expired, bringing relatively pessimistic asset-holders back to the market (*Ofek and Richardson*, 2003). However, this conclusion does not imply that a bubble cannot have a perpetual component with boom-and-bust phases, which we do not model, on top of it. This also reconciles the perpetual bubbles with the experimental findings that inexperienced traders' entry into the market does not cause "bubble-crash phenomena" (*Dufwenberg et al.*, 2005; *Xie and Zhang*, 2016). Such a causal effect is theoretically possible, though, if the inexperienced behave like positive-feedback-investment strategists (noise traders that buy high and sell low) in *de Long et al.* (1990).

Our question about trade between veteran investors and novices is indeed similar to the one of rational speculators versus noise traders in *de Long et al.* (1990). Unlike them, we focus on perpetual bubbles rather than temporary ones and do not assume that newcomers follow the noise traders' "really dumb" positive-feedback strategies. Nevertheless, a part of how we model disagreement between veterans and novices—possibly coarse and nonstatistical learning about history—likewise helped *de Long et al.* (1990) justify noise traders' long-run relevance:

> By the time the new bubble comes along, many investors have forgotten the old one or have been replaced by younger investors who have never experienced the old one at all.

To capture experience acquisition and learning, we adopt (and adapt) the framework of *Harrison and Kreps* (1978), where all traders, unlike in *de Long et al.* (1990), react to current prices. In the former model, traders' fully rational consideration of all available information may not prevent overpricing if their prior beliefs are suitably heterogeneous and short-selling is limited. Our approach is to introduce dispersed timing of market entry and to assume that newcomers form prior beliefs over the future at time of entry but not over the entire history, learning about the past not via updating. This accommodates heterogeneous beliefs (at a given time) even though traders form identical beliefs at time of entry—the histories prior to entry are identical—and learn identically as time goes by: The source of disagreement in our symmetric-information setting is merely the variation in the duration of market participation and probabilistic learning. Our assumption of identical newcomers permits the formal identification of optimism as a function of cohort with that as a function of a single trader's experience: The traders' identical beliefs at time of market entry induce identical fundamental valuations of a single asset as functions of future time (Section 4). When this fundamental valuation function is strictly decreasing—i.e., younger traders value the asset more—our model predicts perpetual bubbles (Section 5). Depending on the shape parameter pertaining to the beliefs, our model accommodates two more (three in total) shapes of the fundamental valuation function: constant and strictly increasing. These last two are in a way control scenarios that allow us to pin down the significance of relatively pessimistic veteran traders as a part of our sufficient condition: When more experienced traders value the asset at least as much as less experienced do, bubbles are absent (Section 6).

We introduce the basic building blocks for our model—the asset, the traders, the dynamics of their fundamental valuations, and their market participation—in Sections 2.1–2.2. It accommodates different trading frequencies (via a parameter), as we describe in Section 2.3 along with the notion of Harrison-Kreps equilibrium, where traders balance between keeping their dividend entitlement and timely liquidation (Section 3). The equilibrium prices are unique but, as we have partially indicated above, can be of three types (Sections 5–6), depending on which traders are the most optimistic (Section 4). If newcomers are, then the highest-valuation status perpetually switches from trader to trader, which is sufficient for bubbles in the (previous) uniform-experience literature too (Section 5.4).

2. Model

Despite being similar in spirit to the papers in the framework of *Harrison and Kreps* (1978), with recent contributions by *Steiner and Stewart* (2015) and *Werner* (2018), our setting is novel and one of the most tractable. We model time as continuous with infinite forward and backward horizons. We identify traders with their market-entry times (one trader per time point). They are risk-neutral and maximize expected discounted returns of holding the asset under uncertainty about the dividend. In Section 2.1 below, we start describing the model in detail from time 0.

2.1. Asset, Traders, and Valuations

At time 0, a trader enters the market for an asset and forms beliefs that the asset pays a dividend of \$1 per unit at uncertain time $\theta \in (0, \infty)$, but for simplicity the truth is that this asset does not pay dividends at all. Our model applies when the asset pays a dividend, but we need to reinterpret many things (for example, the bubble) as conditional on the dividend being unpaid. The total number of units of the asset is finite and strictly positive.

The beliefs of this time-0 entrant about the dividend time $\theta \in (0, \infty)$ are Gamma (α, β), which has density $f_{\alpha,\beta}$, cumulative distribution $F_{\alpha,\beta}$, and hazard function $H_{\alpha,\beta}$ defined, each on $(0, \infty)$ into $\mathbb{R}$, by

$$f_{\alpha,\beta}(\theta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\theta^{\alpha-1}e^{-\frac{\theta}{\beta}}, \; F_{\alpha,\beta}(t) = \int_0^t f_{\alpha,\beta}(\theta)\, d\theta, \text{ and } H_{\alpha,\beta}(t) = \frac{f_{\alpha,\beta}(t)}{1 - F_{\alpha,\beta}(t)},$$

while we let $F_{\alpha,\beta}(0) = 0$. We adopt the interpretation according to which these beliefs incorporate information from the past and current times $(-\infty, 0]$, but we neither model this information processing nor require it to be rational in any sense. In particular, we do not view these beliefs as a Bayesian update of longer-horizon beliefs. Since this is the moment the trader just enters the market, the idea that the trader's Bayesian statistical model does not cover the past is hard to dismiss without finding a grain of truth. This spares the trader forming probabilistic beliefs about no longer random, at least to the extent that they have occurred, events. Admitting that rationality and timing of prior formation are philosophical questions, we remind the reader that the intuition behind the bubbles in question relies on the induced valuations, not on belief formation per se.

To define the trader's fundamental valuation as a measure of the trader's willingness to pay for the asset if obliged to hold it forever, we assume that the trader can borrow and lend at a constant rate $r \in (0, \infty)$. The fundamental valuation is a function of time, through probabilistic learning. Since the trader learns that the asset does not pay, the (conditional) fundamental valuation of the asset at time $t \geq 0$ is the (per-unit) expected discounted dividend

$$V_0(t) = \frac{1}{1 - F_{\alpha,\beta}(t)} \int_t^{\infty} e^{-r(\theta - t)} f_{\alpha,\beta}(\theta)\, d\theta < 1, \tag{1}$$

strictly smaller than the dividend amount. We document basic properties of this valuation function $V_0 : [0, \infty) \to \mathbb{R}$, stating and proving some of which requires the hazard function $H_{\alpha,\beta}$, in Section 4 and Appendix B.

At each time point $\tau \in \mathbb{R} \backslash \{0\}$, an identical trader enters the market after past entrants observe that the asset does not pay the dividend at this time τ and forms beliefs that the asset pays at uncertain time $\theta \in (\tau, \infty)$. This trader's uncertainty is analogous to the time-0 entrant's. Since we will look at a steady-state price (Section 2.3), all traders will find themselves in exactly time-0 entrant's position at entry. This allows us to assume that they form their beliefs and valuations in the same way, starting the same probabilistic learning at entry. For every time-τ entrant, who can also borrow and lend at the rate r, the fundamental valuation of the asset at time $t \geq \tau$, in $t - \tau$ time units from entry, is

$$V_\tau(t) = V_0(t - \tau),$$

equal to the time-0 entrant's in $t - \tau$ time units from entry. This fundamental valuation gives their willingness to pay for the asset in $t - \tau$ units from entry if obliged to hold the asset forever.

2.2. Market Participation

All traders stay in the market for a fixed duration $T \in (0, \infty)$: Every time-τ entrant leaves the market at time $\tau + T$ after the new trader enters, giving us the compact set $[\tau, \tau + T]$ of potential asset-holders at $\tau + T$. For simplicity, we suppose that these finitely lived traders can plan to hold the asset forever for the sake of a paternalistic bequest but do not model intergenerational transfers explicitly.

An alternative is to assume that traders do not know for how long they live, but this approach would call for the inclusion of traders' beliefs about their survival in the calculations of fundamental valuations. In this case, a perpetual bubble would mean that the price always exceeds all traders' survival-adjusted fundamental valuations, as opposed to all traders' pure expectations of discounted returns under our assumption. A third possibility is to endogenize fundamental valuations by using expected discounted returns of holding the asset only all the way until, but with the option to sell right before, the exit moment, known to traders. Here, similarly to the other cases, a bubble would entail plans to sell the asset strictly sooner even though this means lower expected discounted dividends. Although we allow traders to plan to hold the asset forever but to be finitely lived, the bubbly prices in our model are bubbly even relative to shorter holding horizons or departed traders (Section 5.2).

2.3. Trade, Equilibrium Prices, and Their Existence

We focus on a steady-state nominal per-unit price $p \in \mathbb{R}$, which is compelling for our simple stationary set-up and our emphasis on perpetual bubbles. As done in other papers in the framework of *Harrison and Kreps* (1978), we assume from the outset that traders cannot sell the asset short. This allows us to make every trader's demand for the asset, when the trader is in the market, a simple function of the trader's reservation prices, which are expected discounted returns of holding the asset. We allow traders to plan to resell, only conditional on the dividend being unpaid for the asset to be still valuable, the asset at future time points.

The time-0 entrant's expected discounted return of holding a unit of the asset between arbitrary $t \geq 0$ and $s > t$ is the sum $R_0(t, s)$ of the expected discounted dividend

$$D_0(t, s) = \frac{1}{1 - F_{\alpha,\beta}(t)} \int_t^s e^{-r(\theta - t)} f_{\alpha,\beta}(\theta) \, d\theta \tag{2}$$

and resale proceeds

$$S_0(t, s) = \frac{1 - F_{\alpha,\beta}(s)}{1 - F_{\alpha,\beta}(t)} e^{-r(s-t)} p, \tag{3}$$

i.e.,

$$R_0(t, s) = D_0(t, s) + S_0(t, s) < \max\{p, 1\}, \tag{4}$$

smaller than the dividend or p, whichever is greater. As the price is constant, for every time-τ entrant at every time $t \geq \tau$ the expected discounted return of holding a unit of the asset to an arbitrary $s > t$ is

$$R_\tau(t, s) = R_0(t - \tau, s - \tau),$$

defined using the time-0 entrant in $t - \tau$ and $s - \tau$ time units from entry.

We formulate every traders' expected discounted returns for all time points from entry but model trade as discrete with period $\Delta \in (0, T)$: Trade occurs every Δ time units at time points $t \in \Delta\mathbb{Z}$ after new traders enter but before the old traders leave the market at those points t, giving us compact sets $[t - T, t]$ of market participants. We will comment why trade, unlike market entry, is discontinuous at the end of Section 5.3, which will offer some intuition about the continuous case.

When the market is open, every participating trader has separate reservation prices for different durations, including forever, of holding the asset. For every time-τ entrant and at every trading time $t \in [\tau, \tau + T] \cap (\Delta\mathbb{Z})$ the reservation price for holding the asset to an arbitrary future trading time $s \in (t, \infty) \cap (\Delta\mathbb{Z})$ is

$$P_\tau(t, s) = R_\tau(t, s),$$

and the reservation price for holding the asset forever is

$$P_\tau(t, \infty) = V_\tau(t) = \lim_{s \to \infty} R_\tau(t, s).$$

The domains of the reservation prices, unlike those of the expected discounted returns, take into account the discreteness of trade and the plans to hold the asset forever.

The trader also has separate, but identical, simple demand schedules for these different durations of holding the asset. The demand is zero above, arbitrary at, and infinite below the corresponding reservation price. For market clearing, only the total across durations and market participants matters. Thus, the price p clears the market at trading time $t \in \Delta\mathbb{Z}$ if and only if p is the maximum of the cross section of the reservation prices

$$\begin{aligned}\{P_\tau(t, s) : \tau \in [t - T, t], s \in ((t, \infty) \cap (\Delta\mathbb{Z})) \cup \{\infty\}\} \\ = \{P_\tau(0, s) : \tau \in [-T, 0], s \in ((0, \infty) \cap (\Delta\mathbb{Z})) \cup \{\infty\}\},\end{aligned} \tag{5}$$

equal to the cross section at time 0 and itself dependent on p. Due to identity (5) between the two cross sections, which is a consequence of the stationarity of the model, the price p clears the market at 0 if and only if p clears the market at all $t \in \Delta\mathbb{Z}$. Any such price is called a *Harrison-Kreps equilibrium price.*

For a fixed-point version of this definition, notice that the maximization of reservation prices is well-behaved once we view the selling times s in (5) as elements of the one-point compactification $\mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ (see, for example, *Aliprantis and Border*, 2006): Most importantly, we have a continous value function $V : \mathbb{R} \to \mathbb{R}$ defined by

$$V(p) = \max_{\substack{\tau \in [-T, 0] \\ s \in ((0, \infty) \cap (\Delta\mathbb{Z})) \cup \{\infty\}}} P_\tau(0, s),$$

as seen from the Berge Maximum Theorem. Thus, the price p is a Harrison-Kreps equilibrium price if and only if p is the fixed point of this continuous function V, whose values are the maximum reservation prices. Since for $p \leq 0$ we still have positive reservation prices

$$V(p) > 0 \geq p$$

and for $p \geq 1$ (the dividend amount) the price is already strictly greater than the maximum reservation price

$$V(p) < p,$$

a Harrison-Kreps equilibrium price exists and must be in

$$(0, 1) \subset \mathbb{R}. \tag{6}$$

A Harrison-Kreps equilibrium price can exceed all traders' fundamental valuations, as we are going to show in Section 5, but, by (6) above, the price is, like these valuations in (1), always less than one. In other words, the price is fair enough to stay below the one-shot randomly timed dividend payment.

3. Optimal Selling

Every trader's reservation prices reflect the trade-offs apparent in (4) between planning to hold the asset longer to increase the expected discounted dividend (2) and shorter to raise the expected discounted resale proceeds (3). Looking at the time-0 entrant at any time $t \geq 0$ illustrates how the marginal expected discounted dividend and resale proceeds, which in this case, at every selling time $s > t$, are the derivatives

$$\frac{\partial D_0(t,s)}{\partial s} = \frac{1}{1 - F_{\alpha,\beta}(t)} e^{-r(s-t)} f_{\alpha,\beta}(s) \tag{7}$$

and

$$\frac{\partial S_0(t,s)}{\partial s} = -\frac{\partial D_0(t,s)}{\partial s} p - \frac{1 - F_{\alpha,\beta}(s)}{1 - F_{\alpha,\beta}(t)} e^{-r(s-t)} rp, \tag{8}$$

are interestingly interrelated: A 1¢ gain in the expected discounted dividend goes hand in hand approximately with at least p¢ loss in the expected discounted resale proceeds, because the payout of the dividend makes the asset worthless. While the loss also includes the expected discounted interest forgone on \$$p$ due to the marginal sale delay, there is no dividend disbursement delay, and hence no analogous interest forgone in (7). These demand trade-offs depend on and discipline the price through market-clearing conditions. We will use the following consequences of the marginal returns (7)–(8) for the time-0 entrant's time-t-expected-discounted-return function $R_0(t,\cdot)$ on (t,∞) into $\mathbb{R}$:

(C1) when $p < 1$, then a selling time $s > t$ is a stationary point of $R_0(t,\cdot)$ if and only if

$$H_{\alpha,\beta}(s) = \frac{f_{\alpha,\beta}(s)}{1 - F_{\alpha,\beta}(s)} = \frac{rp}{1-p}, \tag{9}$$

i.e., the first-order condition equates the marginal noninterest net gain with the marginal interest loss;

(C2) when $p < 1$ and $H_{\alpha,\beta}$ (in a sense a dividend likelihood) is strictly decreasing, then a stationary point of $R_0(t,\cdot)$, if any, is a maximum and $R_0(t,\cdot)$ has a truncated- or half-bell shape, meaning that:

(a) $R_0(t,\cdot)$ is strictly increasing on $(t,\infty) \cap H_{\alpha,\beta}^{-1}([rp/(1-p),\infty))$ and

$$\lim_{s \to t^+} R_0(t,s) = p;$$

(b) $R_0(t,\cdot)$ is strictly decreasing on $(t,\infty) \cap H_{\alpha,\beta}^{-1}((-\infty, rp/(1-p)])$ and

$$\lim_{s \to \infty} R_0(t,s) = V_0(t);$$

(C3) when $p < 1$ and $H_{\alpha,\beta}$ is strictly decreasing, then every selling time $s > t$ satisfies

$$R_0(t,s) > \min\{p, V_0(t)\},$$

because $R_0(t,\cdot)$ has a (generalized) bell shape in the sense of (C2).

Optimal future selling times of potential asset-holders underpin their current buying decisions and play a key role in determining equilibrium prices. An attractive plan may be to sell in finite time to get a premium above the fundamental valuation $V_0(t)$, which is possible by (C2)–(C3) if the hazard function is strictly decreasing, as when $\alpha < 1$ (see, for example, *Klugman et al.*, 2012): It suffices to have the price between the fundamental valuation ($p \geq V_0(t)$) and the dividend amount ($p < 1$), which, for $t \in [0,T]$, both are necessary conditions for p to be a Harrison-Kreps equilibrium price. In this case, the expected-discounted-return function $R_0(t,\cdot)$ has a maximizing, but potentially infeasible, selling time if $t = 0$, because $\lim_{s\to 0^+} H_{\alpha,\beta}(s) = \infty$, or if $t > 0$ and $H_{\alpha,\beta}(t) > rp/(1-p)$. The trader would only consider holding the asset to trading time points $s \in \Delta\mathbb{Z}$ close to this maximizer.

4. Disagreement across Experience Levels

Before studying equilibria, we demonstrate that our model indeed has room for the heterogeneity of fundamental valuations, and thus, in our setting, beliefs at a given time, prior to trading, if any, at that time. Traders' fundamental valuations depend on time only through experience, and their fundamental valuation as a function of experience coincides with the time-0 entrant's fundamental-valuation function V_0. Thus, the heterogeneity will be there provided that the valuation function V_0 is not constant on admissible experience levels $[0,T]$.

Answering the question, Proposition 1, below, says that a necessary and sufficient condition for the heterogeneity of fundamental valuations is $\alpha \neq 1$. This is the belief-shape-parameter threshold at which the valuation function V_0 is constant and switches from being strictly decreasing, for $\alpha < 1$, to being strictly increasing, when $\alpha > 1$. For a quick intuition about these slopes, we can use the fact that the relative rate of change of the density $f_{\alpha,\beta}$ at every dividend (waiting) time $\theta \in (0,\infty)$ is

$$\frac{f'_{\alpha,\beta}(\theta)}{f_{\alpha,\beta}(\theta)} = \frac{\alpha-1}{\theta} - \frac{1}{\beta}. \tag{10}$$

If $\alpha < 1$, learning that the asset does not pay at θ time units from entry, which truncates the density further marginally, makes it relatively lower close to θ and higher far, through a smaller percentage decrease (10) far. Due to discounting, the valuation $V_0(\theta)$ must decline. An analogous intuition applies to the other two cases. A detailed proof is in Appendix A.

Proposition 1. *The shape of the valuation function $V_0 : [0,\infty) \to \mathbb{R}$ depends on the parameter α as follows:*

(a) if $\alpha < 1$, then V_0 is strictly decreasing;

(b) if $\alpha = 1$, then V_0 is constant;

(c) if $\alpha > 1$, then V_0 is strictly increasing.

As a side note, this formulation with identical newcomers offers, at least partly, a simple model of symmetric-information belief heterogeneity that is acquired and experience-based as opposed to inborn or ability-based. Many economists have discussed the fine lines between heterogeneity of information, prior-beliefs, bounded rationality, and ability in various contexts (we mention *Aumann*, 1976; *Morris*, 1995; *Alaoui and Penta*, 2016). As *Morris* (1995) puts it, for instance, "individuals may have misinterpreted a signal at the beginning of time, and this is what gave them heterogeneous prior beliefs." In our model, traders enter the market at different points in time, are identical in terms of rationality, ability, and information, but, nevertheless, disagree across experience levels. In many contexts in the literature, making these subtle distinctions would indeed matter.

5. Bubble under Pessimistic Old-timers

In our model, our claimed condition for a perpetual bubble holds if and only if $\alpha < 1$, in which case the most experienced are always, not just occasionally, not the most optimistic according to Proposition 1: They are the least optimistic. In the other two scenarios, in violation of the condition, the most optimistic are either as optimistic as everyone else (when $\alpha = 1$) or the most optimistic (when $\alpha > 1$). Now we prove formally that the condition $\alpha < 1$ is sufficient for a perpetual bubble.

5.1. Overshooting Fundamental Valuations

Essentially, the bubble is a corollary of (C3), (6), and the fact that the hazard function $H_{\alpha,\beta}$ is strictly decreasing in this case ($\alpha < 1$): If p is a Harrison-Kreps equilibrium price, then

$$p \geq P_0(0, \Delta) = R_0(0, \Delta) > \min\{p, V_0(0)\} = V_0(0),$$

i.e., the equilibrium price is, on the one hand, strictly above the base, $\min\{p, V_0(0)\}$, of the (generalized) bell $R_0(0, \cdot)$ and ensures, on the other hand, that the base is $V_0(0)$. Since $V_0(0)$ is the maximum valuation by part (a) of Proposition 1, the equilibrium price must always be strictly greater than all traders' fundamental valuations, and hence bubbly, which we now state as Proposition 2.

Proposition 2. *If $\alpha < 1$ and p is a Harrison-Kreps equilibrium price, then every time point $t \geq 0$ satisfies $p > V_0(t)$.*

5.2. More Extreme Possibilities

We view our model as embodying a general overpricing principle under relatively pessimistic experienced investors on the one hand and illustrating just how far-reaching this bubble can be on the other hand. For this, we are going to deduce that, in addition to overshooting the long-term fundamental valuations, the equilibrium price must be strictly above even very short-term expected discounted returns of holding the asset. Notice that we have also shown in Proposition 2 that the equilibrium price is strictly above not only the fundamental valuations $V_0([0,T])$ of any trading time's market participants, but of all departed traders.

We do not make generality claims about such more extreme bubbles. The reason is technical and boils down to the strict monotonicity of $H_{\alpha,\beta}$, which features in the first-order condition (9) and yields a unique maximizer over traders and selling times. We argue through the following instructive sequence of necessary conditions for p to be a Harrison-Kreps equilibrium price, pinpointing a unique candidate for the equilibrium price along the way, when $\alpha < 1$:

(N1) there are times $t^* \in [0,T]$ and $s^* \geq t^* + \Delta$ such that $p = R_0(t^*, s^*)$, i.e., the price, of course, must coincide with some short-term expected discounted return, because holding the asset forever is unprofitable by Proposition 2;

(N2) the (generalized) bell $R_0(t^*, \cdot)$ has a maximizer $\bar{s} \in (t^*, s^*)$, is strictly increasing on $(t^*, \bar{s}]$, and is strictly decreasing on $[\bar{s}, \infty)$—i.e., the graph must spring from the point (t^*, p) upward to be able to pass through (s^*, p);

(N3) every selling time $s \in (t^*, s^*)$ satisfies $p > R_0(t^*, s)$, i.e., the truncated bell $R_0(t^*, \cdot)$ forms an arc over the interval (t^*, s^*);

(N4) $s^* = t^* + \Delta$, since otherwise $t^* + \Delta \in (t^*, s^*)$, and thus p would be strictly greater than the reservation price $P_{-t^*}(0, \Delta) = R_0(t^*, t^* + \Delta)$;

(N5) $p = \max_{t \in [0,T]} R_0(t, t + \Delta)$, because these expected discounted returns are precisely the reservation prices $P_{-t}(0, \Delta)$;

(N6) $t^* = 0$, since otherwise

$$\frac{dR_0(t^*, t^* + \Delta)}{dt} = \frac{\partial R_0(t^*, s^*)}{\partial t} + \frac{\partial R_0(t^*, s^*)}{\partial s} < 0,$$

because

$$\begin{aligned} \frac{\partial R_0(t^*, s^*)}{\partial t} &= rR_0(t^*, s^*) - H_{\alpha,\beta}(t^*) + H_{\alpha,\beta}(t^*) R_0(t^*, s^*) \qquad (11) \\ &= rp - H_{\alpha,\beta}(t^*)(1-p) \\ &< rp - H_{\alpha,\beta}(\bar{s})(1-p) \\ &= 0 \end{aligned}$$

(the first two summands in (11) reflect drops in interest forgone and unconditional probability to get the dividend, while the third adjusts for further truncation) and

$$\frac{\partial R_0(t^*, s^*)}{\partial s} \le 0;$$

(N7) every time point $t \in (0, T]$ satisfies $p > R_0(t, t + \Delta)$, because $t^* = 0$ is a unique maximizer in (N5);

(N8) every time point $t > T$ satisfies $p > R_0(t, t + \Delta)$, because $t \ge \Delta \ge \bar{s}$, and thus $R_0(t, \cdot)$, whose right-hand limit at t is p, is strictly decreasing;

(N9) all time points $t \ge 0$ and $s > t + \Delta$ satisfy $p > R_0(t, s)$, because $t + \Delta \ge \Delta \ge \bar{s}$, and thus $R_0(t, \cdot)$ is strictly decreasing on $[t + \Delta, \infty)$, which means that $p \ge R_0(t, t + \Delta) > R_0(t, s)$.

What we have shown (and include below in Proposition 3) is that the equilibrium price is strictly greater than all expected discounted returns of holding the asset for longer than the market-imposed minimum holding period Δ. In other words, the equilibrium price must always be bubbly even relative to all these (active and departed traders') short holding horizons in addition to infinite ones.

Proposition 3. *If $\alpha < 1$ and p is a Harrison-Kreps equilibrium price, then all time points $t \ge 0$ $(t > 0)$ and $s > t + \Delta$ $(s \ge t + \Delta)$, respectively, satisfy $p > R_0(t, s)$ and we have $p = R_0(0, \Delta)$.*

A useful by-product of this exercise of finding insightful necessary conditions for p to be an equilibrium price is the identification of a unique candidate for such a price, also documented in Proposition 3. It says that the equilibrium price p solves the equation

$$p = \int_0^{\Delta} e^{-r\theta} f_{\alpha,\beta}(\theta)\, d\theta + (1 - F_{\alpha,\beta}(\Delta))\, e^{-r\Delta} p$$

and that in this equilibrium, at every trading $t \in \Delta\mathbb{Z}$, only time-t entrant holds the asset: Past entrants are not willing to hold the asset even for the minimum holding period Δ, as the price is strictly greater than their, but not the time-t entrant's, expected discounted returns of doing so.

5.3. Full-blown Bubble under Frequent Trade

Full-blown bubbles are, loosely defined, those that overshoot all possible beliefs about fundamentals. This admittedly fragile notion is useful for evaluating the size of the bubble at least within our model, where the price threshold for being full-blown is \$1—the certain, but randomly timed, dividend amount. While the Harrison-Kreps equilibrium price is less than one according to (6), in the limit, as trading becomes more and more frequent, the equilibrium price approaches one by the following one-line argument: When $\alpha < 1$, letting $\Delta \to 0^+$ and the equilibrium price respond squeezes the unconstrained optimal selling time $\bar{s}$ in (N2) to zero, pushes $H_{\alpha,\beta}(\bar{s})$ to infinity, and, to satisfy the first-order condition (9), drives the price to one. In this frequent-trade limit, the bubble is full-blown in the sense of the price being equal to the supremum of fundamental valuations over all possible beliefs. We state this limit result below as Proposition 4.

Proposition 4. *Let $\alpha < 1$ and $P(\Delta)$ be the Harrison-Kreps equilibrium price. We have*

$$\lim_{\Delta \to 0^+} P(\Delta) = 1.$$

If we allowed for continuous trade, one might expect an equilibrium price to be this limit, equal to one, of "discrete" Harrison-Kreps equilibrium prices. However, this price would be too high for any trader to ever want to hold the asset to any future time point—strictly above any duration's expected discounted return, i.e. (1) or (4). It would be somewhat hard to make sense of such a continuous-trade approximation, as it would be unclear who holds the asset when one instant succeeds another. Yet, on the other hand, we could simply say that the asset changes hands continuously, just like a flying arrow passes through instants of continuous time. (A well-known Zeno's paradox is that such an arrow is never moving, because at every instant the arrow is in some fixed position; see, for example, *Hamming* (1998).)

5.4. Perpetual Valuation Switching

To emphasize that every time-τ entrant's valuation will be exceeded by some future (inexperienced) entrant's valuation arbitrarily soon after this time point τ if $\alpha < 1$, we dedicate a separate Proposition (No. 5) to this. It is an immediate corollary of part (c) of Proposition 1 and informally the main piece of our intuition behind the perpetual bubble. The previous literature, where all traders had the same experience, allowed every trader to anticipate comparable valuation switching and resulting possibilities to resell the asset for strictly more than the trader's valuation. These are basically the circumstances in which the previous literature with other kinds of belief heterogeneity predicts bubbly prices in the sense of them overshooting the most optimistic valuations. Notice that the valuation switching in our model specification is stronger in the sense that every time-τ entrant's valuation will be exceeded at all, not just some, future time points $\sigma > \tau$ if $\alpha < 1$.

Proposition 5 (Perpetual Switching)**.** *If $\alpha < 1$, then all time points τ and $\sigma > \tau$ satisfy $V_\sigma(\sigma) > V_\tau(\sigma)$.*

In the terminology of *Morris* (1996), Proposition 5 says that at each time point there are no optimists, defined as traders whose valuation is at least as high as all other traders' valuations from that time onwards. *Morris* (1996) coined the term perpetual switching to refer to the nonexistence of optimists after every history. In his model, perpetual (valuation) switching is equivalent to the existence of a speculative bubble. *Morris* (1996) characterizes this notion of an optimist in terms of the traders' prior beliefs about the probability that the asset pays a dividend as opposed to no dividend—i.e., even his uncertainty is very different from ours. In his setting, which also assumes away newcomers in the market, a trader is an optimist if and only if the trader's prior is monotone-likelihood-ratio (MLR) dominant among all trader's heterogeneous priors[1]. One of the goals of *Morris* (1996) was to investigate precisely how different the priors must be for a bubble to arise in a special case of the model of *Harrison and Kreps* (1978). The answer is this necessary and sufficient condition that rules out "that there is a single [MLR dominant] trader whose [prior] density is always increasing at the fastest rate". Over time, the traders' posteriors converge to one another, and the bubble fades away, whereas in our model market newcomers may add fuel to the fire and keep the bubble going forever. Our model allows for disagreement across experience levels and is optimist-free simply when the most experienced investors are not the most optimistic. The former approach is compelling for explaining temporary bubbles such as overpriced IPOs relative to long-run values (*Miller*, 1977) and requires neither bounded rationality nor market newcomers. *Morris* (1996) argues that the assumption of unexplained differences in prior beliefs at initial public offerings is hard to refute, because there have been no opportunities to learn the true data-generating process. In contrast, our assumption of relatively pessimistic most experienced cohort seems more applicable to "seasoned" assets such as stocks of old firms. Bubbles due to presence of permanently overconfident traders in *Scheinkman and Xiong* (2003) are also more relevant to newly issued assets, because over time the traders could realize their overconfidence.

6. Optimistic Old-timers (Bubble-free Case)

Our model is rich enough to separate the wheat from the chaff—i.e., to accommodate bubble-free scenarios, thereby proving that our condition of relatively pessimistic veteran investors is not redundant, as follows. The most experienced are among the most optimistic if and only if $\alpha \geq 1$ (Proposition 1). In this case, the most optimistic fundamental valuation is always $V_0(T)$. Thus, we want to show that $p = V_0(T)$ is a Harrison-Kreps equilibrium price when $\alpha \geq 1$. This would be impossible if the hazard function $H_{\alpha,\beta}$ were strictly decreasing, because then our bubble argument (Section 5.1) would apply almost word for word and yield the contradiction

$$\begin{aligned} V_0(T) = p &\geq P_{-T}(0,\Delta) = R_0(T, T+\Delta) \\ &> \min\{p, V_0(T)\} = V_0(T). \end{aligned}$$

Indeed, in this case $H_{\alpha,\beta}$ is increasing (see again, for example, *Klugman et al.*, 2012). The consequence is that expected discounted returns as functions of selling time are no longer (generalized) bells, but inverted bells, have minima rather than maxima in (C2), with weak monotonicites in place of strict ones. Any $p \in (V_0(T), 1)$ is already above the maximum reservation price

$$V(p) < p,$$

[1] *Werner* (2018) argues that MLR dominance yields optimists in a more abstract framework, retaining from the previous literature short-sales constraints, risk-neutral traders, and learning about the dividend.

as these inverted bells spring from points with vertical coordinate p downward or straight, may change direction only once, and approach less optimistic valuations. Thus, a Harrison-Kreps equilibrium price, which must be at least $V_0(T) = P_{-T}(0,\infty)$, cannot be different from $V_0(T)$ and hence must be bubble-free. This is a part of our Proposition 6, below, while the rest of this proposition is due to the inverted-bell shapes and the fact that the valuation function V_0 is strictly increasing for $\alpha > 1$ and constant for $\alpha = 1$ (Proposition 1). In the latter case, the beliefs $f_{\alpha,\beta}$ reduce to an exponential distribution, the hazard function $H_{\alpha,\beta}$ is constant, expected discounted returns as functions of selling time are monotone (generalized bells), and all reservation prices in (5) equal $p = V_0(T)$.

Proposition 6. *If $\alpha \geq 1$, then the price $p = V_0(T)$ is a unique (bubble-free) Harrison-Kreps equilibrium price. In this equilibrium, asset holdings at every trading time point $t \in \Delta\mathbb{Z}$ depend on the shape parameter $\alpha \geq 1$ as follows:*

(a) if $\alpha > 1$, then only the most experienced trader holds the asset;

(b) if $\alpha = 1$, then any trader can hold the asset.

We did not boast that our sufficient condition for a perpetual bubble is also necessary, as it has turned out in Proposition 6, because we lack robust intuition for the necessity, unlike that for the sufficiency. For instance, with paternalistic traders we would not expect the necessity if beliefs were such that optimism as a function of experience peaked at the highest admissible experience level T: The most experienced would paternalistically expect the next generation to be relatively pessimistic and to be able to sell the inherited assets above their fundamental valuations to more optimistic traders. Besides, we must admit that we abstract away from other potential causes, such as higher-order uncertainty, of price bubbles (*Brunnermeier*, 2001; *Abreu and Brunnermeier*, 2003).

A. Proof of Proposition 1

(a) Since all $t, s \in (0,\infty)$ such that $t < s$ satisfy

$$\int_t^\infty \frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} d\theta = \int_s^\infty \frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(s)} d\theta = \int_t^\infty \frac{f_{\alpha,\beta}(\theta - t + s)}{1 - F_{\alpha,\beta}(s)} d\theta,$$

there is an $\eta \in (t,\infty)$ with

$$\frac{f_{\alpha,\beta}(\eta)}{1 - F_{\alpha,\beta}(t)} = \frac{f_{\alpha,\beta}(\eta - t + s)}{1 - F_{\alpha,\beta}(s)}.$$

Since every $\theta \in [t,\infty)$ satisfies

$$\frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} \cdot \frac{1 - F_{\alpha,\beta}(s)}{f_{\alpha,\beta}(\theta - t + s)} = \frac{1 - F_{\alpha,\beta}(s)}{1 - F_{\alpha,\beta}(t)} e^{\frac{s-t}{\beta}} \left(\frac{\theta}{\theta - t + s}\right)^{\alpha - 1},$$

we have

$$\frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} > \frac{f_{\alpha,\beta}(\theta - t + s)}{1 - F_{\alpha,\beta}(s)}$$

if $\theta < \eta$, and

$$\frac{f_{\alpha,\beta}(\theta)}{1-F_{\alpha,\beta}(t)} \leq \frac{f_{\alpha,\beta}(\theta - t + s)}{1-F_{\alpha,\beta}(s)}$$

if $\theta \geq \eta$. Now

$$\begin{aligned}
V_0(t) - V_0(s) &= \int_t^\infty \frac{f_{\alpha,\beta}(\theta)}{1-F_{\alpha,\beta}(t)} e^{-r(\theta-t)} d\theta - \int_s^\infty \frac{f_{\alpha,\beta}(\theta)}{1-F_{\alpha,\beta}(s)} e^{-r(\theta-s)} d\theta \\
&= \int_t^\infty \frac{f_{\alpha,\beta}(\theta)}{1-F_{\alpha,\beta}(t)} e^{-r(\theta-t)} d\theta - \int_t^\infty \frac{f_{\alpha,\beta}(\theta-t+s)}{1-F_{\alpha,\beta}(s)} e^{-r(\theta-t)} d\theta \\
&= \int_t^\infty \left(\frac{f_{\alpha,\beta}(\theta)}{1-F_{\alpha,\beta}(t)} - \frac{f_{\alpha,\beta}(\theta-t+s)}{1-F_{\alpha,\beta}(s)} \right) e^{-r(\theta-t)} d\theta \\
&> \left(\int_t^\infty \frac{f_{\alpha,\beta}(\theta)}{1-F_{\alpha,\beta}(t)} d\theta - \int_t^\infty \frac{f_{\alpha,\beta}(\theta-t+s)}{1-F_{\alpha,\beta}(s)} d\theta \right) e^{-r(\eta-t)} \\
&= 0,
\end{aligned}$$

proving that V_0 is strictly decreasing on $(0, \infty)$. Since

$$\lim_{t \to 0^+} V_0(t) = V_0(0),$$

every time point $t \geq 0$ satisfies every $s \in (0, \infty)$ satisfies $V_0(s) < V_0(0)$, completing the proof.

(b) Every $t \in [0, \infty)$ satisfies

$$V_0(t) = \frac{1}{1 - \int_0^t \frac{1}{\beta} e^{-\frac{\theta}{\beta}} d\theta} \int_t^\infty \frac{1}{\beta} e^{rt} e^{-\left(r + \frac{1}{\beta}\right)\theta} d\theta = (\beta r + 1)^{-1}.$$

(c) The proof is analogous to that of (a).

B. Fundamental Valuations and Hazard

We denote the cumulative distribution of

$$\text{Gamma}\left(\alpha, \frac{\beta}{\beta r + 1}\right),$$

defined analogously to $F_{\alpha,\beta}$, by $\tilde{F}_{\alpha,\beta}$ and let $\tilde{F}_{\alpha,\beta}(0) = 0$.

Lemma 1. *The valuation function $V_0 : [0, \infty) \to \mathbb{R}$ has the following properties:*

(a) every time point $t \geq 0$ satisfies

$$V_0(t) = \frac{1 - \tilde{F}_{\alpha,\beta}(t)}{1 - F_{\alpha,\beta}(t)} e^{rt} (\beta r + 1)^{-\alpha} < 1; \tag{12}$$

(b) $\lim_{t\to 0^+} V_0(t) = V_0(0)$*;*

(c) at every time point $t > 0$*, the first and second derivatives of* V_0 *are*

$$V_0'(t) = rV_0(t) - H_{\alpha,\beta}(t)(1 - V_0(t))$$

and

$$V_0''(t) = (H_{\alpha,\beta}(t) + r) V_0'(t) + (V_0(t) - 1) H'_{\alpha,\beta}(t)$$

respectively.

Lemma 2. *The hazard function* $H_{\alpha,\beta} : (0,\infty) \to \mathbb{R}$ *has the following properties:*

(a) at every time point $t > 0$*, the derivative is*

$$H'_{\alpha,\beta}(t) = \left(\frac{\alpha - 1}{t} - \frac{1}{\beta}\right) H_{\alpha,\beta}(t) + H^2_{\alpha,\beta}(t)\,; \tag{13}$$

(b) if $\alpha < 1$*, then every time point* $t > 0$ *satisfies* $H_{\alpha,\beta}(t) \geq 1/\beta$*;*

(c) if $\alpha > 1$*, then every time point* $t > 0$ *satisfies* $H_{\alpha,\beta}(t) \leq 1/\beta$*;*

(d) $\lim_{t\to\infty} H_{\alpha,\beta}(t) = 1/\beta$*;*

(e) if $\alpha < 1$*, then every time point* $t > 0$ *satisfies* $H'_{\alpha,\beta}(t) < 0$*;*

(f) if $\alpha > 1$*, then every time point* $t > 0$ *satisfies* $H'_{\alpha,\beta}(t) > 0$*.*

Proof. (b) For all $t' \in (t,\infty)$, the Cauchy Mean-value Theorem yields an $s \in (t,t')$ such that

$$\frac{f_{\alpha,\beta}(t') - f_{\alpha,\beta}(t)}{1 - F_{\alpha,\beta}(t') - (1 - F_{\alpha,\beta}(t))} = \frac{f'_{\alpha,\beta}(s)}{(1 - F_{\alpha,\beta})'(s)} = \frac{1-\alpha}{s} + \frac{1}{\beta} \geq \frac{1}{\beta}.$$

Now

$$H_{\alpha,\beta}(t) = \frac{f_{\alpha,\beta}(t)}{1 - F_{\alpha,\beta}(t)} = \lim_{t'\to\infty} \frac{f_{\alpha,\beta}(t') - f_{\alpha,\beta}(t)}{1 - F_{\alpha,\beta}(t') - (1 - F_{\alpha,\beta}(t))} \geq \frac{1}{\beta},$$

as desired.

(c) The proof is analogous to that of (b).

(e) Suppose, by way of contradiction, that $H'_{\alpha,\beta}(t) \geq 0$. By (13), every stationary point s of $H_{\alpha,\beta}$ satisfies

$$\frac{\alpha - 1}{s} - \frac{1}{\beta} + H_{\alpha,\beta}(s) = 0. \tag{14}$$

Define a function $g : (0,\infty) \to \mathbb{R}$ by letting $g(s)$ be the left-hand side of equation (14). At every $s \in (0,\infty)$, this function g has a derivative

$$g'(s) = \frac{1-\alpha}{s^2} + H'_{\alpha,\beta}(s)\,.$$

Every stationary point s of $H_{\alpha,\beta}$ satisfies $g'(s) = (1-\alpha)/s^2 > 0$, and thus every such stationary point s is a strict local minimum of $H_{\alpha,\beta}$. It follows that every $s \in (t,\infty)$ satisfies $H'_{\alpha,\beta}(s) > 0$, implying that $H_{\alpha,\beta}$ is strictly increasing on (t,∞). By (b) and (d), we have $1/\beta = H_{\alpha,\beta}(t+1)$. Now $s \in (t,t+1)$ satisfy $H_{\alpha,\beta}(s) < 1/\beta$, which contradicts (b).

(f) The proof is analogous to that of (e), but we use (c) instead of (b). □

Lemma 3. *The valuation function $V_0 : [0,\infty] \to \mathbb{R}$ also has the following properties:*

(a) if $\alpha < 1$, then every time point $t > 0$ satisfies $V_0(t) \geq (\beta r + 1)^{-1}$;

(b) if $\alpha > 1$, then every time point $t > 0$ satisfies $V_0(t) \leq (\beta r + 1)^{-1}$;

(c) $\lim_{t\to\infty} V_0(t) = (\beta r + 1)^{-1}$.

Proof. (a) For all $t' \in (t,\infty)$, the Cauchy Mean-value Theorem yields an $s \in (t,t')$ such that, after also applying part (b) of Lemma 2, we have

$$\begin{aligned}
\frac{1-\tilde{F}_{\alpha,\beta}(t') - \left(1-\tilde{F}_{\alpha,\beta}(t)\right)}{(1-F_{\alpha,\beta}(t'))\,e^{-rt'} - (1-F_{\alpha,\beta}(t))\,e^{-rt}} &= \frac{\left(1-\tilde{F}_{\alpha,\beta}\right)'(s)}{(1-F_{\alpha,\beta})'(s)\,e^{-rs} - (1-F_{\alpha,\beta}(s))\,e^{-rs}r} \\
&= \frac{H_{\alpha,\beta}(s)}{H_{\alpha,\beta}(s)+r}(\beta r+1)^{\alpha} \\
&\geq (\beta r+1)^{-1}(\beta r+1)^{\alpha} \\
&= (\beta r+1)^{\alpha-1}.
\end{aligned}$$

The limit

$$\frac{1-\tilde{F}_{\alpha,\beta}(t)}{(1-F_{\alpha,\beta}(t))\,e^{-rt}} = \lim_{t'\to\infty}\frac{1-\tilde{F}_{\alpha,\beta}(t') - \left(1-\tilde{F}_{\alpha,\beta}(t)\right)}{(1-F_{\alpha,\beta}(t'))\,e^{-rt'} - (1-F_{\alpha,\beta}(t))\,e^{-rt}} \geq (\beta r+1)^{\alpha-1}$$

and (12) confirm that $V_0(t) \geq (\beta r+1)^{-1}$.

(b) The proof is analogous to that of (a), but we use part (c) of Lemma 2 instead of its part (b). □

Lemma 4. *For all time points $t \geq 0$ and $s > t$, we have*

$$R_0(t,s) = V_0(t) + \frac{1-F_{\alpha,\beta}(s)}{1-F_{\alpha,\beta}(t)}e^{-r(s-t)}(p - V_0(s)).$$

Lemma 5. *For all time points τ, t, s, and s' such that $t \geq \tau$ and $s,s' \in (t,\infty)$, the following conditions are equivalent:*

(a) $R_\tau(t,s) \geq R_\tau(t,s')$;

(b) $R_\tau(\tau,s) \geq R_\tau(\tau,s')$;

(c) $R_t(t,s-\tau+t) \geq R_t(t,s'-\tau+t)$.

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