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Discussion Paper No. 173

July 30, 2019

Collaborative Research Center Transregio 190 | www.rationality-and-competition.de
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Boolean Representations of Preferences under Ambiguity*

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July 19, 2019

Abstract

We propose a class of multiple-prior representations of preferences under ambiguity where the belief the decision-maker (DM) uses to evaluate an uncertain prospect is the outcome of a game played by two conflicting forces, Pessimism and Optimism. The model does not restrict the sign of the DM's ambiguity attitude, and we show that it provides a unified framework through which to characterize different degrees of ambiguity aversion, as well as to represent context-dependent negative and positive ambiguity attitudes documented in experiments. We prove that our baseline representation, *Boolean expected utility (BEU)*, yields a novel representation of the class of invariant biseparable preferences (Ghirardato, Maccheroni, and Marinacci, 2004), which drops uncertainty aversion from maxmin expected utility (Gilboa and Schmeidler, 1989), while extensions of BEU allow for more general departures from independence.

1 Introduction

A central approach to modeling preferences under ambiguity is based on the idea that the decision-maker (DM) quantifies uncertainty with a set of relevant beliefs (i.e., probability measures) and may use a different belief from this set to evaluate each uncertain prospect. A well-known limitation underlying many such multiple-prior models—notably Gilboa and Schmeidler's (1989) maxmin expected utility model and several of its generalizations[1]—is a restrictive mechanism of belief selection, whereby the DM evaluates each prospect according to the worst possible relevant belief. Behaviorally, this restriction is reflected by

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[1]E.g., Maccheroni, Marinacci, and Rustichini (2006); Chateauneuf and Faro (2009); Strzalecki (2011); Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011); Skiadas (2013).

Schmeidler's (1989) uncertainty aversion axiom, which captures a negative attitude towards ambiguity through a strong form of preference for hedging. Subsequent work has questioned this formalization of ambiguity aversion and proposed several alternative definitions and measures.[2] The experimental literature documents yet more nuanced patterns of ambiguity attitudes, with the same subjects appearing ambiguity-averse in some decision problems but ambiguity-seeking in others, depending on contextual features of each problem (for a survey, see Trautmann and van de Kuilen, 2015).

In this note, we propose a class of multiple-prior representations that provides a unified lens through which to understand different formalizations of ambiguity aversion, as well as the context-dependent negative and positive ambiguity attitudes documented in experiments. To capture a flexible mechanism of belief selection, our representations adopt a "dual self" perspective on ambiguity, by modeling the belief the DM uses to evaluate any given prospect as the outcome of a game between two conflicting forces or selves, henceforth Pessimism and Optimism.[3]

Our baseline representation is a parsimonious generalization of Gilboa and Schmeidler's (1989) maxmin expected utility model. Under *Boolean expected utility (BEU)*,[4] there is a compact collection $\mathbb{P}$ of closed and convex sets of beliefs and an affine utility u such that the DM evaluates each act f according to

$$W_{\mathrm{BEU}}(f) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mathbb{E}_\mu[u(f)].$$

That is, the belief used to evaluate f is the outcome of a sequential zero-sum game: First, Optimism chooses a set of beliefs P from the collection $\mathbb{P}$ with the goal of maximizing the DM's expected utility to f; then Pessimism chooses a belief μ from P with the goal of minimizing expected utility. Maxmin expected utility corresponds to the extreme special case where Optimism has no choice, while the opposite extreme case, maxmax expected utility, provides Pessimism with no choice. Other special cases include Choquet expected utility (Schmeidler, 1989) and α-maxmin.

Our first main result is that BEU represents the class of preferences over Anscombe-

[2]E.g., Epstein (1999); Ghirardato and Marinacci (2002); Baillon, L'Haridon, and Placido (2011); Dow and Werlang (1992); Baillon, Huang, Selim, and Wakker (2018).

[3]The idea that the DM consists of multiple strategic selves with conflicting motives is employed frequently in behavioral economics, for example to model risk preferences and intertemporal choices (e.g., Thaler and Shefrin, 1981; Fudenberg and Levine, 2006; Brocas and Carrillo, 2008).

[4]We borrow this terminology from mathematical analysis (e.g., Ovchinnikov, 2001), where a functional $W : \mathbb{R}^n \to \mathbb{R}$ is said to admit a *Boolean representation* with respect to a family of functionals $\{V_i\}_{i \in I}$ if there is some collection $\mathbb{J}$ of subsets of I such that $W(x) = \max_{J \in \mathbb{J}} \min_{i \in J} V_i(x)$ for all x; that is, denoting the max and min operator by $\vee$ and $\wedge$, $W(x) = \bigvee_{J \in \mathbb{J}} \bigwedge_{i \in J} V_i(x)$ is written as a Boolean polynomial in disjunctive normal form.

Aumann acts that satisfy all of Gilboa and Schmeidler's (1989) axioms except for uncertainty aversion (Theorem 1). Equivalently, the presence of ambiguity is captured solely by relaxing independence to certainty independence, without additionally restricting the DM's ambiguity attitude to be negative (or positive). Obtaining an easy-to-interpret representation for this class of preferences—which are known as *invariant biseparable*—has been considered an important question in the ambiguity literature. Section 4.2 contrasts BEU with existing representations due to Ghirardato, Maccheroni, and Marinacci (2004) (generalized α-maxmin) and Amarante (2009) (Choquet integration over beliefs).

Proposition 1 shows that any BEU preference $\succsim$ uniquely reveals a set of *relevant priors* $C = \bigcup_{P \in \mathbb{P}} P$, which represents the possible outcomes of the belief-selection game.[5] Moreover, C admits a behavioral characterization in terms of the extent to which $\succsim$ departs from independence, in the sense that it coincides with Ghirardato, Maccheroni, and Marinacci's (2004) unanimity representation of the largest independent subrelation of $\succsim$.

Exploiting the fact that BEU allows for flexible attitudes towards ambiguity, our second main contribution is to use BEU to represent and contrast a range of theoretically and experimentally appealing ambiguity attitudes. We begin by showing that the standard comparative notion of ambiguity aversion is represented by a natural preorder over collections $\mathbb{P}$ of sets of beliefs, which captures the relative power allocated to Pessimism in the belief-selection game (Proposition 2). While for a given set of relevant priors, maxmin and maxmax expected utility are maximal and minimal in this order, the result highlights how less extreme allocations of power across the selves can generate a rich hierarchy of intermediate ambiguity attitudes, which we proceed to characterize in Sections 3.2 and 3.3.

First, Theorem 2 shows that several different shades of ambiguity aversion—as captured by varying degrees of preference for hedging—are characterized by the extent of overlap of sets in $\mathbb{P}$. Specifically, Ghirardato and Marinacci's (2002) notion of *absolute ambiguity aversion* (i.e., being more ambiguity-averse than some subjective expected utility preference), which corresponds to a preference for *complete* hedges that fully eliminate uncertainty, is characterized by the intersection of all sets in $\mathbb{P}$ being nonempty. This requires there to be at least one prior that is always available to Pessimism regardless of Optimism's choice, and hence is strictly weaker than uncertainty aversion (i.e., a preference for all hedges), which requires that all relevant priors are always available to Pessimism. Since absolute ambiguity aversion is inconsistent with experimental evidence that subjects are often ambiguity-averse for bets involving moderate odds but *ambiguity-seeking for small odds*, we also consider the even weaker notion of *k-ambiguity aversion* (for some $k = 2, 3, \ldots$) and show that it can accommodate this evidence. This notion imposes a preference for complete hedges only

[5] Uniqueness holds up to convex closure and elimination of redundant (never selected) beliefs.

among any k acts and is characterized by the requirement that the intersection of any k sets in $\mathbb{P}$ is nonempty.

Second, motivated by experimental findings that subjects' ambiguity attitudes may be negative or positive depending on their familiarity with the *source* of uncertainty, we further relax k-ambiguity aversion to a "local" analog, which characterizes the sign of an event-based ambiguity aversion index commonly used in experimental work. While BEU can accommodate source-dependent negative and positive ambiguity attitudes by allowing the sign of this index to vary across events (Proposition 3), we show that this phenomenon is incompatible with its widely used special case, α-maxmin.

Finally, Section 4.1 introduces some natural extensions of BEU that relax certainty independence to weaker axioms, accommodating additional experimental findings. The resulting representations are also Boolean, in the sense that they feature a game between Optimism and Pessimism with more general payoffs. This suggests that Boolean models provide a unified way of representing ambiguity preferences that depart from independence, without restricting the sign of the DM's ambiguity attitude.

2 Boolean Expected Utility

2.1 Setup

Let Z be a set of prizes and let $\Delta(Z)$ denote the space of simple lotteries (that is, probability measures with finite support) over Z. We refer to typical elements $p, q \in \Delta(Z)$ as lotteries. Let S be a finite set of states. An *(Anscombe-Aumann) act* is a mapping $f : S \to \Delta(Z)$. Let $\mathcal{F}$ be the space of all acts, with typical elements f, g, h. For any $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, define the mixture $\alpha f + (1 - \alpha)g \in \mathcal{F}$ to be the act that in each state $s \in S$ yields lottery $\alpha f(s) + (1 - \alpha)g(s) \in \Delta(Z)$. As usual, we identify each lottery $p \in \Delta(Z)$ with the constant act that yields lottery p in each state $s \in S$.

Let $\Delta(S)$ denote the set of all probability measures over S, which we embed in $\mathbb{R}^S$ and endow with the Euclidean topology. We refer to typical elements $\mu, \nu \in \Delta(S)$ as beliefs. Given any act $f \in \mathcal{F}$ and map $u : \Delta(Z) \to \mathbb{R}$, let $u(f)$ denote the element of $\mathbb{R}^S$ given by $u(f)(s) = u(f(s))$ for all $s \in S$, and let $\mathbb{E}_\mu[u(f)] := \mu \cdot u(f)$.

The DM's *preference* over $\mathcal{F}$ is given by a binary relation $\succsim$ on $\mathcal{F}$. As usual, $\succ$ and $\sim$ denote the asymmetric and symmetric parts of $\succsim$.

2.2 Representation

We now introduce our baseline representation, Boolean expected utility. Let $\mathcal{K}(\Delta(S))$ denote the space of all nonempty closed, convex sets of beliefs, endowed with the Hausdorff topology. A ***belief-set collection*** is a nonempty compact collection $\mathbb{P} \subseteq \mathcal{K}(\Delta(S))$; that is, each element $P \in \mathbb{P}$ is a nonempty closed, convex set of beliefs.

Definition 1. A ***Boolean expected utility (BEU)*** representation of preference $\succsim$ consists of a belief-set collection $\mathbb{P}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that

$$W_{\mathrm{BEU}}(f) = \max_{P\in\mathbb{P}} \min_{\mu\in P} \mathbb{E}_\mu[u(f)] \tag{1}$$

represents $\succsim$.[6]

Just as Gilboa and Schmeidler's (1989) maxmin expected utility model, BEU is a multiple-prior model of ambiguity: The DM has in mind a set of relevant beliefs $\bigcup_{P\in\mathbb{P}} P$, and might use a different belief to evaluate each act. But unlike maxmin expected utility, the belief μ used to evaluate any given act f is not necessarily worst-case among all relevant beliefs. Instead, μ is the outcome of a sequential zero-sum game between two conflicting forces or "selves:" First, self 1 ("Optimism") chooses an action $P \in \mathbb{P}$ with the goal of maximizing expected utility to act f; then self 2 ("Pessimism") chooses an action $\mu \in P$ with the goal of minimizing expected utility to f.

As Remark 1 below shows, both the specific form of action sets and the order of moves in (1) are without loss of generality. Note that maxmin expected utility corresponds to the extreme special case of BEU where Optimism's action set is trivial (i.e., $\mathbb{P} = \{P\}$ is a singleton), as in this case (1) reduces to $W(f) = \min_{\mu\in P} \mathbb{E}_\mu[u(f)]$. Likewise, maxmax expected utility, $W(f) = \max_{\mu\in P} \mathbb{E}_\mu[u(f)]$, corresponds to the opposite extreme where Pessimism's action set is always trivial (i.e., $\mathbb{P} = \{\{\mu\} : \mu \in P\}$ is a collection of singletons).

Our first main result is that BEU represents the class of preferences that satisfy all subjective expected utility axioms, except that independence is relaxed to certainty independence:

Axiom 1 (Weak Order)**.** $\succsim$ is complete and transitive.

Axiom 2 (Monotonicity)**.** If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.

Axiom 3 (Nondegeneracy)**.** There exist $f, g \in \mathcal{F}$ such that $f \succ g$.

Axiom 4 (Archimedean)**.** For all $f, g, h \in \mathcal{F}$ with $f \succ g \succ h$, there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h.$$

[6] The functional (1) is well-defined since $\mathbb{P}$ is nonempty and compact.

Axiom 5 (Certainty Independence)**.** For all $f, g \in \mathcal{F}$, $p \in \Delta(Z)$, and $\alpha \in (0,1]$,

$$f \succsim g \Longleftrightarrow \alpha f + (1-\alpha)p \succsim \alpha g + (1-\alpha)p.$$

Theorem 1. *Preference $\succsim$ satisfies Axioms 1–5 if and only if $\succsim$ admits a BEU representation.*

Thus, like maxmin expected utility, BEU captures the possible *presence* of ambiguity by imposing independence only for mixtures with constant acts, i.e., mixtures that apply equally to all states.[7] However, unlike maxmin expected utility, BEU does not additionally impose uncertainty aversion, which reflects a *negative attitude* toward ambiguity through a preference for hedging (see Axiom 6).

Theorem 1 shows that BEU provides a novel, easy-to-interpret representation of the class of preferences that Ghirardato, Maccheroni, and Marinacci (2004) (henceforth GMM) term *invariant biseparable*. In Section 4.2, we contrast BEU with existing representations due to GMM and Amarante (2009). In addition, Section 4.1 shows that natural generalizations of BEU represent classes of preferences that further relax certainty independence.

We prove Theorem 1 in Appendix B.1. We first invoke the well-known fact that $\succsim$ satisfies Axioms 1–5 if and only if $\succsim$ can be represented by $I \circ u$ for some nonconstant affine utility u and a functional $I : \mathbb{R}^S \to \mathbb{R}$ that is monotonic, positively homogeneous, and constant-additive (Appendix A.1 defines these terms). For the sufficiency direction of the proof, we then make use of the Clarke differential $\partial I(\underline{0}) \subseteq \Delta(S)$ of I at the constant vector $\underline{0}$ (Clarke, 1990, see Appendix A.2). The key step, which we discuss in Remark 1(iv) below, is to show that the belief-set collection $\mathbb{P}^*$ given by

$$\mathbb{P}^* := \mathrm{cl}\{P^*_\phi : \phi \in \mathbb{R}^S\} \ \text{ with } \ P^*_\phi := \{\mu \in \partial I(\underline{0}) : \mathbb{E}_\mu[\phi] \geq I(\phi)\} \tag{2}$$

yields a BEU representation of I, i.e., for all $\phi \in \mathbb{R}^S$,

$$I(\phi) = \max_{P \in \mathbb{P}^*} \min_{\mu \in P} \mathbb{E}_\mu[\phi]. \tag{3}$$

Remark 1. (i) **General action sets.** The specific form of action sets for Optimism and Pessimism in (1) is without loss of generality. Indeed, $\succsim$ admits a BEU representation with utility u if and only if there exist *arbitrary* action sets A_1, A_2 and a mapping $\mu : A_1 \times A_2 \to$

[7] See Ghirardato, Maccheroni, and Marinacci (2005), who argue why certainty independence is important for achieving a separation of tastes and beliefs.

$\Delta(S)$ from action profiles to beliefs such that

$$W(f) = \max_{a_1 \in A_1} \min_{a_2 \in A_2} \mathbb{E}_{\mu(a_1,a_2)}[u(f)] \tag{4}$$

is well-defined and represents $\succsim$.[8]

(ii) **Min-max form.** While BEU takes the max-min form in which Optimism is the first mover, it is equivalent to consider representations of the min-max form. That is, $\succsim$ admits a BEU representation if and only if it can be represented by the functional $W(f) = \min_{Q \in \mathbb{Q}} \max_{\mu \in Q} \mathbb{E}_\mu[u(f)]$ for some belief-set collection $\mathbb{Q}$. However, the collection $\mathbb{Q}$ need not coincide with $\mathbb{P}$ in general. See Supplementary Appendix S.2 for more details.

(iii) **Single-self interpretation.** In addition to the dual-self interpretation above, BEU admits a single-self interpretation, whereby the DM optimally selects her own ambiguity preference from a feasible set.[9] Specifically, feasible ambiguity preferences take the maxmin expected utility form $\min_{\mu \in P} \mathbb{E}_\mu[u(f)]$ and depending on f, the DM optimally controls the parameter P, where $\mathbb{P}$ represents the constraints of the subjective optimization.

(iv) **Relationship with mathematics literature.** Equation (3) relates to recent results on the linearization of positively homogeneous functions, which imply that a functional $I : \mathbb{R}^S \to \mathbb{R}$ admits a representation of the form $I(\phi) = \max_{U \in \mathbb{U}} \min_{\ell \in U} \ell \cdot \phi$ for some collection $\mathbb{U}$ of compact, convex subsets of $\mathbb{R}^S$ if and only if I is positively homogeneous, lower semicontinuous, and locally Lipschitz (see the survey by Rubinov and Dzalilov, 2002). Our proof shows that under the additional assumption that I is monotonic and constant-additive, $\mathbb{U}$ can be taken to be a belief-set collection. More importantly, our construction only makes use of beliefs μ in the Clarke differential $\partial I(\underline{0})$, which represents precisely the set of priors considered *relevant* by the DM (see Section 2.3). This requires a different proof approach, which builds partly on a non-smooth generalization of Ovchinnikov (2001), who shows that continuously differentiable functionals I admit Boolean representations in terms of affine functionals whose slopes are gradients of I (see Appendix A.3). ▲

2.3 Relevant Priors

A natural way to identify the DM's set of ***relevant priors*** under BEU is to consider the union $\bigcup_{P \in \mathbb{P}} P$ of all sets in the belief-set collection. This captures all possible outcomes of the belief-selection game between Optimism and Pessimism. To eliminate redundant beliefs

[8] To see this, suppose $(\mathbb{P}, u)$ is a BEU representation of $\succsim$. Then (4) represents $\succsim$ with $A_1 := \mathbb{P}$, $A_2 := \prod_{P \in \mathbb{P}} P$, and $\mu(P, \sigma) := \sigma(P)$ for all $P \in A_1$, $\sigma \in A_2$. Conversely, suppose (4) represents $\succsim$ for some (A_1, A_2, μ, u). Then setting $\mathbb{P} := \{\overline{\text{co}}(\mu(a_1, A_2)) : a_1 \in A_1\}$ yields a BEU representation of $\succsim$.

[9] See Sarver (2018) for an analogous model in the context of risk preferences.

that are never selected, we focus on the *smallest* closed, convex set of beliefs that can arise under any BEU representation. Proposition 1 shows that this set is uniquely identified:

Proposition 1. *Suppose $\succsim$ satisfies Axioms 1–5. There exists a unique closed, convex set $C \subseteq \Delta(S)$ such that*

$$C \subseteq \overline{\text{co}} \bigcup_{P \in \mathbb{P}} P \tag{5}$$

for all BEU representations $(\mathbb{P}, u)$ of $\succsim$ and such that (5) holds with equality for some $(\mathbb{P}, u)$.

We call a BEU representation ***tight*** if (5) holds with equality. To prove Proposition 1 (Appendix B.2), we show that for any BEU representation, $\overline{\text{co}} \bigcup_{P \in \mathbb{P}} P$ includes the Clarke differential $\partial I(\underline{0})$ at $\underline{0}$ of the functional I from the proof of Theorem 1. Since the representation $\mathbb{P}^*$ in (2) satisfies $\overline{\text{co}} \bigcup_{P \in \mathbb{P}^*} P = \partial I(\underline{0})$, this implies that the set of relevant priors C is precisely $\partial I(\underline{0})$ and that $\mathbb{P}^*$ is a tight representation.

An implication of this Clarke-differential characterization of C is that our definition of the DM's relevant priors as the possible outcomes of the belief-selection game is equivalent to the following behavioral definition due to GMM, which is based on quantifying departures from independence. For any preference $\succsim$ satisfying Axioms 1–5, GMM define the ***unambiguous preference*** $\succsim^*$ as the largest independent subrelation of $\succsim$; equivalently, $f \succsim^* g$ means that $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$ holds for all $\alpha \in (0, 1]$ and $h \in \mathcal{F}$.

Note that $\succsim^*$ is incomplete whenever $\succsim$ violates independence. GMM show that $\succsim^*$ admits a unanimity representation à la Bewley (2002) and identify the unique closed, convex set of priors in the unanimity representation as the DM's relevant set of priors.[10] Since GMM show that the latter set again coincides with $\partial I(\underline{0})$, we obtain the following corollary:

Corollary 1. *If $\succsim$ admits a BEU representation with utility u, then the set of relevant priors C is the unique closed, convex set such that*

$$f \succsim^* g \iff \mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\mu[u(g)] \text{ for all } \mu \in C. \tag{6}$$

Remark 2 (Uniqueness)**.** Our results in the remainder of this note apply to *all* BEU representations of a given preference, and thus do not require unique identification of a particular representation. Nevertheless, standard arguments imply that the utility u under BEU is unique up to positive affine transformation. Moreover, Supplementary Appendix S.1 shows that the belief-set collection $\mathbb{P}$ is unique up to "half-space closure," analogous to recent representations featuring collections of sets of utilities (e.g., Hara, Ok, and Riella, 2019). ▲

[10] Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) take an alternative approach by including $\succsim^*$ as part of the primitive. Ghirardato and Siniscalchi (2012) extend GMM's characterization of relevant priors beyond the invariant biseparable class. See Klibanoff, Mukerji, and Seo (2014) for a discussion of the interpretation of C.

3 Ambiguity Attitude

In this section, we highlight that BEU provides a unified framework through which to represent and contrast different attitudes toward ambiguity.

3.1 Comparative Ambiguity Attitude

Recall the standard comparative notion of ambiguity aversion (Ghirardato and Marinacci, 2002), whereby $\succsim_1$ is ***more ambiguity-averse*** than $\succsim_2$ if whenever $f \succsim_1 p$ for some $f \in \mathcal{F}$ and $p \in \Delta(Z)$, then $f \succsim_2 p$. We begin by showing that under BEU, this is represented by a preorder over belief-set collections which captures the relative "power" allocated to Pessimism in the belief-selection game.

Formally, write $\mathbb{P}_1 \sqsupseteq \mathbb{P}_2$ if

$$\text{for all } P_1 \in \mathbb{P}_1 \text{ there exists } P_2 \in \mathbb{P}_2 \text{ with } P_1 \supseteq P_2.$$

To interpret, if $\mathbb{P}_1 \sqsupseteq \mathbb{P}_2$, then for any potential move P_1 of Optimism under $\mathbb{P}_1$, Optimism has a move $P_2 \subseteq P_1$ in $\mathbb{P}_2$ that restricts Pessimism's action set more. Thus, Pessimism's relative power to influence the DM's belief is weaker under collection $\mathbb{P}_2$ than under $\mathbb{P}_1$.[11]

Proposition 2. *Suppose $\succsim_1$, $\succsim_2$ admit BEU representations. The following are equivalent:*

1. *$\succsim_1$ is more ambiguity-averse than $\succsim_2$.*
2. *$\succsim_1$ admits a BEU representation $(\mathbb{P}_1, u_1)$ such that every BEU representation $(\mathbb{P}_2, u_2)$ of $\succsim_2$ satisfies $\mathbb{P}_1 \sqsupseteq \mathbb{P}_2$ and $u_1 \approx u_2$.*

The proof exhibits a representation $\hat{\mathbb{P}}_i$ of $\succsim_i$ that $\sqsupseteq$-dominates any other representation, and shows that $\succsim_1$ is more ambiguity-averse than $\succsim_2$ if and only if $\hat{\mathbb{P}}_1 \sqsupseteq \hat{\mathbb{P}}_2$. Note that Proposition 2 does not assume any relationship between the sets of relevant priors C_1 and C_2 associated with $\succsim_1$ and $\succsim_2$. This is in contrast with GMM's characterization of comparative ambiguity aversion, which requires the assumption that $C_1 = C_2$ (Proposition 12 in GMM).

While for a given set of relevant priors C, maxmin expected utility ($\mathbb{P} = \{C\}$) and maxmax expected utility ($\mathbb{P} = \{\{\mu\} : \mu \in C\}$) represent the most and least ambiguity-averse BEU representations, the following two subsections proceed to characterize a hierarchy of intermediate ambiguity attitudes that correspond to less extreme allocations of power across the two selves.

[11] An alternative, stronger order over belief-set collections is given by set inclusion, $\mathbb{P}_1 \subseteq \mathbb{P}_2$. One can show that this represents comparative ambiguity aversion in the following weaker sense: Suppose $\succsim_1$, $\succsim_2$ admit BEU representations. Then $\succsim_1$ is more ambiguity-averse than $\succsim_2$ if and only if there exist BEU representations $(\mathbb{P}_i, u_i)$ of $\succsim_i$ that satisfy $\mathbb{P}_1 \subseteq \mathbb{P}_2$ and $u_1 \approx u_2$.

3.2 Shades of Ambiguity Aversion

Existing decision-theoretic definitions of ambiguity aversion postulate a preference for hedging, or randomization, but vary in the degree to which they impose this attitude. The seminal axiom in this literature, Schmeidler's (1989) *uncertainty aversion*, postulates that the DM *always* takes up an opportunity to hedge between two equally valued prospects.

Axiom 6 (Uncertainty Aversion)**.** If $f, g \in \mathcal{F}$ with $f \sim g$, then $\frac{1}{2}f + \frac{1}{2}g \succsim f$.

The second standard definition is Ghirardato and Marinacci's (2002) notion of absolute ambiguity aversion, which relies on the comparative definition considered in the previous section. Analogous to the definition of absolutely risk-averse as more risk-averse than a risk-neutral preference, $\succsim$ is said to be ***absolutely ambiguity-averse*** if it is more ambiguity-averse than some nondegenerate subjective expected utility preference.[12]

Our main result in this section contrasts these two formalizations of ambiguity aversion under BEU, as well as the following third notion:

Axiom 7 (k-Ambiguity Aversion)**.** For all $f_1, ..., f_k \in \mathcal{F}$ with $f_1 \sim f_2 \sim \cdots \sim f_k$ and any $p \in \Delta(Z)$,

$$\frac{1}{k}f_1 + \cdots + \frac{1}{k}f_k = p \Longrightarrow p \succsim f_1.$$

Axiom 7 only imposes a preference for *complete* hedging between k equally valued prospects, that is, for hedges that eliminate subjective uncertainty entirely. We say that $\succsim$ satisfies ∞***-ambiguity aversion*** if it satisfies k-ambiguity aversion for all k. This corresponds to the notion of preference for sure diversification used by Chateauneuf and Tallon (2002) to characterize absolute ambiguity aversion under Choquet expected utility. Arguments in Grant and Polak (2013) imply that this characterization extends to BEU; moreover, we note that $|S|$-ambiguity aversion is sufficient for ∞-ambiguity aversion (where $|S|$ is the cardinality of the state space):

Lemma 1. *Suppose $\succsim$ admits a BEU representation. The following are equivalent: (i) $\succsim$ is absolutely ambiguity-averse; (ii) $\succsim$ satisfies ∞-ambiguity aversion; (iii) $\succsim$ satisfies $|S|$-ambiguity aversion.*

We now show that under BEU, the above notions of ambiguity aversion are characterized by the degree of overlap of sets in $\mathbb{P}$, capturing successively less power allocated to Pessimism:

Theorem 2. *Suppose that $\succsim$ admits a BEU representation $(\mathbb{P}, u)$. Then:*

[12] See Epstein (1999) for another approach that takes as its benchmark probabilistic sophistication instead of subjective expected utility.

1. $\succsim$ *satisfies uncertainty aversion if and only if* $\bigcap_{P\in\mathbb{P}} P = C$*;*

2. $\succsim$ *is absolutely ambiguity-averse if and only if* $\bigcap_{P\in\mathbb{P}} P \neq \emptyset$*;*

3. $\succsim$ *satisfies* k*-ambiguity aversion if and only if* $\bigcap_{i=1,\ldots,k} P_i \neq \emptyset$ *for all* $P_1, ..., P_k \in \mathbb{P}$*.*

Uncertainty aversion corresponds to the maximal allocation of power to Pessimism, in the sense that *all* relevant priors $\mu \in C$ are available to Pessimism regardless of which set P Optimism chooses. The game thus boils down to Pessimism choosing a belief $\mu \in C$, yielding maxmin expected utility; indeed, note that if $(\mathbb{P}, u)$ is tight, then $\succsim$ satisfies uncertainty aversion if and only if $\mathbb{P} = \{C\}$.

Absolute ambiguity aversion allocates less power to Pessimism, requiring only that there is *some* prior $\mu \in \bigcap_{P\in\mathbb{P}} P$ that is always available to Pessimism regardless of Optimism's choice. Thus, the DM's valuation of any act f is bounded above by the expected utility $\mathbb{E}_\mu[u(f)]$ of f under prior μ, which implies that $\succsim$ is more ambiguity-averse than the expected utility preference with belief μ and utility u.

Finally, while absolute ambiguity aversion requires the intersection of *all* sets in $\mathbb{P}$ to be nonempty, k-ambiguity aversion imposes this only for any k sets in $\mathbb{P}$. Thus, k-ambiguity aversion further decreases the power allocated to Pessimism, and more so the smaller k. Indeed, whenever k-ambiguity aversion holds at $\mathbb{P}_1$, then any representation $\mathbb{P}_2 \sqsupseteq \mathbb{P}_1$ displays a weakly higher degree of k-ambiguity aversion.

The relevance of further relaxing the DM's negative ambiguity attitude in this manner is underscored by experimental evidence. Indeed, one notable pattern suggesting that subjects' preferences might be better described by k-ambiguity aversion for small k than for large k is *ambiguity seeking for small odds*, which was originally conjectured by Ellsberg (e.g., Ellsberg, 2011) and subsequently confirmed in laboratory experiments:[13]

Example 1 (Ellsberg urn with many colors)**.** Consider an urn with 10 balls with unknown composition from up to 10 different colors. A ball is drawn from the urn and its color observed. State space $S = \{1, \cdots, 10\}$ represents the observed color. For each event $E \subseteq S$, let f_E denote the uncertain bet that pays \$10 if the color of the ball belongs to E and \$0 otherwise, and let p_α denote the objective lottery that pays \$10 with probability α and \$0 otherwise.

When the cardinality of E is 5, this setting is similar to Ellsberg's two-color urn experiment, suggesting a preference for the objective lottery $p_{0.5}$ over the uncertain bet f_E, consistent with 2-ambiguity aversion. However, when E is a singleton event, many subjects prefer

[13] Dillenberger and Segal (2017) show that a version of Segal's (1987) model is consistent with this evidence.

f_E to the corresponding objective lottery $p_{0.1}$ (e.g., Dimmock, Kouwenberg, Mitchell, and Peijnenburg, 2015; Kocher, Lahno, and Trautmann, 2018). Assuming that $f_{\{1\}} \sim \ldots \sim f_{\{10\}}$ by symmetry, this contradicts 10-ambiguity aversion as $p_{0.1} = \dfrac{1}{10} f_{\{1\}} + \cdots \dfrac{1}{10} f_{\{10\}}$. ▲

The following simple example illustrates that BEU allows for flexible degrees of k-ambiguity aversion and hence can accommodate the aforementioned experimental evidence. This contrasts, for instance, with Siniscalchi's (2009) vector expected utility model, which also relaxes uncertainty aversion, but for which 2-ambiguity aversion and ∞-ambiguity aversion are equivalent.[14]

Example 2. Consider a BEU representation $(\mathbb{P}, u)$ of the form $\mathbb{P} = \{P_s : s \in S\}$ where for some fixed $\epsilon \geq 0$,

$$P_s := \{\mu \in \Delta(S) : \mu(s) \geq \epsilon\}$$

for each s. For each $k \leq |S|$, Theorem 2 implies that k-ambiguity aversion is satisfied if and only if $\epsilon \leq \frac{1}{k}$.[15] ▲

3.3 Ambiguity Aversion Index and Source Dependence

While the preceding notions of ambiguity aversion are "global," capturing the DM's attitude towards *any* uncertainty that can be generated in S, the experimental literature commonly takes a "local" approach, measuring the DM's ambiguity attitude relative to specific events or sources of uncertainty.

A primary local measure of ambiguity attitudes is based on the following idea originally proposed by Schmeidler (1989) and subsequently employed in both theoretical work (Dow and Werlang, 1992) and in experiments (Baillon and Bleichrodt, 2015; Baillon, Huang, Selim, and Wakker, 2018). Given any event $E \subseteq S$, we first define its ***matching probability*** $m(E) \in [0, 1]$ by the indifference condition

$$x_E y \sim m(E)\delta_x + (1 - m(E))\delta_y,$$

where $x, y \in Z$ are two outcomes such that $\delta_x \succ \delta_y$ and $x_E y$ denotes the binary act that yields x for all $s \in E$ and y otherwise.[16] Based on this, define the ***ambiguity aversion***

[14] Note that 2-ambiguity aversion is equivalent to Siniscalchi's (2009) Axiom 11, which he shows is equivalent to absolute ambiguity aversion (provided utilities are unbounded).

[15] To see this, suppose $\epsilon \leq \frac{1}{k}$. Then for any $s_1, ..., s_k$, we have $\frac{1}{k}\delta_{s_1} + \cdots \frac{1}{k}\delta_{s_k} \in \cap_{i=1}^k P_{s_i}$, so that k-ambiguity aversion holds. Conversely, if $\epsilon > \frac{1}{k}$, take any distinct $s_1, ..., s_k$. If $\mu \in \bigcap_{i=1}^k P_{s_i}$, then $\mu(s_i) > \frac{1}{k}$ for all $i = 1, ..., k$, contradicting $\mu \in \Delta(S)$. Thus, $\bigcap_{i=1}^k P_{s_i} = \emptyset$, so that k-ambiguity aversion fails.

[16] Under Axioms 1–5, $m(\cdot)$ is well-defined independent of the choice of x, y.

index associated with E by

$$AA(E) := 1 - m(E) - m(E^c). \tag{7}$$

Whereas subjective expected utility implies $AA(E) = 0$ for all E, $AA(E) > 0$ (resp. $AA(E) < 0$) is interpreted as a negative (resp. positive) attitude to ambiguity associated with E. Note that this index can be defined for any event, without imposing symmetry on the state space as is common in urn experiments.

Under BEU, the sign of $AA(E)$ is characterized by the following local analog of the binary intersection condition for 2-ambiguity aversion in Theorem 2:

Lemma 2. *Suppose $\succsim$ admits a BEU representation $(\mathbb{P}, u)$. Then for any $E \subseteq S$,*

$$AA(E) \geq 0 \iff P_E \cap P'_E \neq \emptyset \text{ for all } P, P' \in \mathbb{P},$$

where $P_E := \{\mu(E) : \mu \in P\}$.

As a result, 2-ambiguity aversion implies $AA(E) \geq 0$ for *all* events E. As such, 2-ambiguity aversion may still be too restrictive to accommodate experimental evidence that subjects who display negative ambiguity attitudes for "unfamiliar" events (i.e., when they feel less competent about the relevant domain of uncertainty) sometimes display less negative, or even positive, attitudes for familiar events. For example, among German subjects, Keppe and Weber (1995) find a positive average ambiguity aversion index for bets concerning US geography, but a negative average index for bets concerning German geography. This can be seen as a manifestation of *source dependence*, i.e., the idea that agents' ambiguity attitudes vary across sources of uncertainty.[17] The following is a stylized example in the context of home bias (French and Poterba, 1991):

Example 3 (Home bias)**.** Let $S_H = \{U, D\}$ be a state space specifying whether the domestic stock market goes up ("U") or down ("D"). Similarly, let $S_F = \{U, D\}$ describe the state of the stock market in a foreign country. Consider the product state space $S = S_H \times S_F$, and let $E_H = \{UU, UD\}$ be the event that the domestic stock market goes up, and $E_F = \{UU, DU\}$ be the corresponding event for the foreign stock market. Under source dependence, investors

[17] See, e.g., Heath and Tversky (1991); Fox and Tversky (1995); Abdellaoui, Baillon, Placido, and Wakker (2011); Chew, Miao, and Zhong (2018) for related experimental evidence. Several papers (e.g., Nau, 2006; Chew and Sagi, 2008; Ergin and Gul, 2009; Gul and Pesendorfer, 2015; Cappelli, Cerreia-Vioglio, Maccheroni, Marinacci, and Minardi, 2016) propose formalizations of source dependence based on the idea that the DM is probabilistically sophisticated over prospects that depend on a single common source, but exhibits varying attitudes toward uncertainty across sources. Our focus in this section is a specific form of variation where the DM exhibits negative vs. positive ambiguity attitudes depending on his familiarity with each source.

may display a higher ambiguity aversion index for foreign than domestic stock; indeed, some may reverse the sign, $AA(E_F) > 0 > AA(E_H)$, i.e., be ambiguity-seeking for E_H but ambiguity-averse for E_F.[18] ▲

The following result shows that BEU can accommodate the home bias in Example 3; indeed, it can capture source-dependent negative and positive ambiguity attitudes with respect to any families $\mathcal{E}$ and $\mathcal{F}$ of unfamiliar and familiar events:

Proposition 3. *Fix any disjoint collections $\mathcal{E}$ and $\mathcal{F}$ of events, both of which are closed under complements and do not contain S. There exists a preference $\succsim$ satisfying Axioms 1–5 such that $AA(E) > 0 > AA(F)$ for all $E \in \mathcal{E}, F \in \mathcal{F}$.*

Proposition 3 highlights an important distinction with a special case of BEU, α-maxmin expected utility, which represents preferences by the functional

$$W(f) = \alpha \min_{\mu \in P} \mathbb{E}_\mu[u(f)] + (1-\alpha) \max_{\nu \in P} \mathbb{E}_\nu[u(f)]$$

for some $\alpha \in [0,1]$, nonempty closed, convex set of beliefs P, and nonconstant affine u.

Due to its tractability, α-maxmin is often used in applied theoretical work or for analyzing experimental data.[19] However, while α-maxmin can accommodate flexible degrees of k-ambiguity aversion (based on the same idea as Example 2), Lemma 2 implies that this model is inconsistent with source-dependent negative and positive ambiguity attitudes. Indeed, the sign of the ambiguity index is the *same* for all events and is determined by the value of α:

Corollary 2. *Suppose $\succsim$ admits an α-maxmin representation where P is not a singleton. Then $\alpha \geq 1/2$ (resp. $\alpha \leq 1/2$) if and only if $AA(E) \geq 0$ (resp. $AA(E) \leq 0$) for all E.*

More strongly, one can show that $\alpha \geq \frac{1}{2}$ implies 2-ambiguity aversion while $\alpha \leq \frac{1}{2}$ implies 2-ambiguity seeking (as defined in Supplementary Appendix S.2).

At the same time, we highlight another special case of BEU that retains much of the tractability of α-maxmin, but allows for source-dependent negative and positive ambiguity attitudes as in Example 3. Specifically, consider a simple generalization of α-maxmin that allows for different sets of beliefs P_1 and P_2 for the max and min operator, i.e.,

$$W(f) = \alpha \min_{\mu \in P_1} \mathbb{E}_\mu[u(f)] + (1-\alpha) \max_{\nu \in P_2} \mathbb{E}_\nu[u(f)].$$

[18] In an incentivized field survey among investors, Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg (2019) (Figures 4 and 5) find reversals as in Example 3, where H and F correspond to a domestic and foreign stock market index. They also find a higher population average AA index for E_F than E_H, although the difference is relatively small, as some investors display the opposite reversal (which can also be accommodated by BEU).

[19] Many applications use the representation characterized by Chateauneuf, Eichberger, and Grant (2007), i.e., the special case of α-maxmin where P is a convex combination of a fixed prior π and $\Delta(S)$.

To capture Example 3, we can set $P_1 := \{\mu : \mu(E_H) = \frac{1}{2}\}$ and $P_2 := \{\mu : \mu(E_F) = \frac{1}{2}\}$. This implies $AA(E_H) = \alpha - 1 < 0$ and $AA(E_F) = \alpha > 0$ for $\alpha \in (0, 1)$, thereby generating negative ambiguity attitudes for foreign events and positive attitudes for home events.

Remark 3. While in practice index (7) is typically defined using matching probabilities on binary partitions $\{E, E^c\}$, it can be generalized to arbitrary partitions $\mathcal{E}$ of S by setting

$$AA(\mathcal{E}) = 1 - \sum_{E \in \mathcal{E}} m(E).$$

Given this, k-ambiguity aversion implies that $AA(\mathcal{E}) \geq 0$ for all $\mathcal{E}$ with $|\mathcal{E}| \leq k$.[20] Thus, the aforementioned evidence on ambiguity seeking for small odds suggests the need to allow the sign of $AA(\mathcal{E})$ to depend on the number of events in partition $\mathcal{E}$. Example 2 can accommodate this, as the index satisfies $AA(\mathcal{E}) = 1 - \epsilon|\mathcal{E}|$ for any non-trivial partition $\mathcal{E}$. ▲

4 Discussion

4.1 Generalizations

As we have seen, our baseline model, BEU, corresponds to a relaxation of subjective expected utility where independence is weakened to certainty independence and, equivalently, to dropping uncertainty aversion from Gilboa and Schmeidler's (1989) axioms. The representation adds a maximization stage to Gilboa and Schmeidler (1989), admitting an interpretation in terms of a game between Optimism and Pessimism.

We highlight that this approach generalizes beyond certainty independence, yielding intuitive representations that further relax independence but still do not impose uncertainty aversion. To illustrate, Supplementary Appendix S.3 shows that replacing certainty independence with weak certainty independence (Maccheroni, Marinacci, and Rustichini, 2006) yields a representation of the form

$$W(f) = \max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} \mathbb{E}_\mu[u(f)] + c(\mu), \tag{8}$$

where Optimism first chooses a convex cost function $c : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ from a collection $\mathbb{C}$ that is grounded (i.e., $\max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} c(\mu) = 0$) and Pessimism then chooses a belief subject to this cost.[21] This adds a maximization stage into Maccheroni, Marinacci, and

[20]The proof follows from Lemma C.1 in the appendix.

[21]A special case of (8), imposing the stronger requirement that $\min_{\mu \in \Delta(S)} c(\mu) = 0$ for all $c \in \mathbb{C}$, appears in the conclusion of Castagnoli, Cattelan, Maccheroni, and Tebaldi (in preparation), who note that this

Rustichini's (2006) variational model, which corresponds to the special case that additionally satisfies uncertainty aversion. An even weaker form of independence, which applies only to objective lotteries, leads to a representation with general game payoffs, extending Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio's (2011) uncertainty-averse representation (see Supplementary Appendix S.4).[22]

Further relaxing independence in this manner is motivated by additional experimental evidence. For instance, representation (8), which relaxes the positive homogeneity of I implied by certainty independence while preserving constant-additivity, can accommodate Machina's (2009) paradoxes (see also Baillon, L'Haridon, and Placido, 2011).[23] Another important finding is that ambiguity attitudes can differ for gains and losses, e.g., in urn experiments subjects who are ambiguity-averse for bets with positive payoffs are often ambiguity-seeking when the sign of the bet is reversed (Trautmann and Wakker, 2018). The latter finding is inconsistent with any representation that displays constant-additivity, but can be accommodated by our most general model in Supplementary Appendix S.4.

4.2 Related Literature

This note makes two main contributions to the decision-theoretic literature on preferences under ambiguity (for a survey, see Gilboa and Marinacci, 2016). First, we propose a class of multiple-prior representations, BEU and its extensions, that do not impose the worst-case belief selection mechanism implied by uncertainty aversion and instead model beliefs as the outcome of a game between Pessimism and Optimism. Second, we use BEU to characterize a hierarchy of natural intermediate ambiguity attitudes.

Our first contribution, in particular the finding that BEU represents the class of invariant biseparable preferences, relates to GMM and Amarante (2009). GMM propose the first representation of invariant biseparable preferences, which takes an act-dependent α-maxmin form,

$$W(f) = \alpha(f) \min_{\mu \in C} \mathbb{E}_\mu[u(f)] + (1 - \alpha(f)) \max_{\mu \in C} \mathbb{E}_\mu[u(f)], \tag{9}$$

where C is the set of priors in the Bewley representation (6) of the unambiguous preference $\succsim^*$ and $\alpha(\cdot)$ is a function from acts to $[0, 1]$. Importantly, $\alpha(\cdot)$ must satisfy several restrictions to ensure necessity of the axioms: Specifically, $\alpha(\cdot)$ must be measurable with respect to a

special case is characterized by the following axiom in addition to our axioms: for all $f \in \mathcal{F}$, $p \in \Delta(Z)$ and $\alpha \in (0, 1)$, $f \succsim p \Longrightarrow \alpha f + (1-\alpha)p \succsim p$ (F. Maccheroni, personal communication, June 2019).

[22] Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011) provide an alternative representation of this class of preferences, which generalizes GMM by imposing weaker restrictions on the weight function $\alpha(\cdot)$ in (9) below.

[23] This follows from the fact that Siniscalchi's (2009) vector expected utility model can accommodate these paradoxes and is a special case of (8).

particular derived equivalence relation $\asymp$ over acts and $\alpha(\cdot)$ must be such that the preference represented by (9) is monotonic (see Remark 2 in GMM).[24] Amarante (2009) provides an alternative representation via the functional

$$W(f) = \int_{\Delta(S)} \mathbb{E}_\mu[u(f)]\, d\nu(\mu),$$

which captures a DM who holds first-order beliefs $\mu \in \Delta(S)$ that are probability measures, but faces second-order uncertainty over first-order beliefs that takes the form of a Choquet capacity ν.

BEU is closer to GMM, in that both representations induce act-dependent beliefs that are jointly influenced by Optimism and Pessimism. However, while (9) imposes act dependence exogenously through the weights $\alpha(\cdot)$ on max and min, the act-dependent belief selection under BEU can be interpreted endogenously, as the outcome of a sequential game between Optimism and Pessimism. More importantly, in contrast with the aforementioned restrictions on $\alpha(\cdot)$, necessity of the axioms under BEU requires no additional restrictions on the belief-set collection $\mathbb{P}$.

Our second contribution of using BEU to characterize a hierarchy of intermediate ambiguity attitudes has no counterpart in GMM or Amarante (2009). This is also an important difference from other models that relax uncertainty aversion, including Schmeidler's (1989) Choquet expected utility, Klibanoff, Marinacci, and Mukerji's (2005) smooth model, and models of preferences over utility dispersion (e.g., Siniscalchi, 2009; Grant and Polak, 2013): While some of these papers provide representations of absolute ambiguity aversion, none use their models to characterize weaker degrees of ambiguity aversion.[25]

Related to the structure of BEU, several recent papers employ belief-set or utility-set collections in other contexts. While we maintain the weak order axiom and focus on relaxing independence, Lehrer and Teper (2011) and Nascimento and Riella (2011) (resp. Hara, Ok, and Riella, 2019) represent preferences over acts (resp. lotteries) that violate completeness and/or transitivity. Whereas BEU is a utility representation, these papers provide generalized unanimity representations à la Bewley (2002) and Dubra, Maccheroni, and Ok (2004),

[24] GMM also characterize the special case of (9) where $\alpha(\cdot)$ is constant, i.e., the subclass of α-maxmin representations whose set of priors P coincides with the induced Bewley set C. Eichberger, Grant, Kelsey, and Koshevoy (2011) show that if the state space is finite, this representation reduces to maxmin or maxmax. Siniscalchi (2006) axiomatizes a special case of invariant biseparable preferences that have a piecewise subjective expected utility form.

[25] Absolute ambiguity aversion is equivalent to concavity of the function ϕ that aggregates expected utilities across different priors in Klibanoff, Marinacci, and Mukerji's (2005) smooth model; to non-emptiness of the capacity's core under Choquet expected utility; and to non-positivity of the adjustment function under Siniscalchi's (2009) vector expected utility model.

and the resulting proof methods are quite different. In the context of attitudes to randomization under ambiguity, Ke and Zhang (2019) consider preferences over *lotteries* over acts and propose a representation that adds *minimization* over belief-set collections to maxmin expected utility. When restricted to acts (i.e., degenerate lotteries), their representation is equivalent to Gilboa and Schmeidler (1989).

Beyond the ambiguity literature, BEU is related to Hart, Modica, and Schmeidler (1994), who provide a preference foundation for maxmin values in zero-sum games. They consider a product state space $S = S_1 \times S_2$, where S_1 and S_2 are interpreted as the DM's and opponent's action sets. They characterize when preferences over acts can be represented as the maxmin value of a simultaneous-move zero-sum game,

$$W(f) = \max_{\mu_1 \in \Delta(S_1)} \min_{\mu_2 \in \Delta(S_2)} \sum_{s_1, s_2} \mu_1(s_1)\mu_2(s_2)u(f(s_1, s_2)),$$

which is formally a strict special case of BEU.

Finally, we note the independent and contemporaneous work by Chandrasekher (2019), who characterizes the special case of BEU with finitely many sets of priors by additionally imposing a weak form of uncertainty aversion.[26] His motivation is to study a notion of source dependence which requires the matching probabilities of events belonging to one (e.g., familiar) source to be higher than those of events belonging to another (e.g., unfamiliar) source. This is different from our focus in Section 3.3 on positive vs. negative ambiguity aversion indices for familiar vs. unfamiliar events. Whereas we show that the latter is incompatible with α-maxmin, his notion is in general compatible even with uncertainty aversion.

Appendix: Proofs

A Preliminaries

Throughout this section, we fix any interval $\Gamma \subseteq \mathbb{R}$ and let $U := \Gamma^S$. For any $a \in \mathbb{R}$, let $\underline{a}$ denote the vector in $\mathbb{R}^S$ with $\underline{a}(s) = a$ for all $s \in S$. For any $\phi, \psi \in \mathbb{R}^S$, write $\phi \geq \psi$ if $\phi(s) \geq \psi(s)$ for all s.

[26] His proof approach is quite different from ours, and while as an intermediate step, he obtains a general BEU representation when weak uncertainty aversion is dropped, the resulting representation is not tight, i.e., $\bigcup_{P \in \mathbb{P}} P$ is strictly larger than the set of relevant priors.

A.1 Properties of functionals

Fix any functional $I : U \to \mathbb{R}$. We call I: *monotonic* if $I(\phi) \geq I(\psi)$ for all $\phi, \psi \in U$ with $\phi \geq \psi$; *normalized* if $I(\underline{a}) = a$ for all $a \in \Gamma$; *constant-additive* if $I(\phi + \underline{a}) = I(\phi) + a$ for all $\phi \in U$ and $a \in \Gamma$ with $\phi + \underline{a} \in U$; *positively homogeneous* if $I(a\phi) = aI(\phi)$ for all $\phi \in U$ and $a \in \mathbb{R}_+$ with $a\phi \in U$; and *constant-linear* if I is constant-additive and positively homogeneous. It is easy to see that if $\underline{0} \in \Gamma$, then any constant-linear functional I is normalized.

A.2 Clarke derivative and differential

Consider a locally Lipschitz functional $I : U \to \mathbb{R}$. For every $\phi \in \text{int}U$ and $\xi \in \mathbb{R}^S$, the *Clarke (upper) derivative* of I in ϕ in the direction of ξ is

$$I^\circ(\phi; \xi) := \limsup_{\psi \to \phi, t \downarrow 0} \frac{I(\psi + t\xi) - I(\psi)}{t}.$$

The *Clarke (sub)differential* of I at ϕ is the set

$$\partial I(\phi) := \{\chi \in \mathbb{R}^S : \chi \cdot \xi \leq I^\circ(\phi; \xi), \forall \xi \in \mathbb{R}^S\}.$$

We will frequently invoke the following properties of the Clarke differential. First, if I is locally Lipschitz, then Rademacher's theorem yields a subset $\hat{U} \subseteq \text{int}U$ such that $U \setminus \hat{U}$ has Lebesgue measure zero and I is differentiable on $\hat{U}$. Combining this with Theorem 2.5.1 in Clarke (1990), we obtain the following approximation of the Clarke differential:

Lemma A.1 (Theorem 2.5.1 in Clarke (1990))**.** *Suppose $I : U \to \mathbb{R}$ is locally Lipschitz. Then there exists $\hat{U} \subseteq \text{int}U$ such that $U \setminus \hat{U}$ has Lebesgue measure zero, I is differentiable at each $\psi \in \hat{U}$, and for every $\phi \in \text{int}U$, we have*

$$\partial I(\phi) = \text{co}\{\lim_n \nabla I(\phi_n) : \phi_n \to \phi, \phi_n \in \hat{U}\}. \tag{10}$$

The next result is an "envelope theorem" for Clarke differentials:

Lemma A.2 (Theorem 2.8.6 in Clarke (1990))**.** *Suppose functional $I : U \to \mathbb{R}$ is given by*

$$I(\cdot) = \sup_{t \in T} I_t(\cdot)$$

for some indexed family of functionals $(I_t)_{t \in T}$ with domain U. Assume that there exists some $K > 0$ such that $|I_t(\psi) - I_t(\xi)| \leq K\|\psi - \xi\|$ for every $t \in T$ and $\psi, \xi \in \text{int}U$. Then for every

$\phi \in \mathrm{int}U$, *we have* $\partial I(\phi) \subseteq \mathrm{co}\{\lim_{i\to\infty} \nabla I_{t_i}(\phi_i) : \phi_i \to \phi, t_i \in T, I_{t_i}(\phi) \to I(\phi)\}$.

Last, we note the following relationship between properties of I and its Clarke differential:

Lemma A.3 (Part 1 of Proposition A.3 in GMM)**.** *If* $I : U \to \mathbb{R}$ *is locally Lipschitz, positively homogeneous, and* $\underline{0} \in \mathrm{int}U$*, then* $\partial I(\phi) \subseteq \partial I(\underline{0})$ *for all* $\phi \in \mathrm{int}U$*.*

Lemma A.4 (Parts 2–3 of Proposition A.3 in GMM)**.** *If* $I : U \to \mathbb{R}$ *is locally Lipschitz, monotonic, and constant-additive, then* $\partial I(\phi) \subseteq \Delta(S)$ *for all* $\phi \in \mathrm{int}U$*.*

A.3 Boolean representation of locally Lipschitz I

Throughout this subsection, we assume that $I : U \to \mathbb{R}$ is locally Lipschitz. Let $\hat{U}$ be the generic subset given by Lemma A.1.

Lemma A.6 below shows that, restricted to $\hat{U}$, I admits a Boolean representation in terms of a family of affine functionals whose slopes correspond to gradients of I. This result extends Ovchinnikov (2001), who establishes Lemma A.6 under the assumption that I is continuously differentiable. Our non-smooth generalization is necessary for the proof of Theorem 1, where the utility-act functional I is non-differentiable (except in the case of subjective expected utility). We begin with a preliminary result:

Lemma A.5. *For every* $\phi, \psi \in \hat{U}$ *and* $\epsilon > 0$*, there exists* $\xi \in \hat{U}$ *such that*

$$I(\xi) - I(\psi) + \nabla I(\xi) \cdot (\psi - \xi) \geq 0, \quad I(\xi) - I(\phi) + \nabla I(\xi) \cdot (\phi - \xi) \leq \epsilon.$$

Proof. Take any $\phi, \psi \in \hat{U}$ and $\epsilon > 0$. Let $m := I(\psi) - I(\phi)$. If $\nabla I(\phi) \cdot (\psi - \phi) \geq m$, we can set $\xi = \phi$. Likewise if $\nabla I(\psi) \cdot (\psi - \phi) \geq m$, we can set $\xi = \psi$. It remains to consider the case

$$\nabla I(\phi) \cdot (\psi - \phi), \nabla I(\psi) \cdot (\psi - \phi) < m. \tag{11}$$

Define

$$H(\lambda) := I(\phi + \lambda(\psi - \phi)) - \lambda m - I(\phi)$$

for each $\lambda \in \mathbb{R}$ with $\phi + \lambda(\psi - \phi) \in U$. Since $\phi, \psi \in \hat{U}$, H is differentiable at $\lambda \in \{0, 1\}$, with $H(0) = H(1) = 0$ and $H'(0), H'(1) < 0$ by assumption (11). Hence, H is negative for small enough $\lambda > 0$ and positive for $\lambda < 1$ close enough to 1. Thus, the set $\{\lambda \in (0,1) : H(\lambda) = 0\}$ is nonempty and closed; let λ^* denote its supremum.

Since H is locally Lipschitz, we have $H(\lambda) = \int_{\lambda^*}^{\lambda} H'(\lambda')d\lambda'$ for all $\lambda > \lambda^*$. As $H(\lambda) > 0$ for all $\lambda \in (\lambda^*, 1)$, we can choose $\lambda^{**} \in (\lambda^*, 1)$ close enough to λ^* such that H is differentiable at λ^{**} with $H'(\lambda^{**}) > 0$ and $H(\lambda^{**}) \in (0, \epsilon)$. But then

$$H'(\lambda^{**}) = \lim_{t \to 0} \frac{I(\phi + (\lambda^{**} + t)(\psi - \phi)) - I(\phi + \lambda^{**}(\psi - \phi))}{t} - m > 0,$$

which implies that

$$I^{\circ}(\phi + \lambda^{**}(\psi - \phi); \psi - \phi) - m \geq H'(\lambda^{**}) > 0.$$

Since $I^{\circ}(\xi;\zeta) = \max_{\mu \in \partial I(\xi)} \mu \cdot \zeta$ for any ζ, ξ (e.g., Proposition 2.1.2 in Clarke, 1990), this yields some $\mu \in \partial I(\phi + \lambda^{**}(\psi - \phi))$ such that

$$\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0.$$

By (10), there exists a sequence $\xi_n \to \phi + \lambda^{**}(\psi - \phi)$ such that $\xi_n \in \hat{U}$ for each n and $\lim_n \nabla I(\xi_n) = \mu$. Then

$$\begin{aligned} \lim_n \left(I(\xi_n) - I(\psi) + \nabla I(\xi_n) \cdot (\psi - \xi_n)\right) &= I(\phi + \lambda^{**}(\psi - \phi)) - I(\psi) + (1 - \lambda^{**})\mu \cdot (\psi - \phi) \\ &= H(\lambda^{**}) - (1 - \lambda^{**})m + (1 - \lambda^{**})\mu \cdot (\psi - \phi) > 0 \end{aligned}$$

where the inequality uses the fact that $H(\lambda^{**}) > 0$ and that $\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0$. Similarly,

$$\begin{aligned} \lim_n \left(I(\xi_n) - I(\phi) + \nabla I(\xi_n) \cdot (\phi - \xi_n)\right) &= I(\phi + \lambda^{**}(\psi - \phi)) - I(\phi) - \lambda^{**}\mu \cdot (\psi - \phi) \\ &= H(\lambda^{**}) + \lambda^{**}m - \lambda^{**}\mu \cdot (\psi - \phi) < \epsilon \end{aligned}$$

where the inequality uses $H(\lambda^{**}) < \epsilon$ and $\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0$. Thus, for any large enough n, $\xi_n \in \hat{U}$ is as desired. □

We now establish the Boolean representation of I:

Lemma A.6. *For each $\phi \in \hat{U}$, we have*

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi),$$

where $K_\psi := \{\xi \in \hat{U} : I(\xi) + \nabla I(\xi) \cdot (\psi - \xi) \geq I(\psi)\}$ for all $\psi \in \hat{U}$.

Proof. For each $\phi, \psi \in \hat{U}$ and $\epsilon > 0$, Lemma A.5 yields some $\xi \in K_\psi$ such that $I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \leq I(\phi) + \epsilon$. Thus, $\inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \leq I(\phi)$. Moreover, by definition of K_ϕ,

$\inf_{\xi \in K_\phi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \geq I(\phi)$. Hence, $I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi)$, as required. $\square$

B Proofs for Section 2

B.1 Proof of Theorem 1

We invoke the following standard result:

Lemma B.1 (Lemma 1 in GMM)**.** *Preference $\succsim$ satisfies Axioms 1–5 if and only if there exists a monotonic, constant-linear functional $I : \mathbb{R}^S \to \mathbb{R}$ and a nonconstant affine function $u : \Delta(Z) \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$,*

$$f \succsim g \Longleftrightarrow I(u(f)) \geq I(u(g)). \tag{12}$$

Moreover, I is unique and u is unique up to positive affine transformation.

The necessity proof for Theorem 1 is standard and we omit it. To prove sufficiency, suppose $\succsim$ satisfies Axioms 1–5. Let I and u be as given by Lemma B.1. Consider the collection $\mathbb{P}^*$ given by (2), i.e.,

$$\mathbb{P}^* := \mathrm{cl}\{P^*_\phi : \phi \in \mathbb{R}^S\} \ \text{ with } \ P^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\},$$

where cl denotes the topological closure in $\mathcal{K}(\Delta(S))$ under the Hausdorff topology.

Note that since I is monotonic and constant-linear, it is 1-Lipschitz. Thus, $\partial I(\underline{0}) \subseteq \Delta(S)$ by Lemma A.4, so that each P^*_ϕ is indeed a closed, convex set of beliefs. Moreover, $\mathbb{P}^*$ is compact, as it is a closed subset of the compact space $\mathcal{K}(\Delta(S))$. Thus, $\mathbb{P}^*$ is a belief-set collection. We will show that for all $\phi \in \mathbb{R}^S$,

$$I(\phi) = \max_{P \in \mathbb{P}^*} \min_{\mu \in P} \mu \cdot \phi, \tag{13}$$

which by (12) ensures that $(\mathbb{P}^*, u)$ is a BEU representation of $\succsim$.

Lemma A.1 yields a set $\hat{U} \subseteq \mathbb{R}^S$ such that $\mathbb{R}^S \setminus \hat{U}$ has Lebesgue measure zero and I is differentiable on $\hat{U}$. Moreover, since I is positively homogeneous, Lemma A.3 implies that $\partial I(\phi) \subseteq \partial I(\underline{0})$ for all $\phi \in \mathbb{R}^S$, so that for all $\phi \in \hat{U}$, we have $\mu_\phi := \nabla I(\phi) \in \partial I(\underline{0})$. We will invoke the following lemma:

Lemma B.2. *For each $\phi \in \hat{U}$, $I(\phi) = \mu_\phi \cdot \phi$.*

Proof. Take any $\phi \in \hat{U}$. By positive homogeneity of I, $\alpha\phi \in \hat{U}$ and $\nabla I(\phi) = \nabla I(\alpha\phi)$ for any $\alpha \in (0,1)$. Thus, the function $h : [0,1] \to \mathbb{R}$ defined by $h(\alpha) = I(\alpha\phi)$ is differentiable at every $\alpha \in (0,1)$ and Lipschitz. Hence, $I(\phi) = h(1) - h(0) = \int_0^1 h'(\alpha')d\alpha' = \int_0^1 (\nabla I(\alpha\phi) \cdot \phi)d\alpha' = \phi \cdot \mu_\phi$. $\square$

To complete the proof of (13), first take any $\phi, \psi \in \hat{U}$ and let $K_\psi := \{\xi \in \hat{U} : I(\xi) + \mu_\xi \cdot (\psi - \xi) \geq I(\psi)\}$ be as in Lemma A.6. Then

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \mu_\xi \cdot (\phi - \xi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} \mu_\xi \cdot \phi, \tag{14}$$

where the first equality holds by Lemma A.6 and the second by Lemma B.2. Letting $P_\psi := \{\mu_\xi : \xi \in \hat{U}, \mu_\xi \cdot \psi \geq I(\psi)\}$, Lemma B.2 ensures that $\xi \in K_\psi$ if and only if $\mu_\xi \in P_\psi$. Moreover, (10) implies that $\overline{\mathrm{co}} P_\psi = P^*_\psi$. Combining these two observations with (14) yields

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\mu \in P_\psi} \mu \cdot \phi = \max_{\psi \in \hat{U}} \min_{\mu \in \overline{\mathrm{co}} P_\psi} \mu \cdot \phi = \max_{\psi \in \hat{U}} \min_{\mu \in P^*_\psi} \mu \cdot \phi. \tag{15}$$

Next, take any $\phi, \psi \in \mathbb{R}^S$. Then there exist sequences $\phi_n \to \phi, \psi_n \to \psi$ such that $\phi_n, \psi_n \in \hat{U}$. For each n, pick $\mu_n \in P^*_{\psi_n}$ such that $\min_{\mu \in P^*_{\psi_n}} \mu \cdot \phi_n = \mu_n \cdot \phi_n$ and consider a convergent subsequence (μ_{n_k}) with $\lim_{k\to\infty} \mu_{n_k} = \mu^*$. Note that $\mu^* \in P^*_\psi$: Indeed, for each k, we have $\mu_{n_k} \cdot \psi_{n_k} \geq I(\psi_{n_k})$, which by continuity of I implies $\mu^* \cdot \psi \geq I(\psi)$.

Moreover, for each k, we have $\mu_{n_k} \cdot \phi_{n_k} = \min_{\mu \in P^*_{\psi_n}} \mu \cdot \phi_{n_k} \leq I(\phi_{n_k})$, where the inequality holds by (15). Hence, continuity of I implies $\mu^* \cdot \phi \leq I(\phi)$, so that

$$\min_{\mu \in P^*_\psi} \mu \cdot \phi \leq \mu^* \cdot \phi \leq I(\phi). \tag{16}$$

Since (16) holds for all $\psi \in \mathbb{R}^S$, it follows from the definition of $\mathbb{P}^*$ that

$$\min_{\mu \in P} \mu \cdot \phi \leq I(\phi)$$

holds for all $P \in \mathbb{P}^*$. Finally, applying (16) with $\psi = \phi$ yields $\min_{\mu \in P^*_\phi} \mu \cdot \phi \leq I(\phi) \leq \min_{\mu \in P^*_\phi} \mu \cdot \phi$, where the second inequality holds by definition of P^*_ϕ. Thus,

$$I(\phi) = \min_{\mu \in P^*_\phi} \phi \cdot \mu = \max_{P \in \mathbb{P}^*} \min_{\mu \in P} \mu \cdot \phi,$$

as required. $\square$

B.2 Proof of Proposition 1

We begin with the following lemma:

Lemma B.3. *Consider any functional $I : \mathbb{R}^S \to \mathbb{R}$ and belief-set collection $\mathbb{P}$ such that $I(\phi) = \max_{P\in\mathbb{P}} \min_{\mu\in P} \mu \cdot \phi$ for all $\phi \in \mathbb{R}^S$. Then*

$$\partial I(\underline{0}) \subseteq \overline{\text{co}} \bigcup_{P\in\mathbb{P}} P.$$

Proof. For each $P \in \mathbb{P}$, let $I_P(\phi) := \min_{\mu\in P} \mu \cdot \phi$ for each ϕ. Thus, $I(\phi) = \max_{P\in\mathbb{P}} I_P(\phi)$ for each ϕ. Note that each I_P is 1-Lipschitz and $\partial I_P(0) = P$.

Take any convergent sequence $(\nabla I_{P_i}(\phi_i))$ where $\phi_i \to 0$, $P_i \in \mathbb{P}$, and $\nabla I_{P_i}(\phi_i)$ exists for each i. Then

$$\nabla I_{P_i}(\phi_i) \in \partial I_{P_i}(\phi_i) \subseteq \partial I_{P_i}(0) = P_i$$

where the set inclusion holds by Lemma A.3. Thus, $\lim_i \nabla I_{P_i}(\phi_i) \in \overline{\text{co}} \bigcup_{P\in\mathbb{P}} P$. Hence, the desired conclusion follows by applying Lemma A.2 to I. □

Suppose $\succsim$ satisfies Axioms 1–5. Let I and u be as given by Lemma B.1. For $\mathbb{P}^*$ as in the sufficiency proof of Theorem 1, we have $\overline{\text{co}} \bigcup_{P\in\mathbb{P}^*} P \subseteq \partial I(\underline{0})$. Thus, Lemma B.3 immediately implies that $C = \partial I(\underline{0})$ is the unique closed, convex set satisfying (5) for all BEU representations of $\succsim$, with equality for representation $\mathbb{P}^*$. □

B.3 Proof of Corollary 1

Since the proof of Proposition 1 identifies the set of relevant priors as $C = \partial I(\underline{0})$, Corollary 1 is immediate from the following result in GMM:

Lemma B.4 (Theorem 14 in GMM)**.** *Suppose $\succsim$ satisfies Axioms 1–5 and let I and u be as in Lemma B.1. Then the unique closed, convex set D satisfying*

$$f \succsim^* g \Longleftrightarrow \mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\mu[u(g)] \textit{ for all } \mu \in D$$

is given by $D = \partial I(\underline{0})$.

C Proofs for Section 3

C.1 Proof of Proposition 2

For each preference $\succsim_i$, let utility u_i and functional I_i be as given by Lemma B.1, and note that $\succsim_1$ is more ambiguity-averse than $\succsim_2$ if and only if $u_1 \approx u_2$ and $I_1(\phi) \leq I_2(\phi)$ for all $\phi \in \mathbb{R}^S$.

Consider the belief-set collection $\hat{\mathbb{P}}_i$ defined by

$$\hat{\mathbb{P}}_i = \mathrm{cl}\{\hat{P}^i_\phi : \phi \in \mathbb{R}^S\} \text{ with } \hat{P}^i_\phi = \{\mu \in \Delta(S) : \mu \cdot \phi \geq I_i(\phi)\}. \tag{17}$$

Observe first that $(\hat{\mathbb{P}}_i, u_i)$ is a BEU representation of $\succsim_i$. Indeed, for each ϕ, we have $\max_{P \in \hat{\mathbb{P}}_i} \min_{\mu \in P} \phi \cdot \mu \geq \min_{\mu \in \hat{P}^i_\phi} \phi \cdot \mu \geq I_i(\phi)$ by construction. Conversely, letting $\mathbb{P}^*_i$ be the belief-set collection defined by (2), we have $\max_{P \in \hat{\mathbb{P}}_i} \min_{\mu \in P} \phi \cdot \mu \leq \max_{P \in \mathbb{P}^*_i} \min_{\mu \in P} \phi \cdot \mu = I_i(\phi)$, where the inequality follows from the fact that $\hat{P}^i_\psi \supseteq P^{*i}_\psi$ for each ψ.

Note next that $\hat{\mathbb{P}}_i$ $\sqsupseteq$-dominates all BEU representations of $\succsim_i$. Indeed, consider any BEU representation $\mathbb{P}$ of $\succsim_i$ and any $\hat{P} \in \hat{\mathbb{P}}_i$. By definition of $\hat{\mathbb{P}}_i$, there exists a sequence (ϕ_n) with $\hat{P}^i_{\phi_n} \to \hat{P}$. Since $\mathbb{P}$ represents $\succsim_i$, there exists $P_n \in \mathbb{P}$ for each n such that $\min_{\mu \in P_n} \mu \cdot \phi_n = I_i(\phi_n)$. By definition of $\hat{P}^i_{\phi_n}$, this implies $P_n \subseteq \hat{P}^i_{\phi_n}$, whence

$$\min_{\mu \in P_n} \mu \cdot \phi \geq \min_{\mu \in \hat{P}^i_{\phi_n}} \mu \cdot \phi \tag{18}$$

for all $\phi \in \mathbb{R}^S$. By compactness of $\mathbb{P}$, restricting to a subsequence if necessary, we can assume that $\lim_n P_n = P$ for some $P \in \mathbb{P}$. Then by (18), we have $\min_{\mu \in P} \mu \cdot \phi \geq \min_{\mu \in \hat{P}} \mu \cdot \phi$ for all ϕ. Thus, $P \subseteq \hat{P}$ by the standard property of support functions.

We now prove the equivalence between parts 1 and 2 of the proposition. Suppose first that $\succsim_1$ is more ambiguity-averse than $\succsim_2$, so that $u_1 \approx u_2$ and $I_1 \leq I_2$. Then for all ϕ, the fact that $I_1(\phi) \leq I_2(\phi)$ implies $\hat{P}^1_\phi \supseteq \hat{P}^2_\phi$. By the same argument as in the previous paragraph, it follows that for any $\hat{P}_1 \in \hat{\mathbb{P}}_1$, there exists $\hat{P}_2 \in \hat{\mathbb{P}}_2$ with $\hat{P}_2 \subseteq \hat{P}_1$. Thus, $\hat{\mathbb{P}}_1 \sqsupseteq \hat{\mathbb{P}}_2$. Consider now any BEU representation $(\mathbb{P}_2, u_2)$ of $\succsim_2$. We have $\hat{\mathbb{P}}_1 \sqsupseteq \hat{\mathbb{P}}_2 \sqsupseteq \mathbb{P}_2$, where the latter inequality comes from the $\sqsupseteq$-maximality of $\hat{\mathbb{P}}_2$. Hence, $\hat{\mathbb{P}}_1 \sqsupseteq \mathbb{P}_2$ by transitivity of $\sqsupseteq$, proving part 2.

Conversely, consider the BEU representation $(\mathbb{P}_1, u)$ of $\succsim_1$ described in part 2 and any representation $(\mathbb{P}_2, u)$ of $\succsim_2$. Fix $\phi \in \mathbb{R}^S$, and let P_1 be any element of $\mathbb{P}_1$ such that $I_1(\phi) = \min_{\mu \in P_1} \mu \cdot \phi$. Since $\mathbb{P}_1 \sqsupseteq \mathbb{P}_2$, there exists $P_2 \in \mathbb{P}_2$ with $P_1 \supseteq P_2$, implying $I_2(\phi) \geq \min_{\mu \in P_2} \mu \cdot \phi \geq \min_{\mu \in P_1} \mu \cdot \phi = I_1(\phi)$. Thus, $I_2(\phi) \geq I_1(\phi)$ for all $\phi \in \mathbb{R}^S$, implying that $\succsim_1$ is more ambiguity-averse than $\succsim_2$. $\square$

C.2 Proof of Lemma 1

We combine the proof of Lemma 1 with the proof of Theorem 2 (part 2) below.

C.3 Proof of Theorem 2

Throughout the proof, let I be the functional given by Lemma B.1.

C.3.1 Proof of part 1

To prove the "only if" direction, suppose that $\succsim$ satisfies uncertainty aversion. Since it admits the maxmin expected utility representation of Gilboa and Schmeidler (1989), $I(\phi) = \min_{\mu \in C} \mu \cdot \phi$ holds for all ϕ.

We first show that $\cap_{P\in\mathbb{P}} P \supseteq C$. If not, there exists $P \in \mathbb{P}$ such that $P \not\supseteq C$. By the standard property of support functions, this implies the existence of ϕ such that $\min_{\mu\in C} \phi\cdot\mu < \min_{\mu\in P} \phi \cdot \mu$. This leads to $I(\phi) > \min_{\mu\in C} \mu \cdot \phi$, a contradiction.

We now show that $\cap_{P\in\mathbb{P}} P \subseteq C$. If not, there exists $\mu^* \in \cap_{P\in\mathbb{P}} P \setminus C$. Then there exists ϕ such that $\min_{\mu\in C} \mu\cdot\phi > \mu^*\cdot\phi$. But this implies $I(\phi) \leq \mu^* \cdot \phi < \min_{\mu\in C} \mu\cdot\phi$, a contradiction.

To prove the " if" direction, suppose that $\cap_{P\in\mathbb{P}} P = C$. Take any ϕ. It suffices to show that $I(\phi) = \min_{\mu\in C} \mu \cdot \phi$. Note that by construction of the representation $\mathbb{P}^*$ defined by (2), we have $I(\phi) \geq \min_{\mu\in C} \mu \cdot \phi$. But the representation based on $\mathbb{P}$ yields the inequality $I(\phi) \leq \min_{\mu\in\cap_{P\in\mathbb{P}} P} \mu \cdot \phi = \min_{\mu\in C} \mu \cdot \phi$, which ensures the desired claim.

C.3.2 Proof of part 2 and Lemma 1

We prove the equivalence

$$\begin{aligned}\text{absolute ambiguity aversion} &\Leftrightarrow \infty\text{-ambiguity aversion} \Leftrightarrow |S|\text{-ambiguity aversion} \\ &\Leftrightarrow \bigcap_{P\in\mathbb{P}} P \neq \emptyset,\end{aligned}$$

which implies both part 2 of Theorem 2 and Lemma 1.

The implication *absolute ambiguity aversion* $\Rightarrow$ ∞-*ambiguity aversion* follows from the proofs of Theorem 2a and Corollary 3a in Grant and Polak (2013), which imply the equivalence of absolute ambiguity aversion and ∞-ambiguity aversion for any preference with a normalized, monotonic, continuous, constant-additive, and unbounded utility act functional I (as is the case for BEU). The implication ∞- *ambiguity aversion* $\Rightarrow$ $|S|$- *ambiguity aversion* is trivial.

We now turn to the implication $|S|$-*ambiguity aversion* $\Rightarrow \cap_{P\in\mathbb{P}} P \neq \emptyset$. If $\succsim$ satisfies $|S|$-ambiguity aversion, then by part 3 of the theorem (see the proof below) any BEU representation $(\mathbb{P}, u)$ of $\succsim$ is such that every subcollection of $\mathbb{P}$ of cardinality at most $|S|$ has nonempty intersection. Since each P_i is convex and compact, Helly's theorem implies that the whole collection $\mathbb{P}$ has nonempty intersection.[27] This proves the implication.

Finally, we prove the implication $\bigcap_{P\in\mathbb{P}} P \neq \emptyset \Rightarrow$ *absolute ambiguity aversion*. Suppose that there exists $\mu^* \in \bigcap_{P\in\mathbb{P}} P$ for some BEU representation $(\mathbb{P}, u)$ of $\succsim$. For any $f \in \mathcal{F}$ and any $P \in \mathbb{P}$, this implies that $\min_{\mu\in P} \mu \cdot u(f) \leq \mu^* \cdot u(f)$, and hence $\max_{P\in\mathbb{P}} \min_{\mu\in P} \mu \cdot u(f) \leq \mu^* \cdot u(f)$. As a result,

$$f \succsim p \Longrightarrow \max_{P\in\mathbb{P}} \min_{\mu\in P} \mu \cdot u(f) \geq u(p) \Longrightarrow \mu^* \cdot u(f) \geq u(p) \Longrightarrow f \succsim_{\mu^*} p$$

where $\succsim_{\mu^*}$ is the subjective expected utility preference with belief μ^* and utility function u. Hence, $\succsim$ is more ambiguity-averse than $\succsim_{\mu^*}$, which proves the result.

C.3.3 Proof of part 3

The proof relies on the following lemma.

Lemma C.1. *Suppose that preference $\succsim$ admits a BEU representation $(\mathbb{P}, u)$. Then $\succsim$ satisfies k-ambiguity aversion if and only if*

$$\sum_{i=1}^{k-1} \max_{P_i\in\mathbb{P}} \min_{\mu_i\in P_i} \mu_i \cdot \phi_i \leq \min_{P\in\mathbb{P}} \max_{\mu\in\mathbb{P}} \mu \cdot \sum_{i=1}^{k-1} \phi_i, \qquad \text{for all } \phi_1, \cdots, \phi_{k-1} \in \mathbb{R}^S. \tag{19}$$

Proof. To prove the "if" part, suppose inequality (19) is satisfied. Consider any $f_1, \cdots, f_k \in \mathcal{F}$ such that $f_1 \sim f_i$ for all i and $\frac{1}{k} f_1 + \cdots + \frac{1}{k} f_k = p$ for some $p \in \Delta(Z)$. We have

$$\begin{aligned} I(\frac{1}{k} u(f_k)) &= I(\underline{u(p)} - \sum_{i=1}^{k-1} \frac{1}{k} u(f_i)) = u(p) - \min_{P\in\mathbb{P}} \max_{\mu\in P} \sum_{i=1}^{k-1} \frac{1}{k} u(f_i) \cdot \mu \\ &\leq u(p) - \sum_{i=1}^{k-1} \max_{P_i\in\mathbb{P}} \min_{\mu_i\in P_i} \frac{1}{k} u(f_i) \cdot \mu_i = u(p) - \sum_{i=1}^{k-1} I(\frac{1}{k} u(f_i)), \end{aligned}$$

where the inequality holds by (19). Rearranging yields $\sum_{i=1}^{k} I(\dfrac{1}{k} u(f_i)) \leq u(p)$, which is simply $I(u(f_1)) \leq u(p)$ since $I(u(f_i)) = I(u(f_1))$ for all i. This is turn implies $p \succsim f_1$, and thus $\succsim$ satisfies k-ambiguity aversion.

[27] Recall that $\Delta(S)$ has dimension $|S| - 1$.

To prove the "only if" part, suppose that there exist some vectors $\phi_1, \cdots, \phi_{k-1}$ such that the inequality (19) is violated. By the constant linearity of the max-min and min-max functionals, we can assume without loss of generality that $I(\phi_i) = I(\phi_1)$ for all i, and that each ϕ_i belongs to $[-1,1]^S$.

Let $c \in \mathbb{R}$ be given by $c = -I(-\phi_1 - \ldots - \phi_{k-1}) + I(\phi_1)$, so that $I(\underline{c} - \phi_1 - \cdots - \phi_{k-1}) = I(\phi_1)$. Note that $c \in [-k, k]$. Let $\phi_k \in \mathbb{R}^S$ be defined by $\phi_k = \underline{c} - \phi_1 - \cdots - \phi_{k-1}$, which implies $\phi_1 + \cdots + \phi_k = \underline{c}$. Up to rescaling all the ϕ_is and c by a common factor, this vector ϕ_k also belongs to $[-1,1]^S$. By definition of c, we have $I(\phi_k) = I(\phi_1)$, and

$$\begin{aligned} I(\phi_k) = I(\underline{c} - \sum_{i=1}^{k-1} \phi_i) = c - \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu \cdot \sum_{i=1}^{k-1} \phi_i > c - \sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \mu_i \cdot \phi_i \\ = c - \sum_{i=1}^{k-1} I(\phi_i). \end{aligned}$$

Rearranging yields $\sum_{i=1}^{k} I(\phi_i) > c$, which implies $I(\phi_1) > \frac{c}{k}$.

To conclude the proof, we assume that $u(\overline{z}) \geq 1, u(\overline{z}) \leq -1$ for some outcomes $\overline{z}, \underline{z} \in Z$. (This is without loss of generality by taking a positive affine transformation of u if necessary.) Since each ϕ_i belongs to $[-1,1]^S$, it is possible to find weights (ϵ_i^s) such that the act f_i that maps each state s into the lottery $\epsilon_i^s \delta_{\underline{z}} + (1 - \epsilon_i^s)\delta_{\overline{z}}$ satisfies $u(f_i) = \phi_i$. In addition, the fact that $\sum_{i=1}^{k} u(f_i)$ is a constant vector equal to c shows that $\sum_{i=1}^{k} \frac{1}{k} f_i$ is a constant act that delivers a lottery p supported on $\{\overline{z}, \underline{z}\}$, where $u(p) = \frac{c}{k}$. The collection $(f_1, \cdots, f_k)$ thus satisfies $\frac{1}{k} f_1 + \cdots + \frac{1}{k} f_k = p$, $f_i \sim f_1$ for all i since $I(\phi_i) = I(\phi_1)$, and $f_1 \succ p$ since $I(\phi_1) > \frac{c}{k} = u(p)$. Hence, $\succsim$ does not satisfy k-ambiguity aversion. □

Let us now prove part 3 of the theorem.

Sufficiency. Suppose that $P_1 \cap \cdots \cap P_k \neq \emptyset$ for all $P_1, \cdots, P_k \in \mathbb{P}$. Consider any $P_1, \cdots, P_k$ and some vectors $(\phi_1, \cdots, \phi_{k-1})$. Let $\mu \in P_1 \cap \cdots \cap P_k$. We have

$$\min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i \leq \sum_{i=1}^{k-1} \mu \cdot \phi_i \leq \max_{\mu_k \in P_k} \sum_{i=1}^{k-1} \mu_k \cdot \phi_i$$

where the first inequality is due to the fact that $\mu \in P_i$ for all $i \leq k-1$, and the second inequality is due to the fact that $\mu \in P_k$. Since this is true for any $P_1, \cdots, P_k$, this implies

$$\max_{(P_1, \cdots, P_{k-1}) \in \mathbb{P}^{k-1}} \min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i \leq \min_{P_k \in \mathbb{P}} \max_{\mu_k \in P_k} \sum_{i=1}^{k-1} \mu_k \cdot \phi_i,$$

i.e.,

$$\sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \mu_i \cdot \phi_i \leq \min_{P_k \in \mathbb{P}} \max_{\mu_k \in \mathbb{P}} \mu_k \cdot \sum_{i=1}^{k-1} \phi_i.$$

Thus, by Lemma C.1 $\succsim$ satisfies k-ambiguity aversion.

Necessity. Suppose that there exist $P_1, \cdots, P_k \in \mathbb{P}$ such that $P_1 \cap \cdots \cap P_k = \emptyset$. Consider the sets $A, B \subseteq \mathbb{R}^{S(k-1)}$ defined by

$$A = \{(\mu_1, \cdots, \mu_{k-1}) : \mu_i \in P_i\} \text{ and } B = \{(\mu_k, \cdots, \mu_k) : \mu_k \in P_k\}.$$

The sets A and B are compact and convex. In addition, $A \cap B = \emptyset$ since any $(\mu_k, \cdots, \mu_k) \in A \cap B$ would satisfy $\mu_k \in P_1 \cap \cdots \cap P_k$, which is a contradiction. By the separating hyperplane theorem there exists a vector $\phi = (\phi_1, \cdots, \phi_{k-1}) \in \mathbb{R}^{S(k-1)}$, where each $\phi_i \in \mathbb{R}^S$, such that $\min_{a \in A} a \cdot \phi > \max_{b \in B} b \cdot \phi$, which is equivalent to

$$\min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i > \max_{\mu \in P_k} \sum_{i=1}^{k-1} \mu \cdot \phi_i.$$

Hence,

$$\sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \mu_i \cdot \phi_i \geq \min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i > \max_{\mu \in P_k} \sum_{i=1}^{k-1} \mu \cdot \phi_i \geq \min_{P \in \mathbb{P}} \max_{\mu \in \mathbb{P}} \mu \cdot \sum_{i=1}^{k-1} \phi_i.$$

Thus, by Lemma C.1 $\succsim$ does not satisfy k-ambiguity aversion. $\square$

C.4 Proof of Lemma 2

Note that $m(E) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E)$ while $m(E^c) = 1 - \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu(E)$, and thus $AA(E) = \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu(E) - \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E)$. This implies that $AA(E) \geq 0$ if and only if all $P, P' \in \mathbb{P}$ satisfy $\max_{\mu \in P} \mu(E) \geq \min_{\mu' \in P'} \mu'(E)$, i.e., if and only if $\{\mu(E) : \mu \in P\} \cap \{\mu'(E) : \mu' \in P'\} \neq \emptyset$. $\square$

C.5 Proof of Proposition 3

For any $\succsim$ with BEU representation $(\mathbb{P}, u)$, note that $m(E) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E)$ for all events E. Thus, given $\mathcal{E}$ and $\mathcal{F}$ as in the proposition, it suffices to find $\nu \in \Delta(S)$ and a belief-set collection $\mathbb{P}$ such that $\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E) < \nu(E)$ for all $E \in \mathcal{E}$ and $\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(F) > \nu(F)$ for all $F \in \mathcal{F}$.

Pick any $\beta > 0$ and $\nu \in \Delta(S)$ with $\beta < \min_{s \in S} \nu(s)$. Define $\mathbb{P}$ by $\mathbb{P} = \{P_F : F \in \mathcal{F}\}$, where for each $F \in \mathcal{F}$,

$$P_F := \{\mu \in \Delta(S) : \mu(F) = \nu(F) + \frac{\beta}{2}, \, \mu(E) \in [\nu(E) - \beta, \nu(E) + \beta] \, \forall E \subseteq S\}.$$

Note that each P_F is nonempty: Indeed, pick any $s \in F$ and $s' \in F^c$ (which exist since $F \notin \{S, \emptyset\}$). Then setting $\mu(s) = \nu(s) + \frac{\beta}{2}$, $\mu(s') = \nu(s') - \frac{\beta}{2}$, and $\mu(s'') = \nu(s'')$ for all $s'' \neq s, s''$ yields $\mu \in P_F$. Since P_F is also closed and convex, $\mathbb{P}$ is a well-defined belief-set collection.

By definition of $\mathbb{P}$, $\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(F) \geq \nu(F) + \frac{\beta}{2} > \nu(F)$ for all $F \in \mathcal{F}$. To complete the proof, we show that $\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E) \leq \nu(E) - \frac{\beta}{2} < \nu(E)$ for all $E \in \mathcal{E}$. Consider any $E \in \mathcal{E}$, $F \in \mathcal{F}$. Since $E \neq F$ (as $\mathcal{E}$ and $\mathcal{F}$ are disjoint), we either have (a) $F \setminus E \neq \emptyset \neq E \setminus F$; (b) $E \subsetneq F$; or (c) $F \subsetneq E$. In each case, we show that $\min_{\mu \in P_F} \mu(E) \leq \nu(E) - \frac{\beta}{2}$ by constructing a $\mu \in P_F$ such that $\mu(E) = \nu(E) - \frac{\beta}{2}$:

In case (a), pick $s \in F \setminus E$ and $s' \in E \setminus F$. Then define μ by $\mu(s) = \nu(s) + \frac{\beta}{2}$, $\mu(s') = \nu(s) - \frac{\beta}{2}$, and $\mu(s'') = \nu(s'')$ for all $s'' \neq s, s'$.

In case (b), pick $s \in F \setminus E$, $s' \in E$, and $s'' \in F^c \subseteq E^c$. Then define μ by $\mu(s) = \nu(s) + \beta$, $\mu(s') = \nu(s') - \frac{\beta}{2}$, $\mu(s'') = \nu(s'') - \frac{\beta}{2}$, and $\mu(s''') = \nu(s''')$ for all $s''' \neq s, s', s''$.

In case (c), pick $s \in F$, $s' \in E \setminus F$, and $s'' \in E^c \subseteq F^c$. Then define μ by $\mu(s) = \nu(s) + \frac{\beta}{2}$, $\mu(s') = \nu(s') - \beta$, $\mu(s'') = \nu(s'') + \frac{\beta}{2}$, and $\mu(s''') = \nu(s''')$ for all $s''' \neq s, s', s''$. $\square$

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Supplementary Appendix to "Boolean Representations of Preferences under Ambiguity"

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This supplementary appendix is organized as follows. Section S.1 formalizes the uniqueness properties of BEU representations. Section S.2 focuses on the representation obtained by inverting the order of moves of Optimism and Pessimism and uses this to characterize different degrees of ambiguity seeking. Sections S.3 and S.4 present two generalizations of BEU that correspond to relaxations of certainty independence.

S.1 Uniqueness

For any $\phi \in \mathbb{R}^S$ and $\lambda \in \mathbb{R}$, let $H_{\phi,\lambda} := \{\mu \in \Delta(S) : \mu \cdot \phi \geq \lambda\}$ denote the closed half-space in $\Delta(S)$ that is defined by ϕ and λ. For any belief-set collection $\mathbb{P}$, define its ***half-space closure*** by

$$\overline{\mathbb{P}} := \{H \subseteq \Delta(S) : H \text{ is a closed half-space in } \Delta(S) \text{ and } P \subseteq H \text{ for some } P \in \mathbb{P}\}.$$

Proposition S.1.1. *Suppose $(\mathbb{P}, u)$ is a BEU representation of $\succsim$. Then for any belief-set collection $\mathbb{P}'$ and utility u', $(\mathbb{P}', u')$ is a BEU representation of $\succsim$ if and only if $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$ and $u \approx u'$.*

Below we fix the unique functional $I : \mathbb{R}^S \to \mathbb{R}$ associated with $\succsim$, as given by Lemma B.1. We begin with the following lemma:

Lemma S.1.1. *Suppose $(\mathbb{P}, u)$ is a BEU representation of $\succsim$. Then $\overline{\mathbb{P}} = \{H_{\phi,\lambda} : \phi \in \mathbb{R}^S, \lambda \leq I(\phi)\}$.*

Proof. First, take any $\phi \in \mathbb{R}^S$, $\lambda \in \mathbb{R}$ such that $\lambda \leq I(\phi)$. Since $(\mathbb{P}, u)$ represents $\succsim$, there exists $P \in \mathbb{P}$ such that $\min_{\mu \in P} \mu \cdot \phi = I(\phi)$. Thus, $P \subseteq H_{\phi,I(\phi)} \subseteq H_{\phi,\lambda}$, which implies $H_{\phi,\lambda} \in \overline{\mathbb{P}}$.

Conversely, take any $P \in \overline{\mathbb{P}}$. By definition of $\overline{\mathbb{P}}$, there exist $\phi \in \mathbb{R}^S$, $\lambda \in \mathbb{R}$, and $P' \in \mathbb{P}$ such that $P' \subseteq P = H_{\phi,\lambda}$. Since $(\mathbb{P}, u)$ represents $\succsim$, $I(\phi) \geq \min_{\mu \in P'} \mu \cdot \phi \geq \min_{\mu \in H_{\phi,\lambda}} \phi \cdot \mu$. Hence, $\lambda \leq I(\phi)$. $\square$

Proof of Proposition S.1.1. For the "only if" direction, the fact that $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$ is immediate from Lemma S.1.1 and uniqueness of I. The proof that $u \approx u'$ is standard.

For the "if" direction, by uniqueness of I, it suffices to show that $\max_{P' \in \mathbb{P}'} \min_{\mu \in P'} \mu \cdot \phi = I(\phi)$ for all $\phi \in \mathbb{R}^S$. To show this, observe first that by Lemma S.1.1 and since $\overline{\mathbb{P}} = \overline{\mathbb{P}}'$, there exists $P' \in \mathbb{P}'$ such that $P' \subseteq H_{\phi, I(\phi)}$. This ensures $\min_{\mu \in P'} \mu \cdot \phi \geq I(\phi)$. Suppose next that $\min_{\mu \in P''} \mu \cdot \phi - I(\phi) =: \epsilon > 0$ for some $P'' \in \mathbb{P}'$. Then $H_{\phi, I(\phi)+\epsilon} \supseteq P''$, which implies $H_{\phi, I(\phi)+\epsilon} \in \overline{\mathbb{P}'}$. Since $\overline{\mathbb{P}'} = \overline{\mathbb{P}}$, this contradicts Lemma S.1.1. $\square$

S.2 Minmax BEU representation

While BEU assumes that Optimism plays first and Pessimism plays second, this is equivalent to a model with the opposite order of moves. We omit all proofs for this section, as they can be obtained as minor modifications of the original proofs for BEU.

Theorem S.2.1. *Preference $\succsim$ satisfies Axioms 1–5 if and only if $\succsim$ admits a minmax BEU representation, i.e., there exists a belief-set collection $\mathbb{Q}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that*

$$W(f) = \min_{Q \in \mathbb{Q}} \max_{\mu \in Q} \mathbb{E}_\mu[u(f)]$$

represents $\succsim$.

Our construction of the maxmin BEU representation considered in the text uses the belief-set collection $\mathbb{P}^* = \mathrm{cl}\{P^*_\phi : \phi \in \mathbb{R}^S\}$ with $P^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\}$. Analogously, it can be shown that the belief-set collection $\mathbb{Q}^* := \mathrm{cl}\{Q^*_\phi : \phi \in \mathbb{R}^S\}$ with $Q^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \leq I(\phi)\}$ yields a minmax BEU representation. Paralleling Section 2.3, it is straightforward to show that $C := \partial I(\underline{0})$ again corresponds to the smallest set of priors that is contained in $\overline{\mathrm{co}} \bigcup_{Q \in \mathbb{Q}} Q$ for all minmax BEU representations $\mathbb{Q}$ of $\succsim$, with equality for representation $\mathbb{Q}^*$.

While the different notions of ambiguity aversion are most conveniently characterized using the maxmin BEU representation (cf. Theorem 2), the minmax BEU representation is useful for characterizing their ambiguity-seeking counterparts. Axioms 8 and 9 and Theorem S.2.2 below provide the analogs of Axioms 6 and 7 and Theorem 2, respectively.

Axiom 8 (Uncertainty Seeking)**.** If $f, g \in \mathcal{F}$ with $f \sim g$, then $\frac{1}{2}f + \frac{1}{2}g \precsim f$.

Axiom 9 (k-Ambiguity Seeking)**.** For all $f_1, ..., f_k \in \mathcal{F}$ with $f_1 \sim f_2 \sim \cdots \sim f_k$ and any $p \in \Delta(Z)$,

$$\frac{1}{k}f_1 + \cdots + \frac{1}{k}f_k = p \Rightarrow p \precsim f_1.$$

We say that $\succsim$ is *absolutely ambiguity-seeking* if there exists a nondegenerate subjective expected utility preference that is more ambiguity-averse than $\succsim$. Analogous to Lemma 1, this is characterized by ∞-*ambiguity seeking*, i.e., k-ambiguity seeking for all k.

Theorem S.2.2. *Suppose that $\succsim$ admits a minmax BEU representation $(\mathbb{Q}, u)$. Then:*

1. *$\succsim$ satisfies uncertainty seeking if and only if $\bigcap_{Q\in\mathbb{Q}} Q = C$;*
2. *$\succsim$ is absolutely ambiguity-seeking if and only if $\bigcap_{Q\in\mathbb{Q}} Q \neq \emptyset$;*
3. *$\succsim$ satisfies k-ambiguity seeking if and only if $\bigcap_{i=1,\cdots,k} Q_i \neq \emptyset$ for all $Q_1, \cdots, Q_k \in \mathbb{Q}$.*

S.3 Boolean variational representation

The variational model introduced by Maccheroni, Marinacci, and Rustichini (2006) (henceforth, MMR) relies on the following relaxation of certainty independence, which retains the "location invariance" property of preferences but relaxes the "scale invariance" property; we refer to MMR for a discussion.

Axiom 10 (Weak Certainty Independence)**.** For any $f, g \in \mathcal{F}$, $p, q \in \Delta(Z)$, and $\alpha \in (0,1)$,

$$\alpha f + (1-\alpha)p \succsim \alpha g + (1-\alpha)p \Longrightarrow \alpha f + (1-\alpha)q \succsim \alpha g + (1-\alpha)q.$$

We now show that dropping uncertainty aversion from MMR's axioms corresponds to adding a maximization stage into the variational model. A ***cost collection*** is a collection of functions $c : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ such that each $c \in \mathbb{C}$ is convex and $\mathbb{C}$ is ***grounded*** (i.e., $\max_{c\in\mathbb{C}} \min_{\mu\in\Delta(S)} c(\mu) = 0$).

Theorem S.3.1. *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 10 if and only if $\succsim$ admits a **Boolean variational (BV)** representation, i.e., there exists a cost collection $\mathbb{C}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that*

$$W_{BV}(f) := \max_{c\in\mathbb{C}} \min_{\mu\in\Delta(S)} \mathbb{E}_\mu[u(f)] + c(\mu) \tag{20}$$

is well-defined and represents $\succsim$.

We note that our characterization of the set of relevant priors under BEU generalizes to the Boolean variational model. Specifically, let $\mathrm{dom}(c) := \{\mu : c(\mu) \in \mathbb{R}\}$ denote the effective domain of any cost function. Then there exists a unique closed, convex set C such that $C \subseteq \overline{\text{co}}\left(\bigcup_{c\in\mathbb{C}} \text{dom}(c)\right)$ for all Boolean variational representations of $\succsim$, with equality for the representation $\mathbb{C}^*$ we construct in the proof of Theorem S.3.1 below. Moreover, it can again be shown that C is the Bewley set of the unambigous preference $\succsim^*$. The argument relies on the observation that $C = \overline{\text{co}}\left(\bigcup_{\phi\in\text{int}U} \partial I(\phi)\right)$, where I is the utility act functional obtained in the proof of Theorem S.3.1 and U its domain. Details are available on request.

S.3.1 Proof of Theorem S.3.1

We will invoke the following result from MMR:

Lemma S.3.1 (Lemma 28 in MMR)**.** *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 10 if and only if there exists a nonconstant affine function $u : \Delta(Z) \to \mathbb{R}$ with $U := (u(\Delta(Z)))^S$ and a normalized niveloid $I : U \to \mathbb{R}$ such that $I \circ u$ represents $\succsim$.*

Recall that functional $I : U \to \mathbb{R}$ is a *niveloid* if $I(\phi) - I(\psi) \leq \max_s(\phi_s - \psi_s)$ for all $\phi, \psi \in U$. Lemma 25 in MMR shows that I is a niveloid if and only if it is monotonic and constant-additive.
Based on this result, the necessity direction of Theorem S.3.1 is standard. We now prove the sufficiency direction. Suppose $\succsim$ satisfies Axioms 1–4 and Axiom 10. Let I, u, and U be as given by Lemma S.3.1. Since I is a niveloid, it is 1-Lipschitz. Hence, Lemma A.1 yields a subset $\hat{U} \subseteq \text{int}U$ with $U \setminus \hat{U}$ of Lebesgue measure 0 such that I is differentiable on $\hat{U}$. Define $\mu_\psi := \nabla I(\psi)$ and $w_\psi := I(\psi) - \nabla I(\psi) \cdot \psi$ for each $\psi \in \hat{U}$. By Lemma A.4 and the fact that niveloids are monotonic and constant-additive, $\mu_\psi \in \Delta(S)$ for all $\psi \in \hat{U}$. For each $\psi \in U$, define

$$D_\psi := \{(\mu, w) \in \Delta(S) \times \mathbb{R} : \mu \cdot \psi + w \geq I(\psi)\} \cap \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in \hat{U}\},$$

and let $\mathbb{D} := \{D_\psi : \psi \in U\}$. The following lemma implies that each D_ψ is nonempty; note also that it is closed, convex, and bounded below.

Lemma S.3.2. *For every $\phi, \psi \in U$, $\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi)$ with equality if $\phi = \psi$.*

Proof. First, consider any $\phi, \psi \in \hat{U}$. Let $K_\psi := \{\xi \in \hat{U} : \mu_\xi \cdot \psi + w_\xi \geq I(\psi)\}$ be as in Lemma A.6. Note that $D_\psi = \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in K_\psi\}$, so that

$$\inf_{\xi\in K_\psi} \mu_\xi \cdot \phi + w_\xi = \min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w,$$

where the minimum is attained as D_ψ is closed and bounded below. Thus, Lemma A.6 implies that

$$\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi), \tag{21}$$

where (21) holds with equality if $\psi = \phi$ by definition of D_ψ.

Next, consider any $\phi, \psi \in U$. Take sequences $\phi_n \to \phi$, $\psi_n \to \psi$ such that $\phi_n, \psi_n \in \hat{U}$ for each n, where we choose $\phi_n = \psi_n$ if $\phi = \psi$. For each n, the previous paragraph yields some $(\mu_n, w_n) \in D_{\psi_n}$ such that $\mu_n \cdot \phi_n + w_n = \min_{(\mu,w)\in D_{\psi_n}} \mu \cdot \phi_n + w \leq I(\phi_n)$, with equality if $\phi = \psi$. Thus, for each n, we have $I(\psi_n) - \mu_n \cdot \psi_n \leq w_n \leq I(\phi_n) - \mu_n \cdot \phi_n$. Since $\phi_n \to \phi$, $\psi_n \to \psi$, and I is continuous, this implies that sequence (w_n) is bounded. Thus, up to restricting to a suitable subsequence, we can assume that $(\mu_n, w_n) \to (\mu_\infty, w_\infty)$ for some $(\mu_\infty, w_\infty) \in \Delta(S) \times \mathbb{R}$. Then $(\mu_\infty, w_\infty) \in D_\psi$ and $\mu_\infty \cdot \phi + w_\infty \leq I(\phi)$ by continuity of I, with equality if $\phi = \psi$. Thus, $\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w = \inf_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi)$, with equality if $\phi = \psi$, where the minimum is attained since D_ψ is closed and bounded below. □

Finally, we obtain a Boolean variational representation of $\succsim$ as follows. For each $D \in \mathbb{D}$, define $c_D : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ by $c_D(\mu) := \inf\{w \in \mathbb{R} : (\mu, w) \in D\}$ for each $\mu \in \Delta(S)$, where by convention the infimum of the empty set is ∞. Note that c_D is convex for all D by convexity of D. Moreover, for all $\phi \in U$, $\min_{(\mu,w)\in D} \mu \cdot \phi + w = \min_{\mu\in\Delta(S)} \mu \cdot \phi + c_D(\mu)$. Thus, Lemma S.3.2 implies

$$I(\phi) = \max_{D\in\mathbb{D}} \min_{\mu\in\Delta(S)} \mu \cdot \phi + c_D(\mu) \tag{22}$$

for all $\phi \in U$. Since I is normalized, applying (22) to any constant vector $\underline{a} \in U$, yields $I(\underline{a}) = a + \max_{D\in\mathbb{D}} \min_{\mu\in\Delta(S)} c_D(\mu) = a$. Thus, collection $(c_D)_{D\in\mathbb{D}}$ is grounded. Hence, $\mathbb{C}^* := \{c_D : D \in \mathbb{D}\}$ is a cost collection and $(\mathbb{C}^*, u)$ is a BV representation of $\succsim$ by Lemma S.3.1. □

S.4 Rational Boolean representation

Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) (henceforth, CMMM) maintain uncertainty aversion, but further relax independence to hold only for objective lotteries:

Axiom 11 (Risk Independence)**.** For any $p, q, r \in \Delta(Z)$ and $\alpha \in (0, 1)$,

$$p \succsim q \Longrightarrow \alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r.$$

Dropping uncertainty aversion from CMMM's axioms yields the following Boolean generalization of their representation:

Theorem S.4.1. *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 11 if and only if $\succsim$ admits a **rational Boolean (RB)** representation, i.e., there exists a collection $(G_t)_{t\in T}$ of quasiconvex functions $G_t : \mathbb{R}\times\Delta(S) \to \mathbb{R}\cup\{\infty\}$ that are increasing in their first argument and grounded*[28]

[28] That is, $\max_{t\in T} \inf_{\mu\in\Delta(S)} G_t(a, \mu) = a$ for all a.

and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that

$$W_{RB}(f) := \max_{t \in T} \inf_{\mu \in \Delta(S)} G_t(\mathbb{E}_\mu[u(f)], \mu) \tag{23}$$

is well-defined, continuous, and represents $\succsim$.

S.4.1 Proof of Theorem S.4.1

The following result follows from a minor modification of the proof of Lemma 57 in CMMM:

Lemma S.4.1. *Preference $\succsim$ satisfies Axioms 1–4 and 11 if and only if there exists a nonconstant affine function $u : \Delta(Z) \to \mathbb{R}$ with $U := (u(\Delta(Z)))^S$ and a monotonic, normalized and continuous functional $I : U \to \mathbb{R}$ such that $I \circ u$ represents $\succsim$.*

Based on this result, the necessity direction of Theorem S.4.1 is standard. We now prove the sufficiency direction. Suppose $\succsim$ satisfies Axioms 1–4 and 11. Let I, u, and U be as given by Lemma S.4.1.

Define $D_\psi := \{(\mu, I(\psi) - \mu \cdot \psi) \in \mathbb{R}^S_+ \times \mathbb{R} : \mu \in \mathbb{R}^S_+\}$ for each $\psi \in U$. Note that D_ψ is nonempty and convex. Let $I_\psi(\phi) := \inf_{(\mu,w) \in D_\psi} \mu \cdot \phi + w$ for each $\phi, \psi \in U$.

Take any $\phi, \psi \in U$. Observe that

$$I_\psi(\phi) = \inf_{\alpha > 0, s \in S} I(\psi) + \alpha(\phi_s - \psi_s) = \begin{cases} I(\psi) \text{ if } \phi \geq \psi \\ -\infty \text{ if } \phi \ngeq \psi \end{cases}$$

Thus, $I(\phi) \geq I_\psi(\phi)$ by monotonicity of I, with equality if $\phi = \psi$. That is, for each $\phi \in U$,

$$I(\phi) = \max_{\psi \in U} I_\psi(\phi). \tag{24}$$

For each $\psi \in U$, define a function $G_\psi : \mathbb{R} \times \Delta(S) \to \mathbb{R} \cup \{\infty\}$ by

$$G_\psi(t, \mu) = \sup\{I_\psi(\xi) : \xi \in U, \xi \cdot \mu \leq t\}$$

for each (t, μ). The map is quasi-convex (Lemma 31 in CMMM) and increasing in t.

Lemma S.4.2. *$I_\psi(\phi) = \inf_{\mu \in \Delta(S)} G_\psi(\mu \cdot \phi, \mu)$ for each $\phi, \psi \in U$.*

Proof. Observe that RHS $= \inf_{\mu \in \Delta(S)} \sup\{I_\psi(\xi) : \xi \cdot \mu \leq \phi \cdot \mu\}$. To see that LHS $\leq$ RHS, observe that $I_\psi(\phi) \leq \sup\{I_\psi(\xi) : \xi \cdot \mu \leq \phi \cdot \mu\}$ holds for any $\mu \in \Delta(S)$.

To see that LHS $\geq$ RHS, note first that if $\phi \geq \psi$ then LHS $= I(\psi)$ and RHS $\in \{I(\psi), -\infty\}$, so the inequality clearly holds. If $\phi \ngeq \psi$ then $\phi_s < \psi_s$ for some $s \in S$.

Thus, by taking $\mu = \delta_s$, any ξ with $\xi \cdot \mu \leq \phi \cdot \mu$ satisfies $\xi_s \leq \phi_s$, which implies $\xi \not\gtrsim \psi$, whence $I_\psi(\xi) = -\infty$. $\square$

Setting $T = U$, Lemma S.4.2 and (24) ensure that W_{RB} given by (23) represents $\succsim$ and is continuous. Finally, to check groundedness, note that since I is normalized, we have $a = I(\underline{a}) = \max_{\psi \in U} \inf_{\mu \in \Delta(S)} G_\psi(a, \mu)$ for any $a \in \mathbb{R}$. $\square$