English versus Vickrey Auctions with Loss Averse Bidders

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Abstract

Evidence suggests that people evaluate outcomes relative to expectations. I analyze this expectation-based loss aversion [Kőszegi and Rabin (2006, 2009)] in the context of dynamic and static auctions, where the reference point is given by the (endogenous) equilibrium outcome. If agents update their reference point during the auction, the arrival of information crucially affects equilibrium behavior. Consequently, I show that—even with independent private values—the Vickrey auction yields strictly higher revenue than the English auction, violating the well known revenue equivalence. Thus, dynamic loss aversion offers a novel explanation for empirically observed differences between these auction formats.

Keywords: Vickrey auction, English auction, expectation-based loss aversion, revenue equivalence, dynamic loss aversion, personal equilibrium

JEL classification: D03, D44

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1 Introduction

Auctions are a universal tool to organize sales in markets. At the core of auction theory stand the famous revenue equivalence results. In particular, Vickrey (1961) notes the strategic equivalence between the dynamic English and the static Vickrey auction: if values are independent and private, there is no effect of sequential information and it is a weakly dominant strategy to bid (up to) one's private valuation in both formats.[1] These powerful theoretical predictions, however, stand in contrast to the experimental literature, which mostly finds lower revenues for the English auction.[2] I identify endogenous preferences in form of expectation-based loss aversion as a possible explanation for this phenomenon.

In my model, bidders evaluate any auction outcome relative to their reference point, formed by rational expectations. Consequently, neither in the second-price (Vickrey), nor in the ascending-clock (English) auction it is optimal any more to bid (up to) the own intrinsic valuation. In particular, loss aversion leads to strong overbidding for high types in the Vickrey auction. Moreover, if agents update their reference point with respect to new information, opponents' behavior influences bidders' reference point, and thus their endogenous preferences. Hence, even if valuations for the object are entirely private, sequential information affects the bidding behavior. Consequently, the English and the Vickrey auction are no longer strategically equivalent. I demonstrate that—consistent with most of the experimental evidence—the English auction yields lower revenue. I establish that this effect is driven by a time-inconsistency problem, which dynamic expectation-based loss averse bidders face when forming their bidding strategy.

Following the concept of loss aversion by Kőszegi and Rabin (2006), I assume bidders experience—in addition to classical utility —gain-loss utility from comparing the outcome to their expectations. Further, I assume that bidders assign gains and losses separately to money and good (narrow bracketers). For the ease of exposition, I consider mostly bidders who are only loss averse with respect to the object.[3] If they win the auction, they experience a feeling of elation, increasingly

[1] Myerson (1981) extends the results to show that all main auction formats give rise to the same expected revenue.

[2] For a summary of the experimental literature, see Kagel (1995).

[3] I show in section 6.1 that the main insights generalize to the case where bidders assign gains and losses separately to the money and good dimension.

in the extent to which winning was unexpected. Similarly, they perceive a feeling of loss if they lose, increasingly in their expectations to win. Taking that into account, bidders will overbid their intrinsic valuation. Since losses with respect to expectations weigh stronger than gains, high types—who expect to win—overbid more aggressively than low types in the Vickrey auction.

To model the impact of dynamic information on the reference point in the dynamic English auction, I take the continuous-time limit of Kőszegi and Rabin (2009): every clock increment bidders observe whether opponents drop out from the auction. This information permanently updates expectations about winning the auction and about how much to pay. If the changes in beliefs immediately update the bidders' reference points, they instantaneously perceive gain-loss utility, which means that they assign gains and losses to changes in the reference distribution.

I consider the two extreme cases as benchmarks: if the reference-point updating is sufficiently lagged with respect to changes in beliefs, there is no updating during the auction process and therefore no impact of sequential information. The English auction remains equivalent to the Vickrey auction in that case.

If the new information immediately updates the reference point, however, bidders' utility depends on the observed signals about opponents' bidding strategies during the auction process, even though values are private.

Kőszegi and Rabin interpret the reference point as lagged beliefs. Recent experimental findings, however, suggest that the reference point adjusts quickly to new information. Whether instantaneous reference-point updating is a realistic approximation may depend on the exact auction environment, e.g. the speed at which the price augments, which can differ immensely across different English auctions. Altogether, instantaneous updating constitutes a natural and important benchmark.

Since losses weigh stronger than gains, expected reference dependent utility is always negative. In particular, bidders dislike fluctuation in beliefs. As bidders are forward looking, they will account for these costs when they form their bidding strategy. In principle, an aggressive bid would to some extent insure against belief fluctuations during the auction process. However, as the auction prevails, bidders' beliefs to win the auction eventually decline. They become less attached to the auctioned object, and at the point they would have to bid aggressively, it is time inconsistent to do so. They eventually perceive themselves as a low type with respect to the active bidders in the remaining auction. This leads to only moderate overbidding - similarly as for low types in the Vickrey auction. Therefore, bidding is less aggressive in the English auction with updated reference points.

Since bidders dislike belief fluctuations, they would prefer to refrain from observing the auction process and rather use proxies to bid on their behalf. The logic is related to Benartzi et al. (1995) and Pagel (2016), who explain the equity premium puzzle by loss aversion: since stock prices fluctuate, an investor who regularly checks her portfolio will experience negative reference-dependent utility in expectation. This disutility makes stocks relatively less attractive to bonds.

Lange and Ratan (2010) highlight that in the presence of loss aversion in hedonic dimensions most laboratory results may not be transferable to the field: the effects of loss aversion are mainly driven by the assumption that bidders account losses and gains separately in the money and the good dimension (narrow bracketing). In order to control for private values, most auction experiments, however, use auction tokens, which can be interchanged for money at the end of the experiment. In context of these induced value experiments, bidders might not evaluate gains and losses to tokens and money separately, as they are in fact both money.[4] Since I assume narrow bracketing throughout this paper, my results are more likely to apply to real commodity auctions, rather than to experiments on induced value auctions. It can therefore explain the revenue gap between the Vickrey auction and the English auction in the induced-value experimental literature, only if we assume that bidders don't perceive the tokens as money.

There is surprisingly little experimental literature that compares revenues of the English auction and the Vickrey auction for real commodities.[5] The only laboratory controlled experiment that I am aware of, is conducted by Schindler (2003). She reports 14 percent lower revenues in the English auctions, therefore confirming the findings of the induced-value literature, as well as my theoretical predictions.

The remainder of the paper is structured as follows: Section 2 discusses the

[4]Indeed, Shogren et al. (1994) run Vickrey auctions to sell different goods and show that an endowment effect is strongest for non-market goods with imperfect substitutes.

[5]The only field experiment I am aware of is conducted by Lucking-Reiley (1999), who trades magic cards on an internet auction platform. He finds no significant difference in revenues, though he admits himself that he cannot entirely control for a potential selection bias and endogenous entry.

related literature, Section 3 analyzes the Vickrey auction with loss-averse bidders, while Section 4 analyzes the English auction with loss-averse bidders. In Section 5, I discuss the revenue comparison of both auction formats. Section 6 discusses several extensions, while Section 7 concludes.

2 Related Literature

Kahneman et al. (1990) establish the *endowment effect* that agents' valuation for goods increase with ownership. It has since been experimentally replicated under many different circumstances, for summaries see Camerer (1995) and Horowitz and McConnell (2002). Tversky and Kahneman (1991) propose loss aversion with respect to the status quo to explain the endowment effect.

Kőszegi and Rabin (2006) suggest recent rational expectations as reference point. The hypothesis that expectations play a role in individual's preferences have been supported in recent experiments (Ericson and Fuster (2011) and Abeler et al. (2011)), as well as challenged (Heffetz and List (2014)).[6]

The idea that the reference point is determined by recent beliefs leads to the natural question of the speed of reference-point adjustment. Strahilevitz and Loewenstein (1998) provide early evidence that the time span for which individuals hold beliefs has an impact on the reference point. Gill and Prowse (2012) use a real effort task to measure loss aversion and find that in their framework "the adjustment process is essentially instantaneous". Smith (2012) induces different probabilities of winning an item across groups of individuals. After the uncertainty resolves, he measures the willingness to pay for the item among bidders who have not won. In contrast to Ericson and Fuster (2011), who elicit valuations *before* the uncertainty resolves, Smith finds no significant difference between different groups, which suggests that the reference point is not so much determined by lagged beliefs, but rather adjusts quickly to the new information.[7]

For static environments Kőszegi and Rabin (2006) has arguably become the standard model of reference-dependent preferences, and been successfully applied to various fields, like mechanism design (Eisenhuth (2012)), contract theory (Herweg

[6]For a literature revue on related evidence, see Ericson and Fuster (2014).
[7]Smith's confidence intervals are, however, rather wide.

et al. (2010)), industrial organization (Heidhues and Kőszegi (2008)), and labor markets (Eliaz and Spiegler (2014)). Heidhues and Kőszegi (2014) show that buyers in monopolistic markets may face a similar form of time inconsistency as I establish for bidders in the English auction: ex ante they would like to commit not to buy. If the seller induces low prices with some probability, this plan, however, is time inconsistent. As a result, the consumer ends up buying for a high prices as well. Dato et al. (2017) extend the equilibrium concepts of Kőszegi and Rabin (2006) to strategic interaction in static games.

In the context of auctions with reference-dependent preferences, Lange and Ratan (2010) point out that loss-averse bidders may behave differently in laboratory experiments than in the field; bidders may not bracket narrowly in induced-value experiments. Further, they calculate the equilibrium bidding function of loss averse bidders in the first-price auction and Vickrey auction for a different equilibrium concept than I use in this paper. (For a more detailed discussion of the equilibrium concepts see section 3.)

Ehrhart and Ott (2014) introduce a model of the Dutch and English auction, where sequential information updates the reference point, but—in contrast to Kőszegi and Rabin (2009)—does not induce gain-loss utility. As a result, in equilibrium there is never any feeling of loss in the English auction, since by the time a bidder drops out she expects to lose. Eisenhuth and Ewers (2010) show that the all-pay auction yields higher payoffs than the first-price auction for narrow-bracketing bidders, since loss-averse bidders dislike payment uncertainty.

For dynamic environments Kőszegi and Rabin (2009) propose a model of dynamic loss aversion, where updates of expectations carry reference-dependent utility. This model has so far only been applied sparsely. First applications nevertheless seem promising. Rosato (2014) uses a two-period dynamic model to show that revenues are decreasing in sequential auctions with loss-averse bidders, due to a discouragement effect. To my best knowledge, Pagel is the first to rigorously apply Kőszegi and Rabin (2009) to dynamic problems with a long time horizon. Pagel (2016) shows that dynamic reference-dependent preferences can explain the historical levels of equity premiums and premium volatility in asset prices. Related to the logic in the English auction, loss-averse agents dislike price fluctuations, which makes assets relatively unattractive. Pagel (2017) shows that dynamic reference-dependent preferences can explain empirical observations about saving schemes for life-cycle consumption.

To my best knowledge, my model is the first to analyze strategic interaction of loss-averse players in a dynamic game with more than two periods.

3 The Vickrey Auction

3.1 Auction Rules

The second-price auction or Vickrey auction is a static, sealed-bid auction format. We assume that there are N loss averse bidders participating in the auction for a non-divisible good. Bidder i's valuation θ_i is privately observed and independently drawn from a common distribution

$$\theta_i \sim G,$$

where G has a differentiable density g which is strictly positive on its support $[\theta^{\min}, \theta^{\max}]$, with $0 \leq \theta^{\min} < \theta^{\max}$. After learning their private valuation, every participant submits a sealed bid. The bidder with the highest bid is assigned the object and has to pay the amount of the second highest bid.

3.2 Preferences

I assume that bidders are loss averse in the sense of Kőszegi and Rabin (2006). In addition to classical utility from an endowment $x \in \mathbb{R}$, bidders perceive a feeling of gain or loss, depending on whether the endowment is higher or lower than their reference point $r \in \mathbb{R}$. If we assume that classical utility is linear in x, this means:

$$u(x|r) = x + \mu(x - r),$$

where μ characterizes the gain-loss utility. In the Vickrey auction there are two commodity dimensions—money and good. We assume that bidders are narrow bracketers: utility is additively separable and gains and losses are evaluated separately across the two different dimensions: for any endowment level $x = (x^m, x^g)$

and any reference level $r = (r^m, r^g)$, agents utility is given by

$$u(x|r) = \sum_{k \in \{m,g\}} x^k + \mu_k(x^k - r^k),$$

where we allow for different loss specifications across dimensions.

If the bidder is loss averse she will weigh losses with respect to her reference point stronger than gains. Following Section IV in Kőszegi and Rabin (2006) and most of the literature, I assume μ_k to be a piecewise linear function with a kink at zero:

$$\mu_k(y) = \begin{cases} \eta_k y & y \geq 0 \\ \lambda_k \eta_k y & y < 0, \end{cases}$$

where $\eta_k > 0$, $\lambda_k > 1$, and $\Lambda_k := \lambda_k \eta_k - \eta_k < 1$ for $k \in \{m, g\}$.[8] Because it suffices for demonstrating the novel economic effect and allows for a significantly simpler exposition, I first focus on the case in which bidders are loss averse in the good dimension only, i.e. $\eta_m = 0$. In the extensions, I show that my results generalize to the case where we allow for loss aversion in the money dimension as well.[9]

The key feature in Kőszegi and Rabin (2006) is that the reference point is stochastic and endogenously determined by rational beliefs over future endowment levels. Consider an agent, who faces an uncertain payoff of x in some commodity dimension, which is distributed according to some distribution F. Let the reference point be determined by the agent's beliefs F' about the outcome. A realization $\overline{x}$ of x then yields an ex post utility in this commodity dimension of

$$u(\overline{x}|F') = \overline{x} + \int \mu(\overline{x} - r) dF'(r).$$

Then the ex-ante expected utility of the endowment x is given by

$$U(F|F') := \mathbb{E}u(x, |F') = \int \left(x + \int \mu(x - r) dF'(r) \right) dF(x).$$

[8]The condition $\Lambda < 1$ is referred to as "no dominance of gain-loss utility" by Herweg et al. (2010) It ensures that the dislike for uncertainty isn't too strong. If $\Lambda > 1$ a bidder could potentially prefer a strictly dominated safe outcome to a lottery.

[9]Horowitz and McConnell (2002) conclude in their summary that the endowment effect is "highest for non-market goods, next highest for ordinary private goods, and lowest for experiments involving forms of money." In this sense it may be plausible that loss aversion mainly applies to the good dimension.

If the agent has rational expectations, we have $F = F'$, and the expected utility of the lottery is

$$U(F|F) := \mathbb{E}u(x, |F) = \int \left(x + \int \mu(x - r) dF(r) \right) dF(x).$$

3.3 Equilibrium Concept

I adapt Kőszegi and Rabin's equilibrium concept under uncertainty to allow for interactive decision problems. I take an interim approach in the sense that each bidder i forms her strategy *after* she learns her private valuation θ_i. Fixing all opponents' behavior, we summarize their strategy in the distribution H of the maximal opponent bid. Given θ_i and H, any bid b induces some distribution of auction outcomes and therefore payoff distribution $F^k = F^k(b, \theta_i, H)$ in the respective commodity dimensions $k \in \{m, g\}$.

Definition 1. A bid $b^* \in \mathbb{R}_+$ constitutes an *unacclimated personal equilibrium (UPE)* in the Vickrey auction for bidder i if for all $b \in \mathbb{R}_+$,

$$\sum_{k \in \{m,g\}} U(F^k(b^*, \theta_i, H)|F^k(b^*, \theta_i, H)) \geq \sum_{k \in \{m,g\}} U(F^k(b, \theta_i, H)|F^k(b^*, \theta_i, H)).$$

In other words b^* is a UPE if, given the reference point generated by the action b^*, there is no profitable deviation b. It contrasts the definition of a *choice-acclimating personal equilibrium (CPE)*, where we require

$$\sum_{k \in \{m,g\}} U(F^k(b^*, \theta_i, H)|F^k(b^*, \theta_i, H)) \geq \sum_{k \in \{m,g\}} U(F^k(b, \theta_i, H)|F^k(b, \theta_i, H))$$

for all $b \in \mathbb{R}_+$. Thus, in contrast to the UPE-bidder, a CPE-bidder—which is analyzed in Lange and Ratan (2010)—already internalizes the effects of her deviation on the reference point. I believe the UPE is the appropriate equilibrium concept in the Vickrey auction, mainly for two reasons.

Firstly, I apply the model as proposed by Kőszegi and Rabin who suggest that the UPE is more appropriate if the bidder "anticipates the decision she faces but cannot commit to a choice until shortly before the outcome" (Kőszegi and Rabin (2007)). In auction settings bidders may know her valuation and form expectations long before the auction starts. Bids are, however, typically placed only shortly before the auction uncertainty resolves, and may depend on characteristics specific to the environment, such as the number of bidders actually participating in the auction.

Secondly, the UPE is the static analog of the concept of a personal equilibrium, which will be introduced in Section 4 to analyze the dynamic English auction. In this context one can gather another (dynamic) interpretation for the UPE: the decision maker ex ante forms a plan before the auction actually starts. This plan will determine her reference-point. The plan is a UPE if it is time-consistent in the sense that the decision maker is willing to carry it through at the time of action.

In a joint equilibrium, the first order statistic of the $n-1$ opponent bids H is endogenously determined by the equilibrium bidding strategy. Thus, if $b(\theta)$ constitutes a symmetric increasing equilibrium bidding function, we necessarily have

$$H(b(\theta)) = G^{n-1}(\theta).$$

Definition 2. In the Vickrey auction with n loss averse bidders, an increasing function $b(\theta)$ constitutes a symmetric UPE if for all θ and all b'

$$\begin{aligned}&\sum_{k\in\{m,g\}} U(F^k(b(\theta),\theta,G^{n-1}(b^{-1}(\cdot)))|F^k(b(\theta),\theta,G^{n-1}(b^{-1}(\cdot))))\\&\qquad\geq \sum_{k\in\{m,g\}} U(F^k(b',\theta,G^{n-1}(b^{-1}(\cdot)))|F^k(b(\theta),\theta,G^{n-1}(b^{-1}(\cdot)))).\end{aligned}$$

3.4 The Equilibrium

In this section we restrict attention to agents who are loss averse only in the good dimension. A more elaborate analysis of the general case, which allows for loss aversion in the money dimension is relegated to the extensions. Consider a bidder of type θ who plans to submit a bid of b^*. Given the distribution H of the highest opponent bid, the plan induces the reference distribution to win a utility of θ with probability of $H(b^*)$. Suppressing some notation, the utility of bidding b if planning to bid b^* is given by

$$\begin{aligned}
u(b,\theta|b^*) &:= \sum_{k\in\{m,g\}} U(F^k(b,\theta,H)|F^k(b^*,\theta,H)) \\
&= \underbrace{H(b)}_{\text{Prob to win}} \big(\theta + \underbrace{(1-H(b^*))\mu(\theta)}_{\text{feeling of gain}}\big) + \underbrace{(1-H(b))}_{\text{Prob to lose}} \underbrace{H(b^*)\mu(-\theta)}_{\text{feeling of loss}} + \underbrace{\int_0^b -sH(s)}_{\text{money dimension}} \\
&= \underbrace{\int_0^b (\theta - s)dH(s)}_{\text{classical utility}} + \underbrace{H(b)(1-H(b^*))\mu(\theta) + (1-H(b))H(b^*)\mu(-\theta)}_{\text{total reference-dependent utility}}.
\end{aligned}$$

In any symmetric equilibrium, $b = b^*$ must be the utility maximizing bid, where H is given by opponents' symmetric bidding behavior.

Theorem 1. *The unique symmetric increasing continuously differentiable UPE in the Vickrey auction with n bidders who are loss averse with respect to the good is given by*

$$b(\theta) = \big(1 + \eta(1 - G^{n-1}(\theta)) + \lambda\eta G^{n-1}(\theta)\big)\theta.$$

Note that all types overbid with respect to their intrinsic valuation θ. This should not be too surprising since we have assigned loss aversion only to the good dimension, and therefore made the good relatively more important, compared to money. More interestingly, the degree of overbidding is increasing in the type. The lowest type moderately overbids by

$$b(\theta^{\min}) = (1+\eta)\theta^{\min},$$

while the highest type aggressively overbids by

$$b(\theta^{\max}) = (1+\lambda\eta)\theta^{\max}.$$

The reason is the so called attachment effect: high types believe to win. Not winning would create a feeling of loss, which they try to prevent by placing an aggressive bid. As we will see section 6.1, this intuition remains intact, if we allow for loss aversion in money as well.

4 The English Auction

4.1 The Model

The Auction Format

The English auction format I am considering is sometimes referred to as the "Ascending Clock Auction" or the "Japanese Auction". In contrast to the "Open Outcry Auction" bidding starts at a fixed price and is raised incrementally by the auctioneer each time period. Each bidder signals—for example by raising or dropping her hand—when she wishes to drop out of the auction. Once a bidder dropped out she cannot bid again. The auction ends if there is only one active bidder left. This bidder has to pay the price, at which the last of her opponents dropped out.

For simplicity, we assume that there is no reservation price—the clock starts with a price of zero. The effect of a reserve price is analyzed in extension 6.3.

Preferences

We assume that bidders' intrinsic valuations θ_i for the object are privately observed and independently drawn

$$\theta_i \sim G$$

from a distribution G that has a differentiable and strictly positive density g on a positive support $[\theta^{\min}, \theta^{\max}]$. The distribution G is common knowledge. Bidders are assumed to be loss averse.

I follow Kőszegi and Rabin (2009) in how to model loss aversion in a dynamic discrete-time environment: agents hold rational beliefs about winning the auction and the respective transfers made after the auction is over. Every period, the agent observes, whether any opponents drop out at the current price and thus receives an information signal about the outcome. We denote by F_t^k, the beliefs over final transfers in $k \in \{money, good\}$, as anticipated at time t. As the signal at any time t changes beliefs over the auction outcome, this instantaneously gives rise to psychological gain-loss utility, denoted by $N(F_t^k|F_{t-1}^k)$, separately to changes in money and good.

For the evaluation of gain-loss utility, agents are assumed to assign gains and losses to changes in the respective quantiles of the distribution function. The intuition is that the agents rank possible outcomes from worst to best and then evaluate changes to the worst, the second worst ,..., until the best outcome. Let us denote with $c_{F_t^k}$ the quantile function of F_t^k, which is mathematically just the inverse of F_t^k. Then

$$N(F_t^k|F_{t-1}^k) = \int_0^1 \mu_k(c_{F_t^k}(p) - c_{F_{t-1}^k}(p))dp,$$

where again

$$\mu(x) = \begin{cases} \eta_k \cdot x & x \geq 0 \\ \eta_k \cdot \lambda_k \cdot x & x < 0, \end{cases}$$

$\eta_k > 0$, $\lambda_k > 1$, and $\Lambda_k := \lambda_k \eta_k - \eta_k < 1$.

In other words, during the auction process bidders accumulate information about the auction outcome. They absorb this information in their reference-point, which instantaneously exposes them to (possibly mixed) feelings of gains and losses. The total utility perceived in the auction process is given by the accumulated gain-loss utility and the classical utility from trade if the auction is won. In the following analysis, it is convenient to index the distributions with the current price rather than with the time period. After learning her type θ_i, bidder i forms a bidding strategy, which induces beliefs F_0^m and F_0^g about the auction outcome. If the auction runs for at most T increments of ε, we can write the total utility of the auction as

$$u_i = \sum_{t=1}^{T} \left(N(F_{t\varepsilon}^m | F_{(t-1)\varepsilon}^m) + N(F_{t\varepsilon}^g | F_{(t-1)\varepsilon}^g) \right) + (\theta_i - x)$$

if bidder i wins the auction at a price of x, and as

$$u_i = \sum_{t=1}^{T} \left(N(F_{t\varepsilon}^m | F_{(t-1)\varepsilon}^m) + N(F_{t\varepsilon}^g | F_{(t-1)\varepsilon}^g) \right)$$

if bidder i loses the auction. Note that the upper bound of T in the sum is without loss of generality; if the auction terminates early, all subsequent periods can be regarded as uninformative, and carry no further reference-dependent utility.

Equilibrium Concepts

I concisely sketch the equilibrium concept of Kőszegi and Rabin (2009). For full details and a psychological justification of the specific dynamic modeling choices, I refer to their paper.

Definition 3. An **action plan** specifies an action for every realization of information at every point in time. An action plan constitutes a **personal equilibrium** (PE) if, given the reference point resulting from the plan, it maximizes expected utility at any point in time among all plans that the agent is willing to carry through.

This means in particular:

- The bidder can only make credible plans in the sense that she cannot commit to plans that her future self does not want to carry through at the time of actions. Committing to unfavorable actions could be profitable, because it would manipulate beliefs, and therefore the own reference point.
- In suppressed notation, an action plan that induces a distribution F is an equilibrium if and only if at any point in time $u(F_t|F_t) \geq u(F'_t|F_t)$ for any distribution F'_t that would result from another credible plan.
- Given the opponents' behavior, an agent determines her set of personal equilibria by backward reasoning: she evaluates any action in $T-1$ with respect to her optimal actions in period T, and proceeds backwards.

The only constraint on initial beliefs is that they are rational, given the action plan. In general, there may be multiple personal equilibria.

Definition 4. A personal equilibrium is a **preferred personal equilibrium** if it is the utility maximizing PE at time zero.

The set of personal equilibria depends on the belief about other players' actions. To analyze the interaction between multiple bidders, we focus on symmetric personal equilibria.

Definition 5. A **strategy** $b(\theta)$ assigns to each possible type θ an action plan. A strategy constitutes a **(preferred) symmetric equilibrium** in the English auction if for each type θ and the belief that all opponents bid according to strategy b the action plan $b(\theta)$ constitutes a (preferred) personal equilibrium.

Timing

First bidders privately learn their valuation θ_i for the object. Then each bidder forms an action plan, which prescribes for any time (clock price) and any opponent drop-out history, the decision whether to drop out or to remain. Rational beliefs induced by this action plan form the bidder's reference point. Finally, the auction takes place. Any period during the auction process is characterized by the following timing:

- The price on the clock ascends and bidders simultaneously signal whether they stay in or drop out. If a bidder deviates from her action plan, she updates her reference point according to new rational beliefs. The update instantaneously induces reference-dependent utility

- Bidders observe, whether opponents drop out and update their reference point about payoffs. The update instantaneously gives rise to gain-loss utility.

- If there is at most one bidder remaining active, the auction is terminated. The remaining bidder is assigned the object and pays the current clock price.[10]

4.2 Analysis

Illustrative Example of Updating

This example aims to provide an illustration how gain-loss utility is formed during the auction process, and to show why bidders would always prefer a proxy to bid on their behalf in the English auction—taken behavior of opponents as given.

Consider an English auction with two bidders. Let bidder 1—in the following referred to as the bidder—have a valuation of θ for the object. Assume that the bidder plans to drop out at a price of 8 and knows that the drop-out price of bidder 2 — in the following called opponent—is ex ante uniformly distributed on $[0, 10]$ (we do not consider here, under which circumstances this behavior would be optimal). Ex ante, the bidder has a probability of 0.8 to win the auction and to have a payoff of θ in the good dimension. In the money dimension she faces a probability of 0.2 to pay nothing. Prices between 0 and 8 are uniformly distributed and have a

[10]For mathematical convenience, I abstract from tie breaking rules and assume that the good is not sold, if the remaining bidders drop out simultaneously. With our assumption of continuous density of types, as we let the increment size go to zero, this becomes equivalent to a tie breaking rule by coin-flip.

mass of 0.8 all together (if we assume arbitrary small increments on the clock for mathematical convenience). Thus, the ex ante quantile functions are given by

$$c_{F_0^g}(p) = \begin{cases} 0 & p \leq 0.2 \\ \theta & p > 0.2 \end{cases}$$

in the good dimension, and

$$c_{F_0^m}(p) = \begin{cases} -8 + 10p & p \leq 0.8 \\ 0 & p > 0.8 \end{cases}$$

in the money dimension.

Assume the opponent drops out at a price of 6. While the clock price ascends, the bidder permanently updates her beliefs. Let us look at the good dimension: for any increment below the price of 6, the bidder realizes that the opponent didn't drop out at that price, which reduces her beliefs to win the auction by some small amount. This means that during the auction process she accumulates perceived losses in the good dimension. Figure 1 shows the quantile functions at different clock prices.

At a clock price of 0—that is before the auction starts—the bidder holds her prior belief to win the auction with a probability of 0.8. The respective quantile function is a step function which is zero with probability 0.2, and θ with probability 0.8 (dotted line). At a price of 4 the bidder knows that the opponent hasn't dropped out between 0 and 4. Therefore bidder's updated belief to win is given by the probability that the opponent will drop out at price between 4 and 8, conditional on the fact that he will drop out between 4 and 10. It has thus decreased to two third which is indicated by the dashed quantile function. The medium grey shaded area is proportional to the loss the bidder has accumulated up to the price of 4 as the difference of the initial and current quantile function. Just before the opponent drops out at 6, bidder's belief has further decreased to almost one half—she wins if opponent drops out between 6 and 8, but loses if the opponent drops out between 8 and 10 (solid quantile function with jump at 0.5). The light shaded area shows the additional loss just before a price of 6 is announced. The losses have to be weighted with a factor of $\lambda\eta$. The moment the price increases to 6, the opponent drops out and the bidder wins with certainty. The quantile function jumps to the constant function $c_{F_6^g} = \theta$, inducing a feeling of gain of η times the three combined shaded areas.

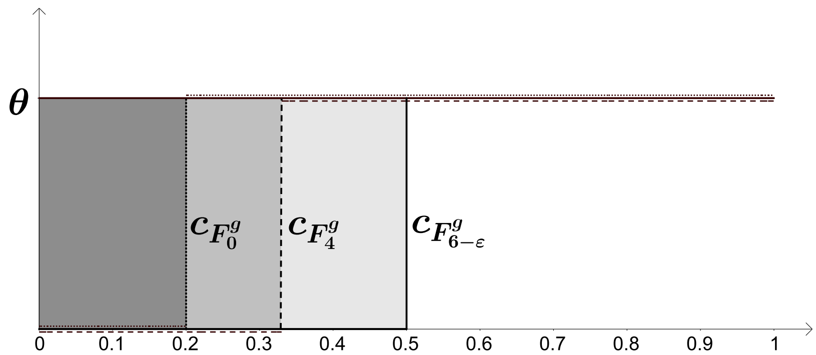

Figure 1: Updating in the English Auction

Thus, the net gain-loss utility in the good dimension is $(0.2\eta + 0.3(\eta - \lambda\eta))\theta = (0.2\eta - 0.3\Lambda)\theta$. Since losses weight stronger than gains, the relief of winning the light gray and medium gray area after all can only partly make up for the disappointment felt during the auction process. If the bidder could use a bidding proxy that enabled her to ignore new information until the auction was over, she would forgo the unpleasant variation in beliefs, which causes disutility of $-0.3\Lambda\theta$. This logic is due to Kőszegi and Rabin (2009), who find that, ceteris paribus, any collapse of information signals weakly increases agents' utility. Note that the use of a proxy in our two-bidder example is equivalent to submitting a sealed maximum bid. The example thus illustrates that——fixing her strategy and other bidders behavior—a loss averse bidder obtains weakly higher utility in the Vickrey auction than in the English auction.

The updating with respect to money is a bit more complex than the updating in the good dimension: if an opponent does not drop out at some price, the probability of losing and paying nothing increases as well as the probability of paying a high price. Nevertheless the same intuition applies: fluctuations in beliefs are costly, and loss-averse bidder would prefer to get all information at once. To summarize:

Corollary 1. *Loss-averse agents would prefer the use of proxies to bid on their behalf in the English auction. Thus, for a given set of bidders' maximal bids, any* *loss-averse bidder receives weakly higher utility in a Vickrey auction than in an English auction.*

Equilibrium Behavior for 2 Bidders

In the following, I analyze the set of equilibria in the English auction with two bidders, who are loss averse in the good dimension, as the increment size goes to zero. In section 6.4, I show that the main insights generalize to the n bidder auction. While the history-dependend strategy space in an n-bidder English auction is huge, it is fairly simple in a two-bidder game. Given type θ, an action plan prescribes the price at which the bidder plans to drop out, provided that the opponent is still active.

Each period the bidder observes whether her opponent remains in the auction. This information permanently updates her reference point, which induces gain-loss utility in each increment. An optimal bidding strategy will take the expected gain-loss utility from news into account.

For calculating the ex-ante expected gain-loss utility, it is more convenient to work with distribution functions rather than with quantile functions. This is possible, since they are inverse functions of each other, and the integral between functions equals the integral between their inverses up to the sign:

Lemma 1. *Let F_1 and F_2 be continuous distributions on an interval $[a, b]$ and let c_{F_1}, c_{F_2} be the respective quantile functions. Then*

$$\int_a^b (F_1(x) - F_2(x))dx = \int_0^1 (c_{F_2}(p) - c_{F_1}(p))dp.$$

With this result, one can look at the expected disutility from news.

Proposition 1. *Assume that a loss-averse bidder's payoff is distributed according to some distribution F_1 with a probability of Δ, and according to distribution F_2 with a probability of $1 - \Delta$. Let $[a, b]$ be the common support of F_1 and F_2. We denote with $F = \Delta F_1 + (1 - \Delta)F_2$ the ex ante distribution of the payoff. Then the ex ante expected reference-dependent utility from learning, whether the true distribution is F_1 or F_2, is given by*

$$\mathbb{E}(N(F_i|F) = -\Delta\Lambda \int_0^1 |c_{F_1}(p) - c_F(p)|dp,$$

or equivalently by

$$\mathbb{E}(N(F_i|F) = -\Delta\Lambda \int_a^b |F(x) - F_1(x)|dx.$$

The intuition for the result is as follows: on average, there is "as much good news as bad news". If gains and losses weighted equally, one would have zero gain-loss utility in average. Since losses loom larger than gains, variation will give us negative utility in expectation where the amount of negative utility is proportional to the expected variation and the loss dominance parameter Λ.

With this result we can calculate the accumulated expected loss due to gain-loss utility, as the increment size goes to zero. Let us denote with F the distribution of the opponent's drop-out price, in the sense that an opponent with drop-out price y remains in the auction at any clock price $t < y$, and drops out at prices $t \geq y$.

Proposition 2. *Consider a loss-averse bidder of type θ in the English auction with increments of ε and one opponent. Let the opponent's drop-out price be distributed according to distribution F with density f. Assume the bidder plans to drop out at x, and the opponent hasn't dropped out until time $t < x$. Then, for ε going to zero, in the limit the ex ante expected marginal gain-loss utility at time t is given by*

$$\ell_t(x, \theta, F) = \frac{-f(t)}{(1 - F(t))^2}(1 - F(x))\Lambda\theta.$$

Expected gain-loss utility for the remaining auction at time t is in the limit given by

$$L_t(x, \theta, F) = \ln\left(\frac{1 - F(x)}{1 - F(t)}\right)\frac{1 - F(x)}{1 - F(t)}\Lambda\theta.$$

Since losses weight stronger than gains, expected gain-loss utility is always negative. Note that the amount of marginal disutility is decreasing in x: an aggressive strategy induces less belief fluctuation at each information update, and thus partly insures against high gain-loss disutility in each increment. There is, however, a countervailing effect on total gain-loss disutility: the higher bidder's drop-out price, the longer she may stay in the auction and be exposed to gain-loss disutility. Figure 2 shows total expected gain-loss disutility at the beginning of the auction for $F \sim U[0, 1]$. We see that losses are the strongest for intermediate bids who face the highest uncertainty. Bidding 0 or 1 induces no uncertainty, and therefore no gain-loss utility.

In the following, we refer to the limit result as we let the increment size go to zero as the *continuous English auction*.[11]

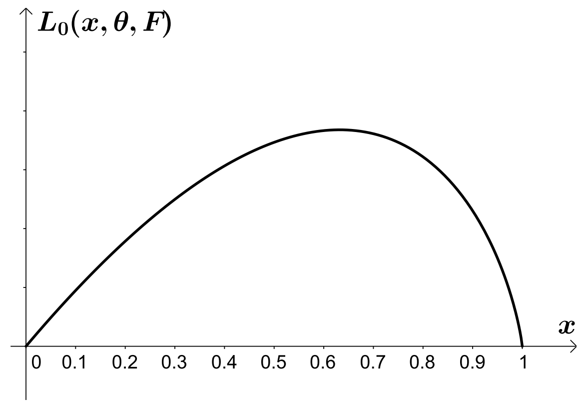

Figure 2: Total Expected Loss for $F \sim U[0, 1]$

With $L_t(x, \theta, F)$ we have established a function for the expected gain-loss utility *on the equilibrium path* for the strategy x. We now calculate the instantaneous gain-loss utility that the bidder perceives, if she decides to *deviate* from strategy x to strategy y at some point in time:

Lemma 2. *Consider a loss averse bidder in an English auction with one opponent. Let the opponent's drop-out price be distributed according to F. If at time t the bidder changes her strategy from dropping out at $x \geq t$ to dropping out at $y \geq t$, this deviation induces an instantaneous gain-loss utility of*

$$N(F_t^y | F_t^x) = \frac{\mu(F(y) - F(x))}{1 - F(t)} \theta.$$

Let us denote with $u_t(y, \theta, F|x)$ for $t \leq x, y$ the remaining expected utility of the agent at time t in the continuous English auction if she deviates at time t from

[11]This notion does not intend to refer to the concept of *continuous games* by Simon and Stinchcombe (1989). One should still regard the game as one with discrete increments on the clock which are, however, arbitrarily small.

strategy x to strategy y. Then, summarizing Proposition 2 and Lemma 2 we obtain

$$u_t(y,\theta,F|x) = \underbrace{\frac{\int_t^y(\theta-s)dF(s)}{1-F(t)}}_{\text{classical utility}} + \underbrace{\frac{\mu(F(y)-F(x))}{1-F(t)}\theta}_{\text{gain/loss from one-time update}} + \underbrace{L_t(y,\theta,F).}_{\substack{\text{expected gain-loss utility}\\ \text{of remaining auction}}}$$

All three terms change if a bidder deviates to another strategy. Note that the deviation utility is non-differentiable at $y = x$, since μ has a kink at zero.

With this notation and the above results, we can restate the condition for a strategy to be an equilibrium as we let the increment size go to zero.

Corollary 2. *In the continuous English auction a bidding strategy x is a personal equilibrium if and only if*

$$u_t(y,\theta,F|x) \leq u_t(x,\theta,F|x)$$

for all $0 \leq t \leq x, y$ and all strategies y that are credible at all times $s > t$.

Since the equilibrium concept restricts to strategies x that the agent wants to carry through at any time, it is in particular necessary that the agent does not want to drop out just before x is reached. This leads to the following constraint on time consistent plans.

Lemma 3. *Consider a loss-averse bidder of type θ in the continuous English auction with one opponent. Let the opponent's drop-out price be distributed according to distribution F with nonzero density f on some positive support $[a,b]$. Then, any time consistent bidding strategy $x \in (a,b)$ satisfies*

$$x \leq (1+\eta)\theta.$$

To understand the significance of this result, it is insightful to look at plans the bidder would choose if she could commit to a bidding strategy before the auction starts. She would not like to deviate from a strategy ex ante if and only if

$$u_0(y,\theta,F|x) \leq u_0(x,\theta,F|x)$$

for all y.

Proposition 3. *If two loss averse bidders could commit ex ante to a bidding strategy in the continuous English auction, the lowest symmetric increasing differentiable equilibrium would satisfy*

$$b(\theta) = \big(1 + \eta - \Lambda(1 + \ln(1 - G(\theta)))\big)\theta.$$

Figure 3 shows the ex ante optimal strategy (solid function) and the boundary of time-consistent strategies (dashed line) for two loss averse bidders.

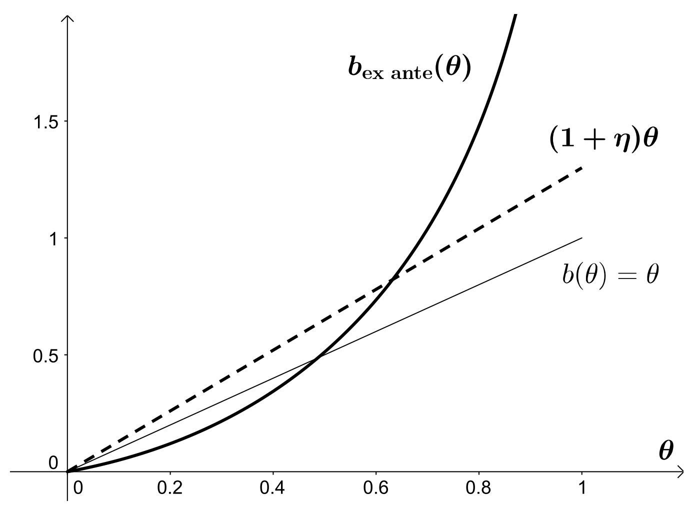

Figure 3: $G(\theta) \sim U[0, 1]$, $\eta = 0.3$, $\lambda = 4$

We see that low types ex ante may wish to underbid, while high types wish to strongly overbid. The intuition here is the same as in the Vickrey auction: bidders want to reduce expected gain-loss utility, and therefore try to reduce the uncertainty about winning. In particular high types would wish to insure with an aggressive bid against belief fluctuations during the auction process.

However, it is time-inconsistent to bid above $x = (1 + \eta)\theta$. Even though a bidder with a high valuation would ex ante like to commit to an aggressive bidding strategy, at the time she has to do so, she is not any more willing to carry that action through: as the auction proceeds, the winning chances for the bidder gradually decline. Thus, she gradually becomes a low type with respect to the remaining auction, and therefore her initial strategy of overbidding becomes less appealing. Just one increment before the bidder's drop out, she perceives the remaining auction similarly as a Vickrey auction, where she has the lowest possible type. Hence, at that point in time, her optimal bidding strategy resembles that of the lowest type in the Vickrey auction, i.e. she bids no more than $x = (1 + \eta)\theta$.

We have so far only considered constraints on equilibrium behavior at time 0 and at time x. It turns out that these are the binding constraints.

Lemma 4. *Consider a loss-averse bidder of type θ in the continuous English auction with one opponent. Let the opponent's drop-out price be distributed according to distribution F with nonzero density f on some positive support $[a, b]$. Then a strategy $x \in (a, b)$ is a PE if and only if*

1. *$x \leq (1 + \eta)\theta$;*
2. *for any $y \in [x, (1 + \eta)\theta]$ we have $u_0(x, \theta, F|x) \geq u_0(y, \theta, F|x)$.*

Theorem 2. *An increasing, almost everywhere differentiable function $b(\theta)$ is a symmetric equilibrium in the continuous English auction with two loss averse bidders if and only if for all θ*

1. *$b(\theta) \leq (1 + \eta)\theta$;*
2. *$b(\theta) \geq \min\left\{(1 + \eta)\theta \ ; \ \left(1 + \eta - \Lambda(1 + \ln(1 - G(\theta)))\right)\theta \ \right\}$.*

Thus, any increasing smooth function in the the gray shaded area of Figure 4 constitutes a symmetric equilibrium.

The thick line indicates the preferred symmetric equilibrium (PPE). Point A, where the PPE hits the boundary of time consistent strategies can be easily determined:

$$\left(1 + \eta - \Lambda(1 + \ln(1 - G(\theta)))\right)\theta = (1 + \eta)\theta$$

if and only if $G(\theta) = 1 - 1/e \approx 0.632$.

Note that the PPE is tangent to $(1 + \eta - \Lambda)\theta$ at the lowest type. Hence there is underbidding for low types if and only if $\eta - \Lambda > 0$, thus if and only if $\lambda > 2$.

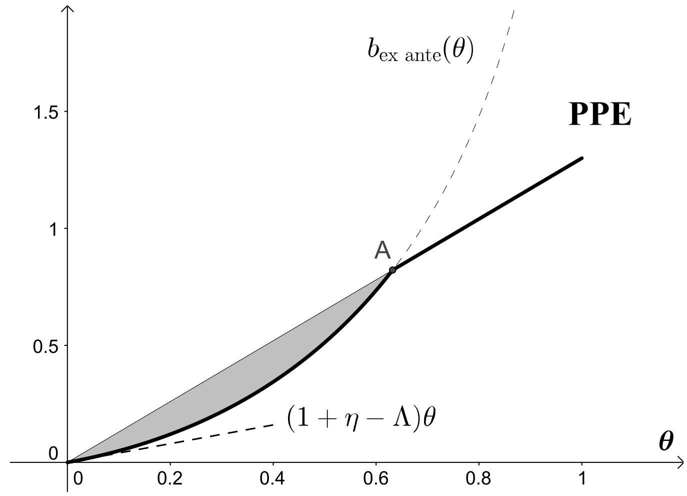

Figure 4: $G(\theta) \sim U[0,1]$, $\eta = 0.3$, $\lambda = 4$

Corollary 3. *The symmetric PPE in the continuous English auction with two loss averse bidders is given by*

$$b_{PPE}(\theta) = \begin{cases} \big(1+\eta-\Lambda(1+\ln(1-G(\theta)))\big)\theta & G(\theta) \leq 1-1/e \\ (1+\eta)\theta & G(\theta) > 1-1/e. \end{cases}$$

Low types underbid their intrinsic valuation θ in the PPE if and only if $\lambda > 2$.

5 Revenue Comparison

The equilibrium bidding function of an English auction with loss-averse bidders strongly depends on the question how quickly new information is absorbed in the reference point.

If the reference point consists of lagged beliefs, and the lag is sufficiently high, new information during the auction process will have no impact on bidders reference point. If values are private, there is therefore no impact of information gathered during the auction process. Each bidder will form her optimal decision with respect to the initial belief, and thus faces the same objective function as in the Vickrey auction—the strategic equivalence between English and Vickrey auction remains.

If bidders, however, update their reference point dynamically with respect to new information, loss-averse bidders bid at most $(1+\eta)\theta$.

The following figure shows the equilibrium bidding function for the Vickrey auction, $b_{\text{Vickrey}}(\theta)$, and the PPE of the English auction with dynamic reference point updating, $b_{\text{English}}(\theta)$. The shaded area indicates the potential other symmetric equilibria in the English auction, which are bounded by the line $(1+\eta)\theta$.

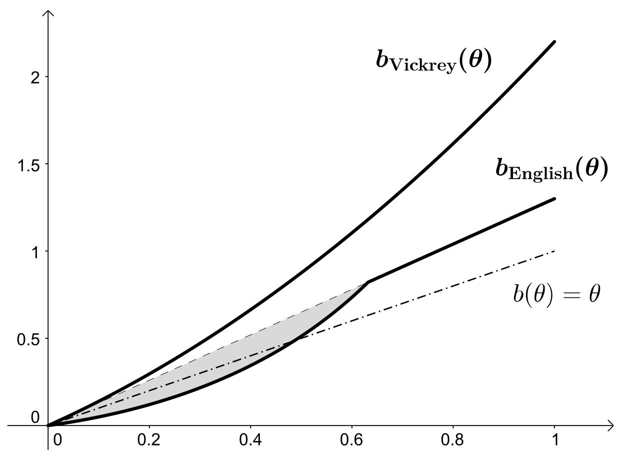

Figure 5: $G(\theta) \sim U[0,1]$, $\eta = 0.3$, $\lambda = 4$

As we have seen in section 3, overbidding with respect to θ is moderate for low types and strong for high types in the Vickrey auction. We can see that $b_{\text{Vickrey}}(\theta)$ at the lowest type is tangent to $(1+\eta)\theta$—the upper bound of equilibria in the English auction. The intuition is that for low types the decision problem in both auction formats becomes increasingly similar: since bidders in the English auction only learn, whether there are opponents with lower valuation than their own, the information difference between the two auction formats at the time the bidder places her (maximal) bid is small for low types.

Since the bidding function in the Vickrey auction satisfies $b_{\text{Vickrey}}(\theta) > (1+\eta)\theta$ for all types $\theta > \theta^{\min}$, it is immediate that the Vickrey auction dominates the English auction with respect to revenue.

Theorem 3. *1. If bidders are loss averse and do not update their reference point during the auction process, the Vickrey auction and the English auction are strategically equivalent: for a given continuous belief on the maximal opponent bid, bidding b is a UPE in the Vickrey auction if and only if bidding up to b is a PE in the English auction.*

2. If bidders are loss averse and update their reference point instantaneously during the auction process, equilibrium bids of the lowest type may coincide for both auction formats. For all other types, the Vickrey auction attains strictly higher revenue than the English auction.

6 Extensions and Robustness

6.1 Loss Aversion in the Money Dimensions

We generalize the baseline model to the case where bidders are loss averse in both commodity dimensions—money and good.

The Vickrey Auction

The utility of a bidder of type θ who places a bid of b but has a reference point as if bidding b^* is given by

$$\begin{aligned}
u(b,\theta|b^*) &:= \sum_{k\in\{m,g\}} U(F^k(b,\theta,H)|F^k(b^*,\theta,H)) \\
&= \int_0^b \left(-s + \int_0^{b^*} \mu_m(t-s)dH(t) + \int_{b^*}^{\infty} \mu_m(-s)dH(t)\right) dH(s) \\
&+ \int_b^{\infty} \left(\int_0^{b^*} \mu_m(t)dH(t) + \int_{b^*}^{\infty} \mu_m(0)dH(t)\right) dH(s) \\
&+ \int_0^b \left(\theta + \int_0^{b^*} \mu_g(0)dH(t) + \int_{b^*}^{\infty} \mu_g(\theta)dH(t)\right) dH(s) \\
&+ \int_b^{\infty} \left(\int_0^{b^*} \mu_g(-\theta)dH(t) + \int_{b^*}^{\infty} \mu_g(0)dH(t)\right) dH(s),
\end{aligned}$$

where H is again the distribution of the maximal opponent bid. The variable s corresponds to the realization of H, the variable t to the reference point. The first of the four summands corresponds to the utility in money if bidder i wins, the second if she loses. Similarly the third summand corresponds to utility in the good dimension if the auction is won, and summand four if the auction is lost.

In equilibrium the order statistic H is again endogenously determined by the opponents' equilibrium bids $b(\theta_{-i})$. Using the opponents' response functions, it is straightforward to calculate the symmetric equilibrium bidding function:

Theorem 4. *The unique symmetric increasing continuously differentiable UPE for n loss averse bidders in the Vickrey auction for commodities is given by*

$$b(\theta) = \frac{1+\eta_g+\Lambda_g G^{n-1}(\theta)}{1+\lambda_m\eta_m}\theta + \int_{\theta^{min}}^{\theta} \frac{\Lambda_m(1+\eta_g+\Lambda_g G^{n-1}(x))}{(1+\lambda_m\eta_m)^2} x \exp\left(\frac{\Lambda_m}{1+\lambda_m\eta_m}(G^{n-1}(\theta)-G^{n-1}(x))\right) dG(x).$$

Note that

$$b(\theta^{\min}) = \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta^{\min},$$

while for any $\theta > \theta^{\min}$

$$b(\theta)) > \frac{1+\eta_g+\Lambda_g G^{n-1}(\theta)}{1+\lambda_m\eta_m}\theta > \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta.$$

In particular, for equally weighted loss aversion in both dimensions, low types underbid, while

$$\begin{aligned} b(\theta^{\max}) &> \frac{1+\eta+\Lambda G^{n-1}(\theta^{\max})}{1+\lambda\eta}\theta^{\max} \\ &= \frac{1+\eta+\Lambda}{1+\lambda\eta}\theta^{\max} \\ &= \theta^{\max} \end{aligned}$$

shows that high types overbid their intrinsic valuation. The intuition is that low types don't expect to win and try to avoid unexpected losses in the money dimension. In contrast, high types expect to win and try to avoid unexpected losses in the good dimension.

The English Auction

We avoid to fully classify the set of symmetric PE again, but rather straightforwardly prove that the revenue ranking between the two auction formats remains intact.[12] The following Lemma parallels Lemma 3.

Lemma 5. *Consider a loss-averse bidder of type θ in the continuous English auction with one opponent. Let the opponent's drop-out price be distributed according to distribution F with nonzero density f on some positive support $[a, b]$. Then, any time consistent bidding strategy $x \in (a, b)$ satisfies*

$$x \leq \frac{1+\eta_g}{1+\lambda_m \eta_m}\theta.$$

Again, the bidders of high type ex ante like to commit excessive bids, but they know that the plan to bid above the threshold of $\frac{1+\eta_g}{1+\lambda_m \eta_m}\theta$ is time-inconsistent. Just one increment before they drop out, their belief to win and pay is virtually zero and—similarly to the lowest type in the Vickrey auction—they trade off the unexpected gain of the good against the unexpected loss in money, which may both occur with very small probability. If loss aversion is equally pronounced in both dimensions, then bidders underbid their intrinsic value θ, since losses weight stronger than gains.

Revenue Comparison

Since in the Vickrey auction we have

$$b_{\text{Vickrey}}(\theta) \geq \frac{1+\eta_g}{1+\lambda_m \eta_m}\theta,$$

with equality only for $\theta^{\min}$, and in the English auction we have

$$b_{\text{English}}(\theta) \leq \frac{1+\eta_g}{1+\lambda_m \eta_m}\theta,$$

it is immediate that the Vickrey auction remains to dominate the English auction with respect to revenue. Figure 6 shows the gray shaded area of potential equilibria in the English auctions, together with its PPE, and the equilibrium in the Vickrey auction. If loss aversion is equally pronounced in both dimensions, there is unambiguously underbidding in the English auction, while in the Vickrey auction low types underbid and high types overbid.

[12] The full derivation of the symmetric equilibrium bidding functions is available on request.

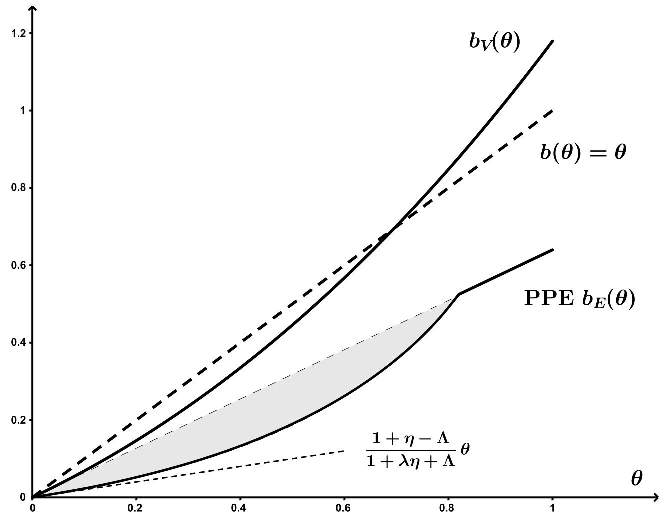

Figure 6: $G(\theta) \sim U[0,1]$, $\eta = 0.4$, $\lambda = 3$

6.2 False Beliefs or Heterogeneous Preferences

So far we have assumed that all participating bidders are loss averse and hold rational beliefs over opponents' behavior. This is not a crucial assumption. Loss-averse bidders will bid higher in the Vickrey auction than in the English auction for *any* continuous belief with full support that they hold over opponents strategies.

Following the analysis of section 4, equation 1 in the proof of Theorem 3.4 states that for any such belief H the bidding function in the Vickrey auction is given by

$$b(\theta) = \big(1 + \eta(1 - H(b(\theta))) + \lambda\eta H(b(\theta))\big)\theta,$$

which shows that

$$b(\theta) > (1+\eta)\theta$$

for all types, who win with positive probability. Contrary, in the English auction Lemma 3 shows that for any such belief

$$b(\theta) \leq (1+\eta)\theta.$$

6.3 Reserve Price

A reserve price is a prominent tool in auctions to guarantee some minimum price. If agents are loss averse, a reserve price will also impact the bidding strategy of the bidders above the reserve price.

Consider a Vickrey auction with n loss-averse bidders. Since the implementation of a reserve price excludes low types from participation, an ex ante announcement of such would have a selection effect on bidders who participate. It would considerably change beliefs about the participating opponents' types. To abstract away from this effect, assume that a reserve price x is announced after the bidders committed to participate, and before bidders form their strategies. Bids below the reserve price remain feasible, but cannot win.

Proposition 4. *Let $b(\theta)$ be the equilibrium bidding function of n loss-averse bidders in the Vickrey auction without reserve price. If bidders are loss averse with respect to money, a public reserve price $\overline{x} > b(\theta^{min})$ increases the equilibrium bid of all bidders with $b(\theta) \geq \overline{x}$.*

Thus, if the object is sold, a reserve price increases revenues, even if it is not binding. To get the intuition for this result, note that the reserve price has no direct effect on the winning probability for bidders with $b(\theta) > \overline{x}$ in any symmetric increasing equilibrium. A reserve price has therefore no impact on loss aversion in the good dimension. However, the belief of paying less than $\overline{x}$ decreases. If bidders are loss averse with respect to money, high prices now induce less loss in the money dimension, with respect to expectations. This reduces expected gain-loss disutility from a high bid.

The same holds for similar reasons in the English auction with loss aversion in money, which we omit to prove here. In the English auction with loss aversion in the good dimension only, a reserve price has again no effect on equilibrium behavior.

Proposition 5. *Consider a continuous English auction with two bidders, who are loss averse in the good dimension. A reserve price of $\overline{x}$ has no effect on an equilibrium bidding function b for any type θ with $b(\theta) > \overline{x}$.*

6.4 Generalization to n bidders

In auctions where bidders face more than one opponent, the set of possible action plans becomes very large. Recall that an action plan prescribes a consistent action for any history and any future contingency at any time. While in the two bidder case the history is rather simple—either the opponent dropped out and the auction is over, or we are still in the auction process—with more bidders the individual decision at each time may in principle depend on the exact timing at which opponents dropped out in the past.

Since each decision must be sequentially optimal, given expectations about the future, one might hope to be able to restrict to Markov perfect equilibria, in the sense that at time t the individual type θ_i and the number of currently active bidders is a sufficient statistic for the optimal decision of bidder i. However, this is not the case. While the set of personal equilibria starting at time t can be determined without looking into the past, the specific equilibrium path will depend on the evolution of beliefs up to time t.

In order to deal with strategies contingent on histories, we define the following notation:

Definition 6. For any n-bidder auction, define for all $k \in \{0, ..., n-2\}$

$$H_k = \{(t_1, ..., t_k) | 0 \leq t_1 \leq, ..., \leq t_k\}$$

as the set of histories / future contingencies with k drop outs at the respective prices $t_1, ..., t_k$, with the convention $H_0 = \{\emptyset\}$.

With this notation, a complete action plan prescribes for each history and future contingency the price at which a bidder of type θ plans to drop out:

Definition 7. A pure strategy action plan prescribes a bidding strategy

$$b : \bigcup_{0 \leq k \leq n-2} H_k \times [\theta^{\min}, \theta^{\max}] \to \mathbb{R}_+,$$

with the restriction that if for any $(t_1, ..., t_k, \theta)$ we have

$$b(t_1, ..., t_k, \theta) > t_k,$$

The latter condition on the bidding function ensures that bidders cannot condition their drop out on events that happen after the drop out.

Again, we restrict attention to differentiable and increasing equilibrium bidding functions in the following sense:

Definition 8. A bidding strategy b in the English auction is differentiable and increasing if for all $(t_1, ..., t_k) \in \bigcup_{0 \leq k \leq n-2} H_k$ the function $b(t_1, ..., t_k, \theta)$ is differentiable and increasing in θ.

Example 1. Consider a continuous English auction with three loss-averse bidders. A complete strategy prescribes for every θ:

- A price $b(\theta)$ at which the bidder drops out if no opponent dropped out before
- For any opponent drop out at some price $t < b(\theta)$, a price $b(t, \theta)$ at which the bidder drops out in the subsequent two-bidder auction

The aim of the example is to illustrate, why the optimal strategy $b(t, \theta)$ for the two-bidder auction following the first drop out depends on t.Suppose that all three bidders bid according to the same symmetric equilibrium bidding strategy $(b(\theta), b(t, \theta))$. Let us focus on the decision problem of a bidder, whose valuation θ is sufficiently high, such that $b(\theta) = (1 + \eta)\theta$ were the only time-consistent strategy in the two-bidder English auction.

Suppose first that an opponent has a valuation of zero and drops out at $t = 0$. For the strategy $b(0, \theta)$ the bidder is now bound by the set of time-consistent strategies of the two-bidder auction, as outlined in Theorem 2. Since she has high beliefs to win, the only time-consistent strategy is $b(0, \theta) = (1 + \eta)\theta$.

Next, we analyze optimal strategies $b(t, \theta)$ for t being smaller, but close to $b(\theta)$. Similar to the two-bidder auction, a bidder with a high winning probability would ex ante like to insure against belief fluctuations with an aggressive strategy. Any strategy for $b(t, \theta)$, however, must be time consistent in the sense that the bidder is willing to stick to it until t. Just before t the belief to win the auction has decreased considerably. The bidder trades off the expected gains from trade against the expected loss from news. The following Lemma states the expected loss at time t for the three bidder case.

Lemma 6. *Consider a continuous English auction with three loss-averse bidders. Assume all bidders follow a symmetric, differentiable, increasing bidding strategy* $(b(\theta), b(t,\theta))$*. Assume further that no bidder dropped out until* $t \in [b(\theta^{min}), b(\theta^{max})]$*. Let* $\theta(t)$ *be defined by* $b(\theta) = t$*. Then expected gain-loss utility at time* t *is given by*

$$L_t(\theta) = -\Lambda\theta \int_{\theta(t)}^{\theta} \frac{2g(s)(1-G(s))}{(1-G(\theta(t)))^2} \Bigg[\underbrace{\frac{G(\theta)-G(s)}{1-G(s)} - \left(\frac{G(\theta)-G(s)}{1-G(s)}\right)^2}_{A} \underbrace{- \ln\left(\frac{1-G(\theta)}{1-G(s)}\right) \frac{1-G(\theta)}{1-G(s)}}_{B} \Bigg] ds$$

The terms of $L_t(\theta)$ are easy to interpret. At time t the conditional marginal probability that the first drop out is of type s is given by $\frac{2g(s)(1-G(s))}{(1-G(\theta(t)))^2}$. In this case, the bidder would update the winning probability from $\left(\frac{G(\theta)-G(s)}{1-G(s)}\right)^2$ to $\frac{G(\theta)-G(s)}{1-G(s)}$ (term A). Further, term B shows the expected loss for the following 2-bidder auction, as calculated in Proposition 2.

Term A indicates an additional source of expected gain-loss disutility, compared to the two bidder auction: even if a bidder loses after all, beliefs to win don't necessarily gradually decline to zero, but might temporarily increase due to one opponent dropping out. This effect leads to more belief fluctuations and worsens bidder's trade-off between expected news disutility and expected gains from trade. As a result, it is no longer time consistent to bid up to $b(t,\theta) = (1-\eta)\theta$ for all t.

Corollary 4. *In any symmetric, increasing, differentiable equilibrium* $(b(\theta), b(t,\theta))$ *of the English auction with three loss-averse bidders, expected news disutility for any* $\theta \in (\theta^{min}, \theta^{max})$ *satisfies*

$$\lim_{t \to b(\theta)} \frac{L_t(\theta)}{\left(\frac{G(\theta)-G(\theta(t))}{1-G(\theta(t))}\right)^2} = -2\Lambda\theta.$$

If $b(t,\theta)$ *is continuous in* t*, then—by time-consistency—*

$$\lim_{t \to b(\theta)} b(t,\theta) \leq (1+\eta-\Lambda)\theta.$$

Since we have argued above that $b(0,\theta) = (1+\eta)\theta$, the corollary illustrates that bidding behavior $b(t,\theta)$ in general depends on opponents' drop-out history t.

Even if the sales price depends on all type realizations, it is immediate that for n bidders the revenue ranking between the two auction format remains: since bidders generically don't share the same valuation, in any symmetric continuous increasing equilibrium they will drop out of the auction consecutively, in order of their types. Eventually, with probability one, the two bidders with the highest valuation will end up in the two-bidder subgame. Here they are bound to the constraints on time-consistent behavior, as analyzed in section 4.2. In particular by Lemma 3, any time-consistent strategy for the two-bidder auction satisfies $b(\theta) \leq (1 + \eta)\theta$.

To summarize:

Corollary 5. *In a symmetric increasing equilibrium of the continuous English auction with n loss-averse bidders, the revenue may depend on all type realizations. For any opponent drop-out history, every bidder's maximal bid is bounded by $b(\theta) \leq (1 + \eta)\theta$. Thus, with n loss-averse bidders, the English auction remains to yield lower revenues than the Vickrey auction.*

Even if the auction outcome for many bidders is similar to the one for two bidders, it is worth noting that individual bidders obtain less utility, compared to two-bidder auctions with the same sales price. To see this, consider—hypothetically—that bidders could choose not to observe individual drop outs, but rather learn in each period, whether *any* opponent is still in the game. The auction would then subjectively resemble an English auction with two bidders, where the opponent's type is drawn from the first order-statistic over all opponents. The key difference is that information is fluctuating much less. As already mentioned earlier and stated in generality in Proposition 1 of Kőszegi and Rabin (2009), the collapse of multiple signals into one will always weakly decrease gain-loss disutility.

7 Conclusion

I studied the effects of expectation-based preferences in dynamic environments, comparing the dynamic English auction to the static Vickrey auction. If the reference point is static and doesn't respond to information, there is no strategic difference between the English auction and the Vickrey auction. If bidders update their reference point instantaneously with respect to new information, however, dynamic information in the English influences bidders endogenous preferences, and thus their bidding strategies. The classical strategic equivalence between the the two auction formats breaks down and the English auction attains strictly lower revenue than the Vickrey auction.

This difference highlights the importance of understanding the evolution of the reference point in dynamic environments. In particular, research about the speed of reference point adaptation with respect to new information is still in its infancy and deserves further study.

The non-equivalence of the two auction formats stands in sharp contrast to the revenue equivalence principles by Vickrey (1961) and Myerson (1981). Indeed, the powerful approach of mechanism design and the revelation principle relies on the assumption that agents' valuations are exogenously given and do not depend on the choice of mechanism. This assumption is violated if bidders have endogenous preferences that depend on expectations induced by the mechanism itself. In particular, if agents update their reference point with respect to new information in a multi-stage mechanism, such a mechanism cannot be replaced by a simple direct mechanism without changing agents' incentives. The failure of the revelation principle naturally leads to the question of optimal mechanism design in dynamic environments with expectation-based loss-averse agents. The study of optimal expectation management in these environments leaves an interesting field for future research.

8 Appendix

Proof of Theorem 3.4. Suppose that all opponents bid according to some increasing, continuously differentiable bidding function $b(\theta)$. Since $G(\theta)$ is a distribution with strictly positive, continuous density g, it follows that the distribution of the maximal opponent bid, $H(x) = G^{n-1}(b^{-1}(x))$, is a differentiable distribution with positive, continuous density $h(x)$ on $[b(\theta^{\min}), b(\theta^{\max})]$ as well.

The bidding function $b(\theta)$ constitutes a UPE if and only if the utility function $u(x, \theta|b(\theta))$ attains its maximum at $x = b(\theta)$ for all θ. Differentiation with respect to x yields

$$\frac{\partial u(x,\theta|b(\theta))}{\partial x} = (\theta - x)h(x) + h(x)(1 - H(b(\theta))\mu(\theta) - h(x)H(b(\theta))\mu(-\theta).$$

By dividing by $h(x)$ and evaluating at $x = b(\theta)$ we obtion the first-order condition

$$0 = (\theta - b(\theta)) + (1 - H(b(\theta)))\eta\theta + H(b(\theta))\lambda\eta\theta.$$

Rearranging yields

$$b(\theta) = \big(1 + \eta(1 - H(b(\theta))) + \lambda\eta H(b(\theta))\big)\theta. \tag{1}$$

Using that $H(b(\theta)) = G^{n-1}(\theta)$ we obtain

$$b(\theta) = \big(1 + \eta(1 - G^{n-1}(\theta)) + \lambda\eta G^{n-1}(\theta)\big)\theta$$

as the unique equilibrium candidate. For sufficiency note first that

$$h(b(\theta)) = \frac{(G^{n-1})'(\theta)}{b'(\theta)} = \frac{(n-1)G^{n-2}(\theta)g(\theta)}{(1 + \eta(1 - G^{n-1}(\theta)) + \lambda\eta G^{n-1}(\theta)) + \Lambda(n-1)G^{n-2}(\theta)g(\theta)\theta}$$

is differentiable since $g(\theta)$ is differentiable. Now it is immediate that

$$\frac{\partial^2 u_i(x, \theta|b(\theta))}{(\partial x)^2}\Big|_{x=b(\theta)} = -h(b(\theta)) + h'(b(\theta))\underbrace{[(\theta - b(\theta)) + (1 - H(b(\theta))\mu(\theta) - H(b(\theta))\mu(-\theta)]}_{=0} < 0.$$

□

Proof of Lemma 1. By the theorem of the integral over inverse functions, we have

$$\int_a^b F_i(x)dx = bF_i(b) - aF_i(a) - \int_0^1 c_{F_i}(p)dp = b - \int_0^1 c_{F_i}(p)dp.$$

Now, it is immediate that

$$\int_a^b (F_1(x) - F_2(x))dx = (b - b) - \int_0^1 c_{F_1}(p)dp + \int_0^1 c_{F_2}(p)dp = \int_0^1 (c_{F_2}(p) - c_{F_1}(p))dp.$$

□

Proof of Proposition 1. By applying Lemma 1, and using the fact that μ is piecewise linear, we can write

$$
\begin{aligned}
\mathbb{E}(N(F_i|F) &= \Delta N(F_1|F) + (1-\Delta)N(F_2|F) \\
&= \Delta \int_0^1 \mu(c_{F_1}(p) - c_F(p))dp + (1-\Delta)\int_0^1 \mu(c_{F_2}(p) - c_F(p))dp \\
&= \Delta \int_a^b \mu(F(x) - F_1(x))dx + (1-\Delta)\int_a^b \mu(F(x) - F_2(x))dx \\
&= \Delta \int_a^b \mu(F(x) - F_1(x))dx + \int_a^b \mu((1-\Delta)F(x) - (1-\Delta)F_2(x))dx \\
&= \Delta \int_a^b \mu(F(x) - F_1(x))dx + \int_a^b \mu((1-\Delta)F(x) - (F(x) - \Delta F_1(x))dx \\
&= \Delta \int_a^b \mu(F(x) - F_1(x))dx + \int_a^b \mu(-\Delta F(x) + \Delta F_1(x))dx \\
&= \Delta \int_a^b \mu(F(x) - F_1(x))dx + \Delta \int_a^b \mu(-F(x) + F_1(x))dx \\
&= \Delta(-\lambda\eta + \eta)\int_a^b |F(x) - F_1(x)|dx \\
&= -\Delta\Lambda \int_a^b |F(x) - F_1(x)|dx \\
&= -\Delta\Lambda \int_0^1 |c_{F_1}(p) - c_F(p)|dp.
\end{aligned}
$$

□

Proof of Proposition 2. Suppose the current clock price is t and the opponent hasn't dropped out yet. If the clock increases in increments of ε, then the conditional probability that the opponent drops out at the next increment is given by

$$\Delta_t := \frac{F(t+\varepsilon) - F(t)}{1 - F(t)}.$$

Given her strategy x and that the opponent hasn't dropped out at t, the bidder faces the conditional probability of $\frac{1-F(x)}{1-F(t)}$ to lose the auction. Thus, if F_t^x denotes the belief about payoffs in the good dimension at time t given strategy x, we have

$$F_t^x(z) = \begin{cases} \frac{1-F(x)}{1-F(t)} & z < \theta \\ 1 & z \geq \theta. \end{cases}$$

If the bidder wins in the next increment, the belief will update to

$$F^x_{t+\varepsilon}(z) = \begin{cases} 0 & z < \theta \\ \\ 1 & z \geq \theta \end{cases}$$

According to Proposition 1, expected gain-loss utility of the increment from t to $t+\varepsilon$ is then given by

$$\mathbb{E}(N(F^x_{t+\varepsilon}|F^x_t) = -\Delta_t \Lambda \int |F^x_t(z) - F^x_{t+\varepsilon}(z)| dz = -\Delta_t \Lambda \frac{1-F(x)}{1-F(t)} \theta.$$

Now, the marginal loss at time t if ε goes to zero reads

$$\ell_t(x, \theta, F) = \lim_{\varepsilon \to 0} \frac{-\Delta_t \Lambda \frac{1-F(x)}{1-F(t)} \theta}{\varepsilon} = \frac{-f(t)}{(1-F(t))^2}(1-F(x))\Lambda\theta.$$

To calculate total expected gain-loss utility starting at time t, note that any information update at time $s > t$ is only informative and carries gain-loss utility if the opponent hasn't already dropped out between t and s, which holds true with the conditional probability $\frac{1-F(s)}{1-F(t)}$. Thus

$$\begin{aligned} L_t(x, \theta, F) &= \lim_{\varepsilon \to 0} \sum_{i=0}^{\lfloor \frac{x-t}{\varepsilon} \rfloor - 1} N(F^x_{t+(i+1)\varepsilon} | F^x_{t+i\varepsilon}) \\ &= \lim_{\varepsilon \to 0} \sum_{i=0}^{\lfloor \frac{x-t}{\varepsilon} \rfloor - 1} -\frac{1-F(t+i\varepsilon)}{1-F(t)} \Delta_{t+i\varepsilon} \Lambda \frac{1-F(x)}{1-F(t+i\varepsilon)} \theta \\ &= \int_t^x \frac{-f(s)}{1-F(s)} \frac{1-F(x)}{1-F(t)} \Lambda \theta ds \\ &= (\ln(1-F(x)) - \ln(1-F(t))) \frac{1-F(x)}{1-F(t)} \Lambda \theta \\ &= \ln\left(\frac{1-F(x)}{1-F(t)}\right) \frac{1-F(x)}{1-F(t)} \Lambda \theta. \end{aligned}$$

□

Proof of Lemma 2. At time t the winning probability is given by the probability that the opponent drops out between t and x, given he didn't drop out before t, thus $\frac{F(x)-F(t)}{1-F(t)}$. Thus, the update changes the probability of getting θ by

$$\frac{F(y)-F(t)}{1-F(t)} - \frac{F(x)-F(t)}{1-F(t)} = \frac{F(y)-F(x)}{1-F(t)}.$$

Hence,

$$N(F_t^y|F_t^x) = \mu\left(\frac{F(y)-F(x)}{1-F(t)}\theta\right) = \frac{\mu(F(y)-F(x))}{1-F(t)}\theta.$$

□

Proof of Lemma 3. The bidder does not want do deviate to a lower strategy at any time t, given plan x if and only if

$$u_t(y,\theta,F|x) \leq u_t(x,\theta,F|x)$$

for all $t \leq y \leq x$. In particular it is necessary that for all $t < x$ the derivative from the left satisfies

$$\begin{aligned} 0 &\leq \lim_{y \nearrow x} \frac{\partial u_t(y,\theta,F|x)}{\partial y} \\ &= \frac{f(x)}{1-F(t)}\left(\theta - x + \lambda\eta\theta - \Lambda\left(1 + \ln\left(\frac{1-F(x)}{1-F(t)}\right)\right)\theta\right). \end{aligned}$$

This expression is well defined, since $F(t) < F(x) < 1$. Now, as t approaches x we get

$$\begin{aligned} 0 &\leq \lim_{t\to x} \frac{f(x)}{1-F(t)}\left(\theta - x + \lambda\eta\theta - \Lambda\left(1 + \ln\left(\frac{1-F(x)}{1-F(t)}\right)\right)\theta\right) \\ &= \frac{f(x)}{1-F(x)}\left(\theta - x + \lambda\eta\theta - \Lambda\theta\right). \end{aligned}$$

Since, by assumption, $f(x) > 0$, this means that necessarily

$$x \leq (1 + \lambda\eta - \Lambda)\theta = (1+\eta)\theta.$$

□

Proof of Proposition 3. Given opponent's strategy F and bidder's type θ, a bid $b(\theta)$

is a personal equilibrium in the auction with commitment if and only if

$$u_0(y,\theta,F|b(\theta)) \leq u_0(b(\theta),\theta,F|b(\theta))$$

for all y. In particular, it is necessary that

$$\lim_{y\searrow b(\theta)} \frac{\partial u_0(y,\theta,F|b(\theta))}{\partial y} \leq 0.$$

Since for $y > b(\theta)$ the utility at time zero reads

$$u_0(y,\theta,F|b(\theta)) = \int_0^y (\theta - s)dF(s) + \eta(F(y) - F(b(\theta)))\theta + \ln(1-F(y))(1-F(y))\Lambda\theta,$$

this necessary condition is equivalent to

$$f(b(\theta))\big(\theta - b(\theta) + \eta\theta - \Lambda(1+\ln(1-F(b(\theta))))\theta\big) \leq 0.$$

In any symmetric equilibrium, the opponent bids according to $b(\theta)$ as well, and therefore we have $F(b(\theta)) = G(\theta)$. From $g(\theta) = f(b(\theta))b'(\theta)$ and the restriction that b is increasing it follows that $f(b(\theta)) > 0$. Hence we have

$$b(\theta) \geq \big(1 + \eta - \Lambda(1 + \ln(1 - G(\theta)))\big)\theta$$

for any equilibrium candidate. It remains to verify that

$$b(\theta) = \big(1 + \eta - \Lambda(1 + \ln(1 - G(\theta)))\big)\theta$$

is a personal equilibrium, given opponent's response $b(\theta)$. For this it is sufficient to show that

$$\frac{\partial u_0(y,\theta,F|b(\theta))}{\partial y} \leq 0$$

for all $y > b(\theta)$, and

$$\frac{\partial u_0(y,\theta,F|b(\theta))}{\partial y} \geq 0$$

for all $y < b(\theta)$. Note that we can without loss of generality restrict to $y \in [b(\theta^{\min}), b(\theta^{\max})]$.

For any such y there exists some $\tilde{\theta}$ with $y = b(\tilde{\theta})$, since the bidding function is continuous.

Consider first $y > b(\theta)$, thus $\tilde{\theta} > \theta$. Then

$$\begin{aligned}
\frac{\partial u_0(y,\theta,F|b(\theta))}{\partial y}\Big|_{y=b(\tilde{\theta})} &= f(b(\tilde{\theta}))\big(\theta - b(\tilde{\theta}) + \eta\theta - \Lambda(1+\ln(1-F(b(\tilde{\theta}))))\theta\big) \\
&< f(b(\tilde{\theta}))\big(\tilde{\theta} - b(\tilde{\theta}) + \eta\tilde{\theta} - \Lambda(1+\ln(1-F(b(\tilde{\theta}))))\tilde{\theta}\big) \\
&= \lim_{y \searrow b(\tilde{\theta})} \frac{\partial u_0(y,\tilde{\theta},F|b(\theta))}{\partial y} \\
&= 0.
\end{aligned}$$

Similarly, for $y < b(\theta)$, thus $\tilde{\theta} < \theta$ we have

$$\begin{aligned}
\frac{\partial u_0(y,\theta,F|b(\theta))}{\partial y}\Big|_{y=b(\tilde{\theta})} &= f(b(\tilde{\theta}))\big(\theta - b(\tilde{\theta}) + \lambda\eta\theta - \Lambda(1+\ln(1-F(b(\tilde{\theta}))))\theta\big) \\
&> f(b(\tilde{\theta}))\big(\tilde{\theta} - b(\tilde{\theta}) + \eta\tilde{\theta} - \Lambda(1+\ln(1-F(b(\tilde{\theta}))))\tilde{\theta}\big) \\
&= \lim_{y \searrow b(\tilde{\theta})} \frac{\partial u_0(y,\tilde{\theta},F|b(\theta))}{\partial y} \\
&= 0.
\end{aligned}$$

□

Proof of Lemma 4. Consider a bidding strategy x.

Claim 1: If and only if $x \leq (1+\eta)\theta$, it is at no time $t < x$ profitable to deviate to a lower strategy $y \in [t, x)$.

Proof: the "only if" has been proved in Lemma 3. For the "if", assume that $x \leq (1+\eta)\theta$. Consider a deviation at some time $t < x$ from x to $y \in [t, x)$. We first look at the change in expected gain-loss disutility: term A can be interpreted as the change due to different expectations at each time between t and y, while term

B is forgone gain-loss disutility, since the auction necessarily ends at y:

$$
\begin{aligned}
& L_t(y,\theta,F) - L_t(x,\theta,F) \\
=& \Lambda\theta\left(\ln\left(\frac{1-F(y)}{1-F(t)}\right)\frac{1-F(y)}{1-F(t)} - \ln\left(\frac{1-F(x)}{1-F(t)}\right)\frac{1-F(x)}{1-F(t)}\right) \\
=& \Lambda\theta\left(\int_t^y \frac{-f(s)}{1-F(s)}ds\frac{1-F(y)}{1-F(t)} - \int_t^x \frac{-f(s)}{1-F(s)}ds\frac{1-F(x)}{1-F(t)}\right) \\
=& \Lambda\theta\left(\int_t^y \frac{-f(s)}{1-F(s)}ds\frac{1-F(y)}{1-F(t)} - \int_t^y \frac{-f(s)}{1-F(s)}ds\frac{1-F(x)}{1-F(t)} - \int_y^x \frac{-f(s)}{1-F(s)}ds\frac{1-F(x)}{1-F(t)}\right) \\
=& \Lambda\theta\left(\underbrace{\int_t^y \frac{-f(s)}{1-F(s)}ds\frac{F(x)-F(y)}{1-F(t)}}_{A} - \underbrace{\int_y^x \frac{-f(s)}{1-F(s)}ds\frac{1-F(x)}{1-F(t)}}_{B}\right) \\
\leq& \Lambda\theta\int_y^x \frac{f(s)}{1-F(s)}ds\frac{1-F(x)}{1-F(t)} \\
<& \Lambda\theta\int_y^x f(s)ds\frac{1-F(x)}{(1-F(x))(1-F(t))} \\
=& \Lambda\theta\frac{F(x)-F(y)}{1-F(t)}
\end{aligned}
$$

Now we have

$$
\begin{aligned}
u_t(y,\theta,F|x) - u_t(x,\theta,F|x) &= \frac{1}{1-F(t)}\left(-\int_y^x(\theta-s)dF(s) + \mu(F(y)-F(x))\theta + \Lambda\theta(F(x)-F(y))\right) \\
&< \frac{F(x)-F(y)}{1-F(t)}\left(-\theta + x - \lambda\eta\theta + \Lambda\theta\right) \\
&= \frac{F(x)-F(y)}{1-F(t)}\left(-(1+\eta)\theta + x\right) \\
&\leq 0.
\end{aligned}
$$

Thus, there is no profitable deviation to $y < x$ at any time, which concludes the proof of claim 1.

Claim 1 directly shows the necessity of 1. for any PE. Certainly, 2. is necessary as well.

Claim 2: If it is not profitable to deviate to a strategy $y > x$ at time $t = 0$, then it is not profitable at any time $t \leq x$.

Proof: It is not profitable to deviate to a strategy $y > x$ at time t if and only if

$$0 \geq u_t(y, \theta, F|x) - u_t(x, \theta, F|x)$$

Now,

$$\begin{aligned}
&u_t(y, \theta, F|x) - u_t(x, \theta, F|x) \\
=&\frac{1}{1-F(t)}\left(\int_x^y (\theta - s)dF(s) + \mu(F(y) - F(x))\theta\right) + \Lambda\theta\left(\frac{1-F(y)}{1-F(t)}\ln\left(\frac{1-F(y)}{1-F(t)}\right) - \frac{1-F(x)}{1-F(t)}\ln\left(\frac{1-F(x)}{1-F(t)}\right)\right) \\
=&\frac{1}{1-F(t)}\Bigg(\int_x^y (\theta - s)dF(s) + \mu(F(y) - F(x))\theta \; ... \\
&\quad ... + \Lambda\theta((1-F(y))\ln(1-F(y)) - (1-F(x))\ln(1-F(x)) + (F(y)-F(x))\ln(1-F(t)))\Bigg).
\end{aligned}$$

Note that the expression in the big brackets is decreasing in t. Thus, if it is negative for $t = 0$, then it is as well negative for all $t > 0$. Hence, if

$$0 \geq u_0(y, \theta, F|x) - u_0(x, \theta, F|x)$$

then

$$0 \geq u_t(y, \theta, F|x) - u_t(x, \theta, F|x)$$

for all $t > 0$, which concludes the proof of claim 2.

Now we are ready to show sufficiency: assume 1. and 2. hold. Then by claim 1 it can't be profitable to deviate to a lower strategy at any time. To show that there is no profitable deviation to a higher strategy, take any time-consistent strategy $y \geq x$. By claim 1 this necessarily means $y \in [x, (1+\eta)\theta]$. From 2. it follows that $u_0(x, \theta, F|x) \geq u_0(y, \theta, F|x)$. Then, by claim 2, the agent does not want to deviate to a higher strategy at any time, and x is indeed a PE. $\square$

Proof of Theorem 2. Take some increasing equilibrium function. By Lemma 4, it satisfies $b(\theta) \leq (1+\eta)\theta$ for all $\theta \in (\theta^{\min}, \theta^{\max})$. If $b(\theta) < (1+\eta)\theta$ for some θ, then—again by Lemma 4—any $y \in [x, (1+\eta)\theta]$ satisfies $u_0(x, \theta, F|x) \geq u_0(y, \theta, F|x)$. This means that

$$\lim_{y \searrow x} \frac{\partial u_0(y, \theta, F|x)}{\partial y} \leq 0,$$

which—as we have seen in the proof of Proposition 3—straightforwardly solves to

$$b(\theta) \geq \big(1+\eta-\Lambda(1+\ln(1-G(\theta)))\big)\theta$$

in equilibrium. This shows that any increasing equilibrium satisfies 1. and 2. for all $\theta \in (\theta^{\min}, \theta^{\max})$. By continuity it also holds for all $\theta \in [\theta^{\min}, \theta^{\max}]$.

Conversely, assume that $b(\theta)$ satisfies 1. and 2. By Lemma 4 it only remains to show that for any

$$y \in [b(\theta), (1+\eta)\theta]$$

we have

$$u_0(b(\theta), \theta, F|b(\theta)) \geq u_0(y, \theta, F|b(\theta)).$$

This condition is trivially satisfied for any θ with $b(\theta) = (1+\eta)\theta$. Consider therefore θ with $b(\theta) < (1+\eta)\theta$. It suffices to show that

$$\frac{\partial u_0(y, \theta, F|b(\theta))}{\partial y} \leq 0$$

for all $y \in [b(\theta), (1+\eta)\theta]$. Let $\tilde{y}$ be any of such y. Since

$$b(\theta^{\max}) = (1+\eta)\theta^{\max} > (1+\eta)\theta \geq \tilde{y} \geq b(\theta),$$

and b is continuous, there exists some $\tilde{\theta} \geq \theta$ with $b(\tilde{\theta}) = \tilde{y}$. Now,

$$\begin{aligned}
\frac{\partial u_0(y, \theta, F|b(\theta))}{\partial y}\Big|_{y=\tilde{y}} &= [(1+\eta)\theta - \tilde{y} - \Lambda\theta(1+\ln(1-F(\tilde{y})))]f(\tilde{y}) \\
&= [\underbrace{(1+\eta-\Lambda(1+\ln(1-F(b(\tilde{\theta})))))}_{>0}\theta - b(\tilde{\theta})]f(b(\tilde{\theta})) \\
&\leq [\underbrace{(1+\eta-\Lambda(1+\ln(1-F(b(\tilde{\theta})))))\tilde{\theta}}_{\leq b(\tilde{\theta})} - b(\tilde{\theta})]f(b(\tilde{\theta})) \\
&\leq 0.
\end{aligned}$$

$\square$

Proof of Corollary 3. We have

$$\big(1+\eta-\Lambda(1+\ln(1-G(\theta)))\big)\theta \leq (1+\eta)\theta$$

if and only if $-(1+\ln(1-G(\theta))) \leq 0$, which is equivalent to $G(\theta) \leq 1-1/e$. Therefore, by Theorem 2, a fuction $b(\theta)$ is a symmetric equilibrium if and only if

- $b(\theta) \in \big[(1+\eta-\Lambda(1+\ln(1-G(\theta))))\theta, (1+\eta)\theta\big]$ for $G(\theta) \leq 1-1/e$, and
- $b(\theta) = (1+\eta)\theta$ for $G(\theta) > 1-1/e$.

We determine the utility maximizing equilibrium on the interval where $G(\theta) \leq 1-1/e$. Bidder's expected utility of a bid x is

$$\begin{aligned} u_0(x,\theta,F|x) &= \int_0^x (\theta-s)dF(s) + L_t(x,\theta,F) \\ &= \int_0^x (\theta-s)dF(s) + \Lambda\theta\ln(1-F(x))(1-F(x)). \end{aligned}$$

Thus, for any $x \geq \big(1+\eta-\Lambda(1+\ln(1-G(\theta)))\big)\theta$

$$\begin{aligned} \frac{\partial u_0(x,\theta,F|x)}{\partial x} &= (\theta-x)f(x) - \Lambda\theta(1+\ln(1-F(x)))f(x) \\ &\leq (\theta-(1+\eta-\Lambda(1+\ln(1-G(\theta))))\theta)f(x) - \Lambda\theta f(x) \\ &\leq (\theta-(1+\eta-\Lambda)\theta)f(x) - \Lambda\theta f(x) \\ &= -f(x)\eta\theta \\ &< 0. \end{aligned}$$

This shows that the lowest x among all equilibrium strategies yields the highest utility.

Finally, since for the PPE

$$b(\theta^{\min}) = \big(1+\eta-\Lambda(1+\ln(1-G(\theta^{\min})))\big)\theta^{\min} = (1+\eta-\Lambda)\theta^{\min},$$

there is underbidding for low types in the PPE if and only if

$$0 > \eta-\Lambda = 2\eta-\lambda\eta,$$

hence if and only if $\lambda > 2$. $\square$

Proof of Theorem 3. For (1) we show that without interim update the equilibrium concepts of the static UPE and the dynamic PE coincide. Given type θ and a continuous belief H on the maximal opponent bid, bidding (up to) b induces the same payoff belief (and therefore reference point) $F^k(b, \theta, H)$ for $k \in \{money, good\} = \{m, g\}$ in the Vickrey and the English auction. Consider a bidder in the English auction who plans to bid up to b but deviates during the auction process, such that the final payoff in dimension $k \in \{m, g\}$ is distributed according to F. If there is no interim updating during the auction, the bidder updates her reference point only once when the auction is terminated. Integrating the utility in dimension k for each possible auction outcome yields expected utility of

$$\begin{aligned}
U_{\text{English}}(F|F^k(b,\theta,H)) &= \int (x + N(\mathbb{1}_{[x,\infty)}|F^k(b,\theta,H)))dF(x) \\
&= \int \left(x + \int_0^1 \mu(x - c_F(p))dp\right) dF(x) \\
&= \int \left(x + \int_{-\infty}^{\infty} \mu(x - c_F(F(s)))dF(s)\right) dF(x) \\
&= \int \left(x + \int_{-\infty}^{\infty} \mu(x - s)dF(s)\right) dF(x) \\
&= U_{\text{Vickrey}}(F|F^k(b,\theta,H)).
\end{aligned}$$

Thus, equally for the UPE concept in the Vickrey auction and the PE concept in the English auction, an action b is an equilibrium if and only if for all distributions (F^m, F^g) that are induced by a deviation strategy we have

$$\sum_{k \in \{m,g\}} U(F^k(b,\theta,H)|F^k(b,\theta,H)) \geq \sum_{k \in \{m,g\}} U(F^k|F^k(b,\theta,H)).$$

The subtle difference lies in the fact that a bidder in the Vickrey auction is constrained to deviations $\hat{b} \in \mathbb{R}^+$, while a bidder in the English auction with multiple opponents can use complex history dependent deviation strategies, leading to are larger set of potential price distributions than in the Vickrey auction. Clearly, if action b is optimal with respect to all possible deviations in the English auction, it is in particular optimal with respect to deviations to all history-independent strategy $\hat{b} \in \mathbb{R}^+$. Thus, if bidding up to b is a PE in the English auction, then bidding

b is a UPE in the Vickrey auction. For the converse, assume that b is a UPE in the Vickrey auction and let (F^m, F^g) be the payoff distribution of some deviation strategy in the English auction. Since H is continuous, there is some $\hat{b}$ such that $F^g(\hat{b}, \theta, H) = F^g$. Further, since strategy $\hat{b}$ wins the auction if and only if the maximal opponent strategy is below $\hat{b}$, it is the most cost effective strategy that wins with probability $H(\hat{b})$. Thus the distribution F^m induces weakly higher costs than $F^m(\hat{b}, \theta, H)$ in the sense of first-order stochastic dominance. It follows that

$$U(F^m|F^k(b,\theta,H)) \leq U(F^m(\hat{b},\theta,H)|F^k(b,\theta,H)) \leq U(F^m(b,\theta,H)|F^k(b,\theta,H)),$$

and since consequently

$$\sum_{k\in\{m,g\}} U(F^k(b,\theta,H)|F^k(b,\theta,H)) \geq \sum_{k\in\{m,g\}} U(F^k|F^k(b,\theta,H)),$$

the strategy b is a PE in the English auction.

For (2) note that by Theorem 4 the equilibrium bidding function for the Vickrey auction is given by

$$b_{\text{Vickrey}}(\theta) = (1 + \eta + \Lambda G^{n-1}(\theta))\theta,$$

whereas any equilibrium bidding function in the English auction with instantaneous reference point updating by Lemma 4 satisfies

$$b_{\text{English}}(\theta) \leq (1 + \eta)\theta.$$

Since, by assumption, $G^{n-1}(\theta)$ is strictly increasing, we have $G^{n-1}(\theta) > 0$ for all $\theta > \theta^{\min}$, and the claim follows. $\square$

Proof of Theorem 4. The structure of the proof is similar to Lange and Ratan

(2010). The utility function can be simplified to

$$
\begin{aligned}
u(b,\theta|b^*) = &\int_0^b (\theta - s)dH(s) \\
&+ \int_0^b \int_0^{b^*} \mu_m(t-s)dH(t)dH(s) + (1-H(b^*))\int_0^b \mu_m(-s)dH(s) \\
&+ (1-H(b))\int_0^{b^*} \mu_m(t)dH(t) \\
&+ H(b)(1-H(b^*))\mu_g(\theta) + H(b^*)(1-H(b))\mu_g(-\theta).
\end{aligned}
$$

Suppose that all opponents bid according to some increasing, continuously differentiable bidding function $b(\theta)$. Since $G(\theta)$ is a distribution with strictly positive, continuous density g, distribution of the maximal opponent bid $H(x) = G^{n-1}(b^{-1}(x))$ is a differentiable distribution with positive, continuous density $h(x)$ on $[b(\theta^{\min}), b(\theta^{\max})]$ as well. The bidding function $b(\theta)$ constitutes a UPE if and only if the utility function $u(x,\theta|b(\theta))$ attains a maximum at $x = b(\theta)$ for all θ. Differentiation of the utility function with respect to x yields

$$
\begin{aligned}
\frac{\partial u(x,\theta|b(\theta))}{\partial x} =&(\theta - x)h(x) + \int_0^{b(\theta)} \mu_m(t-x)h(x)dH(t) + (1-H(b(\theta)))\mu_m(-x)h(x) \\
&- h(x)\int_0^{b(\theta)} \mu_m(t)dH(t) + h(x)(1-H(b(\theta)))\eta_g\theta + h(x)H(b(\theta))\lambda_g\eta_g\theta.
\end{aligned}
$$

By dividing by $h(x)$ and evaluating at $x = b(\theta)$, we obtain the first-order condition

$$
\begin{aligned}
0 \overset{!}{=}&(\theta - b(\theta)) + \int_0^{b(\theta)} \mu_m(t-b(\theta))dH(t) + (1-H(b(\theta))\mu_m(-b(\theta)) \\
&- \int_0^{b(\theta)} \mu_m(t)dH(t) + (1-H(b(\theta)))\eta_g\theta + H(b(\theta))\lambda_g\eta_g\theta \\
=&(\theta - b(\theta)) - \lambda_m\eta_m\int_0^{b(\theta)} (b(\theta)-t)dH(t) + (1-H(b(\theta)))(-\lambda_g\eta_g b(\theta)) \\
&- \eta_m\int_0^{b(\theta)} tdH(t) + (1-H(b(\theta)))\eta_g\theta + H(b(\theta))\lambda_g\eta_g\theta,
\end{aligned}
$$

which simplifies to

$$
0 = (1+\eta_g)\theta - (1+\lambda_m\eta_m)b(\theta) + \Lambda_m\int_0^{b(\theta)} tdH(t) + \Lambda_g H(b(\theta))\theta. \qquad (2)
$$

Using that $H(b(\theta)) = G^{n-1}(\theta)$ we can rewrite this equation to

$$0 = (1+\eta_g)\theta - (1+\lambda_m\eta_m)b(\theta) + \Lambda_m \int_0^\theta b(s)dG^{n-1}(s) + \Lambda_g G^{n-1}(\theta)\theta.$$

Differentiation with respect to θ yields

$$0 = (1+\eta_g) - (1+\lambda_m\eta_m)b'(\theta) + \Lambda_m b(\theta)(G^{n-1})'(\theta) + \Lambda_g (G^{n-1}(\theta)\theta)'.$$

The rearranged equation

$$b'(\theta) = \frac{\Lambda_m (G^{n-1})'(\theta)}{1+\lambda_m\eta_m} b(\theta) + \frac{1+\eta_g+\Lambda_g(\theta G^{n-1}(\theta))'}{1+\lambda_m\eta_m}$$

is a first-order linear differential equation, which solves to

$$b(\theta) = \exp\left(\frac{\Lambda_m}{1+\lambda_m\eta_m} G^{n-1}(\theta)\right)\left(\int_0^\theta \frac{1+\eta_g+\Lambda_g(xG^{n-1}(x))'}{1+\lambda_m\eta_m} \exp\left(-\frac{\Lambda_m}{1+\lambda_m\eta_m} G^{n-1}(x)\right)dx + C\right),$$

where C is the constant of integration. Since $G(x) = 0$ for $x \leq \theta^{min}$, we have

$$b(\theta^{\min}) = \exp(0)\left(\int_0^{\theta^{\min}} \frac{1+\eta_g}{1+\lambda_m\eta_m}\exp(0)dx + C\right) = \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta^{\min} + C.$$

To determine C, we insert $\theta^{\min}$ into equation (2) and obtain that

$$0 = (\theta^{\min} - b(\theta^{\min})) + (-\lambda_m\eta_m b(\theta^{\min})) + \eta_g\theta^{\min},$$

or equivalently

$$b(\theta^{\min}) = \frac{1+\eta_g}{1+\lambda_m\eta_m}\theta^{\min},$$

which shows that $C = 0$. Now we can use partial integration in order to rewrite the solution into

$$b(\theta) = \frac{1+\eta_g+\Lambda_g G^{n-1}(\theta)}{1+\lambda_m\eta_m}\theta + \int_0^\theta \frac{\Lambda_m(1+\eta_g+\Lambda_g G^{n-1}(x))}{(1+\lambda_m\eta_m)^2} x \exp\left(\frac{\Lambda_m}{1+\lambda_m\eta_m}(G^{n-1}(\theta) - G^{n-1}(x))\right) dG(x).$$

Since $G(x) = 0$ for all $x \leq \theta^{\min}$, we finally have

$$b(\theta) = \frac{1+\eta_g+\Lambda_g G^{n-1}(\theta)}{1+\lambda_m\eta_m}\theta + \int_{\theta^{\min}}^\theta \frac{\Lambda_m(1+\eta_g+\Lambda_g G^{n-1}(x))}{(1+\lambda_m\eta_m)^2} x \exp\left(\frac{\Lambda_m}{1+\lambda_m\eta_m}(G^{n-1}(\theta) - G^{n-1}(x))\right) dG(x).$$

For sufficiency note first that $b'(\theta)$ is differentiable, since $g(\theta)$ is—by assumption—differentiable. It follows that

$$h(b(\theta)) = \frac{(G^{n-1})'(\theta)}{b'(\theta)}$$

is differentiable as well. Now it is immediate that

$$\begin{aligned}
&\left.\frac{\partial^2 u(x,\theta|b(\theta))}{(\partial x)^2}\right|_{x=b(\theta)} \\
=&\frac{\partial}{\partial x}\left(h(x)\frac{\partial u(x,\theta|b(\theta)/\partial x)}{h(x)}\right)\Bigg|_{x=b(\theta)} \\
=&h'(b(\theta))\underbrace{\left(\frac{\partial u(x,\theta|b(\theta)/\partial x)}{h(x)}\right)\Bigg|_{x=b(\theta)}}_{=0} \\
&+h(b(\theta))\underbrace{\left[-1+\int_0^{b(\theta)}-\lambda_m\eta_m dH(t)-\lambda_m\eta_m(1-H(b(\theta)))\right]}_{<0} \\
<& \; 0.
\end{aligned}$$

□

Proof of Lemma 5. Assume the clock increases in increments of ε and the bidder plans to bid up to $x \in (a,b)$. Assume the clock price is $x-\varepsilon$, and the opponent has not dropped out yet. We analyze bidders incentives to bid at x given her plan to do so.

Let $\Delta = \Delta(\varepsilon) = \frac{F(x)-F(x-\varepsilon)}{1-F(x-\varepsilon)}$ be the probability that the opponent drops out at x, given he is still in at $x-\varepsilon$. This means the bidder beliefs to win the auction and get a payoff of $(\theta, -(x-\varepsilon))$ with probability Δ. If the bidder bids at x she receives a utility of

$$u(x,\theta,F|x) = \underbrace{\Delta(\theta-(x-\varepsilon))}_{\text{classical utility}} + \Delta\underbrace{(1-\Delta)(\eta_g\theta-\lambda_m\eta_m(x-\varepsilon))}_{\text{gain-loss of winning the auction}} + (1-\Delta)\underbrace{\Delta(-\lambda_g\eta_g\theta+\eta_m(x-\varepsilon))}_{\text{gain-loss of losing the auction}}.$$

If she drops out before bidding x, she receives

$$u(x-\varepsilon,\theta,F|x) = \underbrace{\Delta(-\lambda_g\eta_g\theta+\eta_m(x-\varepsilon))}_{\text{gain-loss of losing the auction}}.$$

If bidding up to x is time consistent, then

$$u(x, \theta, F|x) \geq u(x - \varepsilon, \theta, F|x).$$

This is equivalent to

$$\Delta[\theta - (x - \varepsilon) + (1 - \Delta)(\eta_g \theta - \lambda_m \eta_m (x - \varepsilon)) - \Delta(-\lambda_g \eta_g \theta + \eta_m (x - \varepsilon))] \geq 0.$$

Since F has a positive density, we have $\Delta > 0$, and it follows

$$(1 + \eta_g)\theta - (1 + \lambda_m \eta_m)(x - \varepsilon) + \Delta(\Lambda_g \theta + \Lambda_m (x - \varepsilon)) \geq 0.$$

Since F has no atoms, $\lim_{\varepsilon \to 0} \Delta(\varepsilon) = 0$. Thus, in the limit as the increment size goes to zero, we obtain

$$(1 + \eta_g)\theta - (1 + \lambda_m \eta_m)x \geq 0,$$

or equivalently

$$x \leq \frac{1 + \eta_g}{1 + \lambda_m \eta_m} \theta.$$

□

Proof of Proposition 4. I sketch the main steps of the proof. If a bidder wins the auction, he has to pay $\max\{b, \overline{x}\}$ with b being the maximal opponent bid. Given opponents' strategies, let $H_{RP}(b)$ be the distribution of the maximal opponent bid with reserve price $\overline{x}$. By replacing s with $\max\{s, \overline{x}\}$ and t with $\max\{t, \overline{x}\}$ in the utility function in section 6.1, the utility of a bidder of type θ who bids b with a reference point as if bidding b^* is

$$
\begin{aligned}
u(b,\theta|b^*) = &\int_0^b \left(-\max\{s,\overline{x}\} + \int_0^{b^*} \mu_m(\max\{t,\overline{x}\} - \max\{s,\overline{x}\})dH_{RP}(t)\right) dH_{RP}(s) \\
&+ \int_0^b \int_{b^*}^{\infty} \mu_m(-\max\{s,\overline{x}\})dH_{RP}(t)dH_{RP}(s) \\
&+ \int_b^{\infty} \left(\int_0^{b^*} \mu_m(\max\{t,\overline{x}\})dH_{RP}(t) + \int_{b^*}^{\infty} \mu_m(0)dH_{RP}(t)\right) dH_{RP}(s) \\
&+ \int_0^b \left(\theta + \int_0^{b^*} \mu_g(0)dH_{RP}(t) + \int_{b^*}^{\infty} \mu_g(\theta)dH_{RP}(t)\right) dH_{RP}(s) \\
&+ \int_b^{\infty} \left(\int_0^{b^*} \mu_g(-\theta)dH_{RP}(t) + \int_{b^*}^{\infty} \mu_g(0)dH_{RP}(t)\right) dH_{RP}(s).
\end{aligned}
$$

Following the derivation of the necessary condition for a symmetric increasing equilibrium[13] in the proof of Theorem 4 with this modified utility function, we obtain for all θ with $b_{RP}(\theta) \geq \overline{x}$ the following modification of equation (2):

$$
0 = (1+\eta_g)\theta - (1+\lambda_m\eta_m)b_{RP}(\theta) + \Lambda_g H_{RP}(b_{RP}(\theta))\theta + \Lambda_m \int_0^{b_{RP}(\theta)} y dH_{RP}(y) + H_{RP}(\overline{x})\overline{x}.
$$

Rearranging yields

$$
b_{RP}(\theta) = \frac{1}{1+\lambda_m\eta_m}\left((1+\eta_g)\theta + \Lambda_g H_{RP}(b_{RP}(\theta)) + \Lambda_m \int_{\overline{x}}^{b_{RP}(\theta)} y dH_{RP}(y) + H_{RP}(\overline{x})\overline{x}\right).
$$

Let $\overline{\theta}$ be defined by $b(\overline{\theta}) = \overline{x}$. We need to show that $b(\theta) < b_{RP}(\theta)$ for any $\theta \geq \overline{\theta}$. Assume otherwise, and let $\tilde{\theta} = \min\{\theta \in [\overline{\theta}, \theta^{\max}] | b(\theta) \geq b_{RP}(\theta)\}$. The minimum

[13]The proof of existence of a symmetric increasing continuous equilibrium bidding function $b_{RP}(\theta)$ with reserve price, and its uniqueness for θ with $b(\theta) \geq \overline{x}$ is omitted. It is a modification of Proof 4.

exists by continuity of b and b_{RP}. Now we have

$$\begin{aligned}
b(\tilde{\theta}) &= \frac{1}{1+\lambda_m\eta_m}\left((1+\eta_g)\tilde{\theta} + \Lambda_g H(b(\tilde{\theta})) + \Lambda_m \int_0^{b(\tilde{\theta})} y dH(y)\right) \\
&< \frac{1}{1+\lambda_m\eta_m}\left((1+\eta_g)\tilde{\theta} + \Lambda_g H(b(\tilde{\theta})) + \Lambda_m \int_{b(\overline{\theta})}^{b(\tilde{\theta})} y dH(y) + H(\overline{x})\overline{x}\right) \\
&= \frac{1}{1+\lambda_m\eta_m}\left((1+\eta_g)\tilde{\theta} + \Lambda_g G^{n-1}(\tilde{\theta}) + \Lambda_m \int_{\overline{\theta}}^{\tilde{\theta}} b(s) dG^{n-1}(s) + G^{n-1}(\overline{\theta})\overline{x}\right) \\
&\leq \frac{1}{1+\lambda_m\eta_m}\left((1+\eta_g)\tilde{\theta} + \Lambda_g G^{n-1}(\tilde{\theta}) + \Lambda_m \int_{\overline{\theta}}^{\tilde{\theta}} b_{RP}(s) dG^{n-1}(s) + G^{n-1}(\overline{\theta})\overline{x}\right) \\
&= \frac{1}{1+\lambda_m\eta_m}\left((1+\eta_g)\tilde{\theta} + \Lambda_g H_{RP}(b_{RP}(\tilde{\theta})) + \Lambda_m \int_{b_{RP}(\overline{\theta})}^{b_{RP}(\tilde{\theta})} y dH_{RP}(y) + H_{RP}(b_{RP}(\overline{\theta}))\overline{x}\right) \\
&\leq \frac{1}{1+\lambda_m\eta_m}\left((1+\eta_g)\tilde{\theta} + \Lambda_g H_{RP}(b_{RP}(\tilde{\theta})) + \Lambda_m \int_{\overline{x}}^{b_{RP}(\tilde{\theta})} y dH_{RP}(y) + H_{RP}(\overline{x})\overline{x}\right) \\
&= b_{RP}(\tilde{\theta}),
\end{aligned}$$

a contradiction. □

Proof of Proposition 5. For any given opponent strategy distribution F, the implementation of a reserve price $\overline{x}$ is perceived by the bidder as if playing against a distribution

$$F_{RP}(z) = \begin{cases} 0 & z < \overline{x} \\ F(z) & z \geq \overline{x}. \end{cases}$$

In particular $F_{RP}(z) = F(z)$ for all $z \geq \overline{x}$. Following Lemma 4, a strategy $x > \overline{x}$ is a PE if and only if

1. $x \leq (1+\eta)\theta$

2. For any $y \in [x, (1+\eta)\theta]$ we have $u_0(x, \theta, F|x) \geq u_0(y, \theta, F|x)$.

Since for any $y, x > \overline{x}$ we have

$$u_0(y, \theta, F_{RP}|x) = u_0(y, \theta, F|x),$$

these conditions remain unchanged under a reserve price of $\overline{x}$. Therefore, the set of symmetric equilibria for two loss-averse bidders remains unchanged as well. □

Proof of Lemma 6. From the perspective of a representative bidder, we denote with

$F(x)$ the distribution of prices, at which a particular opponent drops out, i.e. $F(b(\theta)) = G(\theta)$. Similarly we denote with $F_t(x)$ the distribution of drop-out prices of the remaining opponent, given the other opponent drops out at t. Since the remaining opponent j didn't drop out until t, his type θ_j necessarily satisfies $\theta_j > \theta(t)$, and therefore

$$F_t(b(t,\theta)) = Prob(\theta_j \leq \theta | \theta_j > \theta(t)) = \frac{G(\theta) - G(\theta(t))}{1 - G(\theta(t))}.$$

If we denote with $L_{2,t}$ expected gain-loss utility in the two-bidder subgame following an opponent's drop out at price t, then by Proposition 2

$$\begin{aligned} L_{2,t}(\theta) &= \ln\left(\frac{1 - F_t(b(t,\theta))}{1 - F_t(t)}\right)\frac{1 - F_t(b(t,\theta))}{1 - F_t(t)}\Lambda\theta \\ &= \ln(1 - F_t(b(t,\theta))(1 - F_t(b(t,\theta))\Lambda\theta \\ &= \ln\left(\frac{1 - G(\theta)}{1 - G(\theta(t))}\right)\frac{1 - G(\theta)}{1 - G(\theta(t))}. \end{aligned}$$

For the 3-bidder auction leading to the first drop out, consider first price increments of ε. Suppose the clock is at price s and both opponents are still remaining. Since we restrict to symmetric increasing bidding functions, a bidder of type θ wins the auction if and only if both opponents have a type lower that θ. Given that they didn't drop out until s, this holds true with probability $\left(\frac{G(\theta)-G(\theta(s))}{1-G(\theta(s))}\right)^2$.

The probability that a particular opponent j drops out at the next increment is

$$\Delta(s) = \frac{F(s+\varepsilon) - F(s)}{1 - F(s)}.$$

At the next increment $s + \varepsilon$ there are three possibilities:

- With probability $(\Delta(s))^2$ both opponents drop out. The bidder wins with certainty, which induces a gain of $\left(1 - \left(\frac{G(\theta)-G(\theta(s))}{1-G(\theta(s))}\right)^2\right)\eta\theta$.
- With probability $2\Delta(s)(1-\Delta(s))$ exactly one opponent drops out. The bidder updates her belief to win, which induces a gain of $\left(\frac{G(\theta)-G(\theta(s+\varepsilon))}{1-G(\theta(s+\varepsilon))} - \left(\frac{G(\theta)-G(\theta(s))}{1-G(\theta(s))}\right)^2\right)\eta\theta$.
- With probability $(1 - \Delta(s))^2$ no opponent drops out, which induces a loss of $\left(\left(\frac{G(\theta)-G(\theta(s+\varepsilon))}{1-G(\theta(s+\varepsilon))}\right)^2 - \left(\frac{G(\theta)-G(\theta(s))}{1-G(\theta(s))}\right)^2\right)\lambda\eta\theta$.

Since F is continuous, $\Delta(s)$ approaches zero, as the increment size goes to zero. Therefore, in the limit for the continuous English auction, the probability that both opponents drop out at the same time is of second order and has no impact on expected gain-loss utility. Applying Proposition 1, expected gain-loss utility in the increment from s to $s+\varepsilon$ for small ε with both opponents being active approaches

$$L_{s+\varepsilon}(\theta)-L_s(\theta) = -2\Delta(s)(1-\Delta(s))\left(\frac{G(\theta)-G(\theta(s+\varepsilon))}{1-G(\theta(s+\varepsilon))} - \left(\frac{G(\theta)-G(\theta(s))}{1-G(\theta(s))}\right)^2\right)\Lambda\theta.$$

As the increment size goes to zero, in the limit the marginal expected gain-loss utility with both opponents being active at time s is given by

$$\ell(s)(\theta) = \frac{-2f(s)}{1-F(s)}\left(\frac{G(\theta)-G(\theta(s))}{1-G(\theta(s))} - \left(\frac{G(\theta)-G(\theta(s))}{1-G(\theta(s))}\right)^2\right)\Lambda\theta.$$

At time t, the probability that time $s>t$ is reached without at least one opponent drop out is $\left(\frac{1-F(s)}{1-F(t)}\right)^2$. Consequently the marginal probability of a drop out at s—which triggers the 2-bidder auction with expected loss $L_{2,s}$—is

$$\frac{\partial}{\partial s}\left(\frac{(1-F(s))^2}{(1-F(t))^2}\right) = \frac{2f(s)(1-F(s))}{(1-F(t))^2}.$$

Putting the two sources of gain-loss utility together and integrating over s yields

$$\begin{aligned}
L_t(\theta) =& \int_t^{b(\theta)} \left(\left(\frac{1-F(s)}{1-F(t)}\right)^2 \ell(s) + \frac{2f(s)(1-F(s))}{(1-F(t))^2} L_{2,s}(\theta)\right) ds \\
=& -\Lambda\theta \int_t^{b(\theta)} \frac{2f(s)(1-F(s))}{(1-F(t))^2}\left(\frac{G(\theta)-G(\theta(s))}{1-G((\theta(s))} - (\frac{G(\theta)-G(\theta(s))}{1-G(\theta(s))})^2\right) ds \\
&+ \Lambda\theta \int_t^{b(\theta)} \frac{2f(s)(1-F(s))}{(1-F(t))^2} \ln\left(\frac{1-G(\theta)}{1-G(\theta(s))}\right) \frac{1-G(\theta)}{1-G(\theta(s))} ds
\end{aligned}$$

Since $F(s) = G(\theta(s))$ and consequently $f(s) = g(\theta(s))/b'(\theta(s))$, integration by substitution yields

$$L_t(\theta) = -\Lambda\theta \int_{\theta(t)}^{\theta} \frac{2g(s)(1-G(s))}{(1-G(\theta(t)))^2}\left[\frac{G(\theta)-G(s)}{1-G(s)} - \left(\frac{G(\theta)-G(s)}{1-G(s)}\right)^2 - \ln\left(\frac{1-G(\theta)}{1-G(s)}\right)\frac{1-G(\theta)}{1-G(s)}\right] ds$$

□

Proof of Corollary 4. Define

$$\delta(s) = \frac{G(\theta) - G(s)}{1 - G(s)}$$

Since for $\theta < \theta^{\max}$ we have $\delta(s) < 1$, and we can use the power series of the logarithm to rewrite

$$L_t(\theta) = -\Lambda\theta \int_{\theta(t)}^{\theta} \frac{2g(s)(1-G(s))}{(1-G(\theta(t)))^2} \big[\delta(s) - (\delta(s))^2 - (-\delta(s) - \frac{\delta(s)^2}{2} - \frac{\delta(s)^3}{3} ...)(1-\delta(s))\big] ds$$

Since $\lim_{s\to\theta} \delta(s) = 0$, we have

$$\begin{aligned}
&\lim_{t\to b(\theta)} \frac{L_t(\theta)}{\left(\frac{G(\theta)-G(\theta(t))}{1-G(\theta(t))}\right)^2} \\
&= \lim_{t\to b(\theta)} -\Lambda\theta \int_{\theta(t)}^{\theta} \frac{2g(s)(1-G(s))}{(G(\theta)-G(\theta(t))^2} \big[\delta(s) - (\delta(s))^2 - (-\delta(s) - \frac{\delta(s)^2}{2} ...)(1-\delta(s))\big] ds \\
&= \lim_{(\theta(t)\to\theta} -\Lambda\theta \int_{\theta(t)}^{\theta} \frac{2g(s)(1-G(s))}{(G(\theta)-G(\theta(t))^2} \big[\delta(s) - (\delta(s))^2 - (-\delta(s) - \frac{\delta(s)^2}{2} ...)(1-\delta(s))\big] ds \\
&= \lim_{\theta(t)\to\theta} -\Lambda\theta \int_{\theta(t)}^{\theta} \frac{2g(s)(1-G(s))}{(G(\theta)-G(\theta(t))^2} 2\delta(s) ds \\
&= \lim_{\theta(t)\to\theta} -2\Lambda\theta \int_{\theta(t)}^{\theta} \frac{2g(s)(G(\theta)-G(s))}{(G(\theta)-G(\theta(t))^2} ds \\
&= \lim_{\theta(t)\to\theta} -2\Lambda\theta \left[\frac{-(G(\theta)-G(s))^2}{(G(\theta)-G(\theta(t))^2}\right]_{\theta(t)}^{\theta} \\
&= \lim_{\theta(t)\to\theta} -2\Lambda\theta \\
&= -2\Lambda\theta
\end{aligned}$$

Now, since $b(t,\theta)$ is continuous in t, $\lim_{t\to b(\theta)} b(t,\theta)$ exists. We prove the threshold of time-consistent behavior for $(\theta^{\min}, \theta^{\max})$ by contradiction. For the boundaries it follows by continuity. Assume that there is some $\overline{\theta} \in (\theta^{\min}, \theta^{\max})$ with

$$\lim_{t\to b(\overline{\theta})} b(t, \overline{\theta}) > (1 + \eta - \Lambda)\overline{\theta}.$$

Since $b(t,\theta)$ is continuous there is some $\hat{t} < b(\overline{\theta})$ and $\hat{\theta} \in [\theta(\hat{t}), \overline{\theta}]$, such that

$$b(t,\theta) > (1 + \eta - \Lambda)\overline{\theta}$$

for all $t \in [\hat{t}, b(\overline{\theta})]$, $\theta \in [\hat{\theta}, \overline{\theta}]$. This implies that the sales price for the good exceeds $(1+\eta-\Lambda)\overline{\theta}$ if no bidder drops out until $\hat{t}$. If $b(t,\theta)$ is a time-consistent strategy, then at time $\hat{t}$ a bidder of type $\overline{\theta}$ must weakly prefer this strategy to an instantaneous drop out. Since at time $\hat{t}$ her belief to win is $\left(\frac{G(\overline{\theta})-G(\theta(\hat{t}))}{1-G(\theta(\hat{t}))}\right)^2$, this condition reads

$$-\lambda\eta\overline{\theta}\left(\frac{G(\overline{\theta})-G(\theta(\hat{t}))}{1-G(\theta(\hat{t}))}\right)^2 < \left(\frac{G(\overline{\theta})-G(\theta(\hat{t}))}{1-G(\theta(\hat{t}))}\right)^2 (\overline{\theta}-(1+\eta-\Lambda)\overline{\theta}) + L_{\hat{t}}(\theta),$$

with strict inequality since the price strictly exceeds $(1+\eta-\Lambda)\overline{\theta}$. This is equivalent to

$$L_{\hat{t}}(\overline{\theta}) > -2\Lambda\overline{\theta}\left(\frac{G(\overline{\theta})-G(\theta(\hat{t}))}{1-G(\theta(\hat{t}))}\right)^2,$$

a contradiction for $\hat{t}$ sufficiently close to $b(\overline{\theta})$. □

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