Common Information-processing Irrationality as Trade Creator

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Common Information-processing Irrationality as Trade Creator *

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We show that a common (identical across investors) irrationality in information processing can be enough to create nontrivial trade, using one of standard partial-equilibrium environments. We can attribute this trade to their common irrationality because we strip the investors and their circumstances of all heterogeneities but purely age (in a sense experience), make investment horizon age-independent, and keep all information complete. The common irrationality in our model takes the form of a somewhat non-Bayesian information processing. The resulting trade between such essentially identical individuals with the very same irrationality in their information processing can also feature different kinds of mispricing.

1 Introduction

We contribute an insight to the theory of why people trade goods and assets—a fundamental economic phenomenon. Existing explanations in general attribute trade to people being different in one way or another. This paper is about a new kind of trade under a common information-processing irrationality, where the only intrinsic difference between investors is purely age without even any consequences for investment horizon, kept infinite on purpose. We deliberately make all information complete and strip the investors of endowment, specialization, feasibility (including investment horizon), and information-processing heterogeneities. We also make the investors be uncertain only about the possibly infinite terminal time, when an asset pays a dividend, so that the realized history looks always the same and all investors find themselves in exactly the same position at birth. Finally, we let every investor repeat the same somewhat non-Bayesian

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information processing every time the asset does not pay the dividend—a behavioral twist enough, as it turns out, to create trade.

If investors were Bayesian, their prior beliefs about what the dividend payment time is would cover the entire past, present, and future relative to their birth times. Upon learning that the asset did not pay the dividend before they were born, they would update their beliefs. This is where we make them non-Bayesian—with respect to what happens before they are born. Contrary to the Bayesian paradigm, we make them form prior beliefs only about what happens after they are born. They look at what happened before they were born, see the same thing past generations saw when they looked at what happened before they were born, and form analogous prior beliefs. Next they learn what happens over their lifetimes in the standard Bayesian way, i.e., they update their prior beliefs every time they personally see the dividend being unpaid. These analogous prior beliefs must necessarily be subjective in the sense that we cannot base them on relative frequencies, as the only information is that the one-shot dividend remains unpaid.

While Bayesianism in general does not distinguish between rational and irrational prior beliefs (*Gilboa et al.*, 2012; *Gilboa*, 2015), in this semi-Bayesian approach we can deem some prior beliefs irrational and the remaining ones semirational. We declare irrationality if imagining that one goes back in time, looks at what happened before that time, sees the same thing as at present, and forms analogous prior beliefs at that time does not lead to the same posterior beliefs at present as the prior. The reason is that this amounts to the prior failing a kind of test against history, but we do not take a stance on whether passing such a test corresponds to full rationality or not and simply use the term semirationality. What distinguishes irrationality here is that it makes an older investor's posterior beliefs, coming from the same prior beliefs as a younger investor is forming at present, disagree with these younger investor's prior beliefs at present. Now trade under this irrationality should not come as a surprise because it entails disagreement between investors of different age if they live long enough. Two of the many overlapping generations of investors may disagree even though both have complete information and process information the same semi-Bayesian way, just starting at different times. No matter which (dynamic) model of trade one now embeds this irrationality in, general-equilibrium or not, game-theoretic or not, one should expect to obtain trade due to the existence of disagreement under this irrationality. Modeling this formally allows us to additionally study the possibility of mispricing, along with trade, when essentially identical investors have complete information and process information the same way.

One of the simplest approaches is to assume short-sales constraints and risk neutrality to model the asset's price implicitly simply as what the most optimistic investor (generation) is willing to pay, since less optimistic ones cannot sell it short (solution concept of *Harrison and Kreps*, 1978). This has been a popular choice for studying trade and pricing under disagreement. It occurs in existing models due to heterogeneities of various kinds, such as heterogeneous prior beliefs (*Harrison and Kreps*, 1978; *Morris*, 1996; *Adam and Marcet*, 2011), heterogeneous beliefs about the distribution of the signal (*Harris and Raviv*, 1993; *Bolton et al.*, 2006), heterogeneous information ownership with overconfidence in own information (*Scheinkman and Xiong*, 2003), and heterogeneous coarse reasoning (*Steiner and Stewart*, 2015). We adapt this partial-equilibrium approach to overlapping generations, but the source of disagreement in our model—semi-Bayesian information processing—is identical across investors.

Earlier disagreement models, including those with other solution concepts, encompassed, according to *Hong and Stein* (2007), three broad disagreement mechanisms: gradual information flow, limited attention, and heterogeneous priors. While we may classify our semi-Bayesian information processing as limited attention to what happened before one was born, we focus on everyone's attention being limited in the same way.

The consequence of identically irrational prior beliefs in our semi-Bayesian approach is that experience heterogeneity, as measured by age, may also transform them into disagreement and disagreement into trade. On top of that, there are several perspectives from which this trade between essentially identical individuals can even feature mispricing. Firstly, we benchmark equilibrium prices under irrational prior beliefs against those under rational prior beliefs. Secondly, we compare the equilibrium price with the most optimistic investor's (generation's) estimate of the present value of dividends, keeping the specification of the model fixed. Finally, we study overpricing in the sense of the equilibrium price being essentially as if there were no uncertainty about and waiting for this asset's one-shot dividend. Such overpricing occurs in the continuous-trade limit if the irrationality makes investors form prior beliefs that are close to, but do not really assign, probability one to the waiting time till the one-shot dividend turning out to be zero. This irrationality makes young investors be extraordinarily optimistic about the dividend and willing to pay for the asset as if the waiting time till the dividend actually were zero. While their irrational prior beliefs assign a small strictly positive probability to the waiting time being far from zero, they offset it by the anticipation that in that case a similar fool will be instantly willing to pay them for the asset just as much. To describe the equilibrium price here differently, it is as high as to be equal to the uncertain and randomly timed one-shot liquidating dividend. This overpricing is weaker, though, than in related greater fool models, where the buyer anticipates that somebody will be willing to pay even more in the future, and where the price may be higher than the one-shot liquidating dividend with probability one (*Allen et al.*, 1993; or recently, e.g., *Liu and Conlon*, 2018). Our "identical younger fool" story is one of the perspectives on the joint possibility of mispricing and trade between essentially identical individuals we present in Section 5.

The paper is organized as follows. Section 2 presents the model, including the semi-Bayesian investor learning component. Section 3 formalizes and discusses the distinction between irrational and semirational prior beliefs in our semi-Bayesian approach. Section 4 shows that having the same information-processing irrationality as everyone else in the sense of forming irrational prior beliefs can be enough to create trade. Section 5 reaches an analogous conclusion about mispricing. Section 6 concludes.

2 Model

The model has six elements. They comprise a one-shot dividend payment, a mechanism to borrow and lend, a deterministic investor entry-and-exit process, a semi-Bayesian investor learning component, a deterministic trade-timing process, and a solution concept. Risk neutrality and short-sales constraints will automatically be required (*Morris* (1996) discusses the necessity) by the simple solution concept we are going to use, as motivated in the introduction (Section 1). Let us introduce these model components one by one.

Dividend An asset pays a one-shot dividend of \$1 at terminal time $\tau \in (-\infty, \infty]$, where we think of the dividend at infinity as no dividend. Investors do not know (and are uncertain about) the dividend time τ until it comes, but intrinsically τ is not random.

Borrowing and Lending Investors can borrow and lend at a constant rate $r \in (0, \infty)$, which will serve as a discount rate. This side of the market does not have any constraints.

Entry-and-exit Process An investor-lifetime parameter $T \in (0, \infty]$ completely determines the entry-and-exit process. At each time $t \in (-\infty, \tau)$, an investor enters the market (who we refer to as the t entrant), while the $t - T$ entrant, unless $T = \infty$, exits the market.

Semi-Bayesian Learning If investors were Bayesian, they would entertain prior beliefs over all admissible dividend payment times $(-\infty, \infty]$, covering the entire past, present, and future relative to their entry times. But, instead, at each $t \in (-\infty, \tau)$ the t entrant starts priorless, checks what happened before t, sees no dividend, and then forms a prior, analogous to all $s \in (-\infty, \tau)$, over remaining admissible dividend payment times $(t, \infty]$. Analogous means that all such t and s entrants' priors over remaining admissible dividend payment times are as if they come from some common "prior" over dividend waiting times $(0, \infty]$. In other words, it is as if all such t and s entrants think that the dividend waiting time is a particular random variable[1] W taking values in $(0, \infty]$. This representative dividend waiting time W is a primitive of our model and completely determines everything about the semi-Bayesian investor learning component of the model. Firstly, at each time $t \in (-\infty, \tau)$ the t entrant's prior is that the dividend payment time is $W + t$. Secondly, after entry investors update their beliefs in the standard Bayesian way every time the asset does not pay the dividend. That is, at each time $s \in (-\infty, \tau)$, posterior expectations (if well-defined) of the entrant of an arbitrary time $t \in (-\infty, s)$ of functions of dividend payment time are expectations conditional on the event that $W + t > s$. For the purposes of interpreting this semi-Bayesian learning, it is natural to think of the difference $s - t$ between the present time s and the entry time t as a (consequential) experience level rather than just age. For the conditioning events corresponding to this experience level to have nonzero probabilities in relevant cases, we make the following assumption on W:

[1] We consider extended-valued random variables. For a formal definition, see *Chung* (2001). For another application of extended-valued waiting times in economics, see *Khan and Stinchcombe* (2015). We denote the probability measure on events expressed in terms of the random variables by P and adopt the convention $e^{-\infty} = 0$.

Assumption 1. For all (experience levels) $x \in [0,T]\backslash\{\infty\}$, the (prior) probability that $W > x$ is not zero: $\mathrm{P}(W > x) \neq 0$.

Trade-timing Process A trade-frequency parameter $\Delta \in (0,T)$ completely determines the trade-timing process. Trade occurs every Δ time units at time points

$$\{0, \Delta, -\Delta, 2\Delta, -2\Delta, 3\Delta, -3\Delta, \dots\} \cap (-\infty, \tau).$$

Solution Concept The unknown of the solution concept in our formulation is the asset's price $p \in [0,1]$ prevailing over the period until the true dividend time τ and by construction resolving the question of who holds the asset, when, and whether it changes hands. The concisest form of the solution concept is essentially just an equation in p, which we state first and then explain in detail where it comes from:

Definition 1 (Solution Concept in Concise Form)**.** We say that $p^* \in [0,1]$ is an equilibrium price (until the true dividend payment time τ) if the maximization problem on the right-hand side of

$$p^* = \max_{\substack{x\in[0,T]\backslash\{\infty\} \\ y\in\{\Delta,2\Delta,3\Delta,\dots,\infty\}}} \mathrm{E}\left(e^{-r(W-x)} I_{\{W\leq x+y\}} + p^* e^{-ry} I_{\{W>x+y\}} \Big| W > x\right) \tag{1}$$

has a solution and p^* satisfies the resulting condition (1).

Interpretation of Solution Concept We think of x and y, respectively, as an experience level, as in Assumption 1, at which someone might buy the asset and a duration for which someone might hold the asset. The indicator functions partition the range of the dividend waiting time W into within-y and beyond-y parts relative to this buying experience level x. In the former case the investor gets the dividend of \$1 in $W - x$ time units, while in the latter scenario the investor sells the asset at the price p^* in y time units. The conditioning is on the event that the waiting time for this one-shot dividend is greater than the already accumulated experience level x of this investor in question. The expectation is the expected discounted return, conditional on that event, of an investor with experience x of buying the asset and either getting the dividend within or else selling at the price p^* in y time units. Any maximizer (x^*, y^*) of the right-hand side of (1) gives us a prediction of who holds the asset in the equilibrium and when: at each $t \in (-\infty, \tau)$, the entrant of time $t - x^*$ does. Condition (1) implies that by this investor's time-t beliefs the expected discounted return of buying the asset at time t and selling at the equilibrium price p^* at $t + y^*$, unless $t + y^* \geq \tau$, is precisely p^*. One more consequence of condition (1) is that we cannot increase this expected discounted return above p^* by changing the possibly infinite holding duration or by taking another investor's beliefs. In the sense of these two implications this solution concept requires the equilibrium price to be what someone risk-neutral with the most optimistic posterior beliefs given the infinite investment horizon is willing to pay. To make sense of this deliberate assumption of the infinite investment horizon, we either think of the infinite-lifetime case $T = \infty$, or we think of the finite-lifetime case $T < \infty$ and investors that leave paternalistic bequests. The role of this assumption is to eliminate differences in remaining investment horizons as something that creates trade in our model, because our point is about trade between essentially identical individuals with the same irrationality. The idea is that, depending on their age, some investors may still be more optimistic than others if the very same irrationality makes them form, in some sense, irrational prior beliefs (Section 3). The adopted solution concept requires that the most optimistic investors find funds to hold all of the asset, which can cost at most \$1, and less optimistic ones do not, for whatever reason, engage in short-selling of the asset. Under these assumptions, the equilibrium price cannot differ from what the most optimistic investors are willing to pay, as that would mean either shortage or excess supply. Finally, we explicitly state what happens in what order within a particular nonterminal time: first the asset does not pay, then an investor arrives, then trade takes place, and then, unless $T = \infty$, an investor exits.

3 Irrational and Semirational Scenarios

From an investor's perspective, hypothetically, the semi-Bayesian approach—forming prior beliefs only about what happens after the investor is born—has both weaknesses and strengths. The main weakness is arguably that the semi-Bayesian approach brings arbitrariness when it comes to processing information about what happened before the investor was born. The only restriction imposed—that any two investors with the same information about what happened before they were born, possibly at different times, reach the same conclusions at those times—is just to eliminate heterogeneities. On the other hand, the most natural strength is perhaps that it is easier to form prior beliefs, albeit only about the future, when one already has some information about the past. In many situations, it makes sense to use relevant past relative frequencies as prior probabilities about the future. Such considerations challenge the view (*Gilboa et al.* (2012) and *Gilboa* (2015) give many arguments) that rationality implies Bayesianism. But it would be hard to digest our point, if we did not deem such scenarios irrational, that identical individuals apart from age differences might disagree and prefer trade to no trade. What clarifies things is that, depending on how arbitrary the prior beliefs that W induces at a time are relative to what happened before that time, they either do or do not disagree. To capture this distinction, we deem the induced prior beliefs either irrational or semirational, based on how arbitrary they are relative to the past. The idea is that investors can look to history to check how good W is as a model of the future, but with W being a waiting time until a once-in-history dividend payment, that is subtle. Two things they can still notice when they look at what happened before they were born are a long time without change and many older investors that have been updating W for some time. One criterion for W to be good as a model of the future is that the older investors, who start out with the same W, should still agree with it. A stronger criterion for W to be a good model of the future is that imagining that one also goes back in time even more and starts out with W at that time induces the same beliefs at present as W. This is what the following definition requires, allowing one to go back in time as long as the probability of the dividend from that perspective is still positive, and essentially the well-known probabilistic property called lack of memory:

Definition 2. We say that W induces semirational prior beliefs if all (go-back times) $x \in [0, \infty)$ such that $\mathrm{P}(W > x) \neq 0$ and dividend waiting times $w \in (0, \infty)$ satisfy

$$\mathrm{P}(W > w) = \mathrm{P}(W - x > w | W - x > 0). \tag{2}$$

Otherwise, we say that W induces irrational prior beliefs.

Given how we motivated and defined irrational prior beliefs, we may interpret them as limited attention to what happened before one was born. In a special case presented in Sections 4 and 5, we will interpret them as a misbelief, uncorrected due to limited attention, that with strictly positive probability the asset never pays the dividend.

To give that kind of interpretation, we need the characterization of semirationality stated in Proposition 1 below. Its mathematics is very familiar from probability theory (see, e.g., *Billingsley*, 1995) and theory of Cauchy equations (see, e.g., *Falmagne and Doble*, 2015, Section 3.2), but we need to keep in mind that our waiting time W is extended-valued.

Proposition 1. *W induces semirational prior beliefs if and only if either $W = \infty$ with probability one or W is finite-valued with probability one and has an exponential distribution.*

Very few choices of W induce semirational prior beliefs. Those that do are permissive enough to make the (prior) probability that the asset ever pays either zero or one, but nothing in between. Those that put this probability in between induce irrational prior beliefs, and so do most of those that make this probability one, as exponential distributions are the only exception according to Proposition 1. With this terminology, the point of this paper is that semi-Bayesian learning with irrational prior beliefs can be enough to create nontrivial trade even though everyone learns the same way.

4 Trade

We want to illustrate trade between essentially identical individuals with the very same irrationality when the departure from semirationality sounds as innocuous and natural as possible. In view of its characterization in Proposition 1, we think of irrationality with a possible interpretation as forming a misbelief that with strictly positive probability the asset never pays the dividend. For this, we think of a choice of W that on the one hand makes the probability that the asset never pays almost zero, and on the other hand has an exponential distribution conditionally on ever paying. We refer to this by saying, using a common term, that W has an incomplete exponential distribution (for other well-studied examples of incomplete distributions, see *Khan and Stinchcombe*, 2015). An incomplete exponential distribution is in fact fairly close to both types of semirationality as per Proposition 1. But rather than considering such a choice of W in isolation, we first use it to identify a certain key property (Proposition 2) and then look at an arbitrary choice of W with this property. It says that the older the investor, who started out with the same W, is at present, the more pessimistic this investor's posterior beliefs over dividend payment times are in a sense of first-order stochastic dominance. It is convenient to use the following analogue of first-order stochastic dominance for associated posterior beliefs over extended reals (dividend waiting times):

Definition 3. We say that W induces strongly first-order stochastically dominant posterior beliefs at an (experience level) $x \in [0,T]\backslash\{\infty\}$ relative to (another experience level) $x' \in [0,T]\backslash\{\infty\}$ if all dividend waiting times $w \in (0,\infty)$ satisfy

$$\mathrm{P}\left(W - x > w \middle| W - x > 0\right) > \mathrm{P}\left(W - x' > w \middle| W - x' > 0\right). \tag{3}$$

In this case, we write $x <_W x'$ ("$<$" because this dominance reflects pessimism about the dividend waiting time) and say that x exhibits strong first-order stochastic dominance over x'.

Using this terminology, for incomplete exponential distributions higher experience levels exhibit strong first-order stochastic dominance (pessimism) over lower ones, as formally stated in Proposition 2 below. The key to this property lies in comparison with exponential distributions without incompleteness, which assign probability zero to never paying. Without incompleteness the left-hand side of (3) is x-independent (Proposition 1) and the posterior probability that the asset never pays can never increase away from zero. In contrast, with incompleteness the posterior probability that the asset never pays increases with experience, but conditionally on ever paying the experience effect is absent, just like without incompleteness (as made precise by formula (5) in the proof of Proposition 2).

Proposition 2 (Stochastic Dominance under Incomplete Exponential Distribution)**.** *Suppose that* $0 < \mathrm{P}(W = \infty) < 1$ *and conditionally on* $W < \infty$ *the distribution of* W *is exponential, i.e., there is a* $\lambda \in (0,\infty)$ *such that all* $w \in (0,\infty)$ *satisfy*

$$\mathrm{P}\left(W \leq w | W < \infty\right) = 1 - e^{-\lambda w}. \tag{4}$$

If (experience levels) $x, x' \in [0,T]\backslash\{\infty\}$ *are such that* $x > x'$, *then* x *exhibits strong first-order stochastic dominance (pessimism) over* x', *i.e.,* $x <_W x'$.

Proof. At every $x \in [0,T]\backslash\{\infty\}$, the posterior dividend waiting time continues having an incomplete exponential distribution with the same exponential part but different (updated) probabilities of ever paying: all $w \in (0,\infty)$ satisfy

$$\frac{\mathrm{P}\left(W - x \leq w | W - x > 0\right)}{\mathrm{P}\left(W < \infty | W - x > 0\right)} = 1 - e^{-\lambda w}. \tag{5}$$

This makes it clear that the left-hand side of (3) is strictly increasing in x, completing the proof. □

Now we want to illustrate trade between essentially identical individuals when the very same irrationality makes generations disagree in the sense of stochastic dominance in general as under an incomplete exponential distribution. For this, Proposition 3 below verifies that pessimism at higher experience levels in the sense of stochastic dominance guarantees pessimism in the sense of being unwilling to pay for the asset as much as less experienced under a range of resale prices. This covers our whole set of admissible prices $[0,1]$, which we can see intuitively from this stochastic-dominance property as follows. Just as first-order stochastic dominance of one money lottery over another means that expected utility of the first is at least as high as that of the second if one values more over less, so can we compare expected values of functions of, possibly infinite, waiting time. This is what the conditional expectations in the definition of equilibrium (Definition 1) are, with the functions being $f : (0,\infty] \to \mathbb{R}$ of the form

$$f(w) = \begin{cases} e^{-rw} & \text{if } w \leq y, \\ p^* e^{-ry} & \text{if } w > y, \end{cases}$$

so that

$$\mathrm{E}\left(e^{-r(W-x)} I_{\{W \leq x+y\}} + p^* e^{-ry} I_{\{W > x+y\}} \middle| W > x\right) = \mathrm{E}\left(f\left(W-x\right) \middle| W > x\right). \tag{6}$$

Such an f maps dividend waiting time into the present value of the return in that waiting time. To be able to compare expected values of such functions of waiting time using the theory of stochastic dominance, we simply need a negative answer to the following question: "Can an investor value a longer dividend waiting time over a shorter one, in the sense of f not being decreasing (say, because the investor might value being able to cash in on resale, but only after some time)?" The answer to this in our case is indeed negative for all admissible prices, because we do not include prices high enough for an affirmative answer in the set of admissible prices $[0,1]$. In other words, in our model investors value shorter dividend waiting times over longer ones, and thus one would expect monotonicity of the expectation (6) in x under stochastic dominance:

Proposition 3. *If for all (experience levels) $x, x' \in [0,T] \backslash \{\infty\}$ it is true that*

$$x > x' \implies x <_W x', \tag{7}$$

then for all prices $p^* \in [0,1]$ and durations $y \in (0,\infty]$ the (real) mapping

$$x \mapsto \mathrm{E}\left(e^{-r(W-x)} I_{\{W \leq x+y\}} + p^* e^{-ry} I_{\{W > x+y\}} \middle| W > x\right)$$

on the set of experience levels $[0,T] \backslash \{\infty\}$ is strictly decreasing.

Proof. All $x \in [0,T] \backslash \{\infty\}$ satisfy

$$\begin{aligned} &\mathrm{E}\left(\left(e^{-r(W-x)} - e^{-ry}\right) I_{\{W \leq x+y\}} \middle| W > x\right) \\ &= \lim_{n\to\infty} \sum_{m=1}^{2^n} \frac{1-e^{-ry}}{2^n} \mathrm{P}\left(e^{-r(W-x)} - e^{-ry} \geq \frac{m\left(1-e^{-ry}\right)}{2^n} \middle| W > x\right) \end{aligned}$$

$$= \left(1 - e^{-ry}\right) \int_0^1 \mathrm{P}\left(e^{-r(W-x)} - e^{-ry} \geq z\left(1 - e^{-ry}\right) \Big| W > x\right) dz,$$

and thus, by (7) and (3), for all $x, x' \in [0, T] \backslash \{\infty\}$ such that $x > x'$ we have

$$\begin{aligned} &\mathrm{E}\left(\left(e^{-r(W-x)} - e^{-ry}\right) I_{\{W \leq x+y\}} \Big| W > x\right) \\ &< \mathrm{E}\left(\left(e^{-r(W-x')} - e^{-ry}\right) I_{\{W \leq x'+y\}} \Big| W > x'\right). \end{aligned} \tag{8}$$

The conclusion follows because, on the one hand, inequality (8) means that

$$\begin{aligned} &\mathrm{E}\left(e^{-r(W-x)} I_{\{W \leq x+y\}} \Big| W > x\right) - \mathrm{E}\left(e^{-r(W-x')} I_{\{W \leq x'+y\}} \Big| W > x'\right) \\ &< e^{-ry}\left(\mathrm{P}\left(W \leq x + y | W > x\right) - \mathrm{P}\left(W \leq x' + y \big| W > x'\right)\right) \end{aligned}$$

and, on the other hand, using (3) we obtain

$$\begin{aligned} &\mathrm{E}\left(p^* e^{-ry} I_{\{W > x+y\}} \big| W > x\right) - \mathrm{E}\left(p^* e^{-ry} I_{\{W > x'+y\}} \big| W > x'\right) \\ &\leq -e^{-ry}\left(\mathrm{P}\left(W \leq x + y | W > x\right) - \mathrm{P}\left(W \leq x' + y \big| W > x'\right)\right). \end{aligned}$$

□

Proposition 3 shows that pessimism at higher experience levels in the sense of stochastic dominance ensures pessimism in the sense of being unwilling to pay for the asset as much as less experienced in equilibrium, if any, too. An immediate corollary is that in equilibrium under such stochastic dominance the asset changes hands every time the market opens, since the only maximizing experience level is zero:

Corollary 1 (Trade in Equilibrium)**.** *Under the hypothesis* (7) *of Proposition 3, the only maximizing (experience level) x in the definition of equilibrium (Definition 1) when p^* is an equilibrium price is 0.*

In other words, since the only maximizing experience level is zero, at each $t \in (-\infty, \tau)$ only the t entrant—the newcomer—holds the asset. This formally confirms that this irrationality creates trade in equilibrium (we verify existence below) even though everyone is irrational the same way. As time goes by, until the asset pays the dividend, whoever is the least experienced when the market reopens takes the asset over from someone who once also was the least experienced. No admissible price can make an investor with strictly positive experience value the asset as highly as the least experienced investor.

The existence of an equilibrium price within that set of admissible prices $[0, 1]$ is what we verify in Proposition 4 below by solving explicitly for such an equilibrium price. As far as only the existence is concerned, though, by the above monotonicity in experience (Proposition 3), simply an application of the Tarski Fixed-point Theorem to the mapping

$$p^* \mapsto \max_{y \in \{\Delta, 2\Delta, 3\Delta, \ldots, \infty\}} \mathrm{E}\left(e^{-rW} I_{\{W \leq y\}} + p^* e^{-ry} I_{\{W > y\}}\right)$$

on $[0,1]$ into itself would do, but not explicitly. We do more to additionally study the possibility of mispricing, along with trade, when investors are essentially identical, have complete information, and process information the same semi-Bayesian way. Our explicit solution simply follows from the same monotonicity in experience (Proposition 3) in conjunction with the recursive structure of our model, as particularly captured by equation (10) in the proof of Proposition 4. It says that waiting for the (one-shot) dividend for no longer than y time units and reselling the asset if it does not pay by then yields the same expected discounted return as a shorter holding cut-off with extension conditional on the dividend being still unpaid. The buyer's equilibrium trading strategy takes precisely this form in our interpretation of the solution concept. The equilibrium price given below by Proposition 4 simply solves the least experienced investor's (the buyer's) break-even equation under this investor's equilibrium trading strategy. We see what exactly this strategy is from the equilibrium maximizing holding duration (cut-off) also given by Proposition 4. The strategy is to wait for the dividend for no longer than till the next opening of the market and to resell the asset if it does not pay by then. The intuition is that the buyer simply knows that if by then the asset does not pay the dividend then less experienced investors will be more optimistic (Proposition 3) and willing to pay more than her.

Proposition 4 (Equilibrium)**.** *Under the hypothesis* (7) *of Proposition 3, we have the following results:*

(a) an equilibrium price $p^* \in [0,1]$ exists, and the solution of the equation

$$p^* = \mathrm{E}\left(e^{-rW} I_{\{W\le\Delta\}} + p^* e^{-r\Delta} I_{\{W>\Delta\}}\right) \tag{9}$$

for p^* is an equilibrium price;

(b) the only maximizer in the definition of equilibrium (Definition 1) when p^* is the equilibrium price given by (9) *is* $(0,\Delta)$.

Proof. The solution p^* is an admissible price, i.e., $p^* \in [0,1]$, because

$$0 \le \mathrm{E}\left(e^{-rW} I_{\{W\le\Delta\}} + 0e^{-r\Delta} I_{\{W>\Delta\}}\right) \text{ and } 1 \ge \mathrm{E}\left(e^{-rW} I_{\{W\le\Delta\}} + 1e^{-r\Delta} I_{\{W>\Delta\}}\right).$$

All $y \in [\Delta, \infty]$ satisfy

$$\begin{aligned} &\mathrm{E}\left(e^{-rW} I_{\{W\le y\}} + p^* e^{-ry} I_{\{W>y\}}\right) \\ &\quad = p^* + e^{-r\Delta}\,\mathrm{P}\left(W>\Delta\right)\left(\mathrm{E}\left(e^{-r(W-\Delta)} I_{\{W\le y\}} + p^* e^{-r(y-\Delta)} I_{\{W>y\}} \middle| W>\Delta\right) - p^*\right). \end{aligned} \tag{10}$$

Letting $y = 2\Delta$ and noting that by (9) and Proposition 3 we have

$$\mathrm{E}\left(e^{-r(W-\Delta)} I_{\{W\le 2\Delta\}} + p^* e^{-r\Delta} I_{\{W>2\Delta\}} \middle| W>\Delta\right) < p^*,$$

we see that

$$\mathrm{E}\left(e^{-rW} I_{\{W\le 2\Delta\}} + p^* e^{-r(2\Delta)} I_{\{W>2\Delta\}}\right) < p^*.$$

Now by induction, for every nonzero natural number n, letting $y = (n+1)\,\Delta$ in (10) and noting that by Proposition 3 we have

$$\begin{aligned}&\mathrm{E}\left(e^{-r(W-\Delta)} I_{\{W \le (n+1)\Delta\}} + p^* e^{-rn\Delta} I_{\{W > (n+1)\Delta\}} \middle| W > \Delta\right) \\ &\quad < \mathrm{E}\left(e^{-rW} I_{\{W \le n\Delta\}} + p^* e^{-rn\Delta} I_{\{W > n\Delta\}}\right),\end{aligned}$$

we see that

$$\mathrm{E}\left(e^{-rW} I_{\{W \le (n+1)\Delta\}} + p^* e^{-r(n+1)\Delta} I_{\{W > (n+1)\Delta\}}\right) < p^*.$$

Furthermore, letting $y = \infty$ in (10) and using Proposition 3 again, we obtain

$$\begin{aligned}&\mathrm{E}\left(e^{-rW} I_{\{W \le \infty\}} + p^* e^{-r\cdot\infty} I_{\{W > \infty\}}\right) \\ &\quad < p^* + e^{-r\Delta}\,\mathrm{P}\,(W > \Delta)\left(\mathrm{E}\left(e^{-rW} I_{\{W \le \infty\}} + p^* e^{-r\cdot\infty} I_{\{W > \infty\}}\right) - p^*\right),\end{aligned}$$

and thus

$$\mathrm{E}\left(e^{-rW} I_{\{W \le \infty\}} + p^* e^{-r\cdot\infty} I_{\{W > \infty\}}\right) < p^*,$$

too. So all $y \in \{2\Delta, 3\Delta, \ldots, \infty\}$ satisfy

$$\mathrm{E}\left(e^{-rW} I_{\{W \le y\}} + p^* e^{-ry} I_{\{W > y\}}\right) < p^*.$$

Applying Proposition 3 one last time, we readily confirm that this p^* given by (9) is an equilibrium price (part (a)) such that the only maximizer in the definition of equilibrium (part (b)) is $(0, \Delta)$. ☐

The above-mentioned intuition cannot fail to remind us of the intuition behind speculative overpricing in seminal applications of the solution concept of *Harrison and Kreps* (1978), but here we can view this as underpricing relative to a natural benchmark. This is one of the ideas that we present below in Section 5 to show different perspectives from which this trade between essentially identical individuals can even feature mispricing.

5 Mispricing

Here we add a second point—mispricing—to what information-processing irrationalities can do even when everyone has the same irrationality. We start with underpricing, in a special case of our trade results (Section 4), in which we can interpret the irrationality as forming a misbelief that with strictly positive probability the asset never pays. This is because reducing the probability of never paying to zero can turn the beliefs from irrational to semirational when conditionally on ever paying the distribution is exponential (Propositions 1 and 2). Not surprisingly, the equilibrium price becomes higher in the semirational case as the probability of never paying turns to zero, indicating underpricing relative to this natural choice of a benchmark, as formalized below in part (c) of Proposition 5. In this semirational benchmark scenario, disagreement disappears, and everyone is equally willing to pay the equilibrium price, given by the same equation (9), to hold the asset (parts (a) and (b) of Proposition 5).

Proposition 5. *Suppose that* $0 \le \mathrm{P}(W = \infty) < 1$ *and conditionally on* $W < \infty$ *the distribution of* W *is exponential, i.e., there is a* $\lambda \in (0, \infty)$ *such that all* $w \in (0, \infty)$ *satisfy* (4)*. We have the following results:*

(a) regardless of whether $\mathrm{P}(W = \infty) > 0$ *or not, an equilibrium price* $p^* \in [0, 1]$ *exists, and the solution of equation* (9) *for* p^* *is an equilibrium price;*

(b) if $\mathrm{P}(W = \infty) = 0$*, then nonmaximizers in the definition of equilibrium (Definition 1) when* p^* *is the equilibrium price given by (a) are absent;*

(c) the lower the (prior) probability $\mathrm{P}(W = \infty)$ *that the asset never pays the dividend, the greater the equilibrium price* p^* *given by (a).*

Proof. (a)–(b) We already know (a) for the case of $\mathrm{P}(W = \infty) > 0$ from Propositions 2 and 4. The remaining case of $\mathrm{P}(W = \infty) = 0$ is an exponential distribution (without incompleteness), which means that W induces semirational prior beliefs (Proposition 1). Semirationality means that no matter how old the investor, who started out with the same W, is at present, the posterior beliefs over dividend waiting times are the same in the sense of (2). This implies that the conditional expectations in the definition of equilibrium (Definition 1) do not depend on x (experience) in view of their representation in (6), in contrast to strict monotonicity in the other case (Proposition 3). Now we can verify that p^* given by (a) is an equilibrium price as in the proof of Proposition 4, except that we need to use equalities instead of some of the inequalities, so that (b) also holds.

(c) Using (4), we can rewrite (9) as

$$p^* = \mathrm{P}(W < \infty)\,\mathrm{E}\left(e^{-rW} I_{\{W \le \Delta\}} \middle| W < \infty\right) + p^* e^{-r\Delta}\left(1 - \mathrm{P}(W < \infty)\left(1 - e^{-\lambda\Delta}\right)\right)$$

and also infer that here the conditional expectation $\mathrm{E}\left(e^{-rW} I_{\{W \le \Delta\}} \middle| W < \infty\right)$ does not depend on W, the probability $\mathrm{P}(W < \infty)$ in particular. Now the conclusion follows from the Implicit-function Theorem and the fact that the solution p^* satisfies

$$\begin{aligned} &\frac{\mathrm{E}\left(e^{-rW} I_{\{W \le \Delta\}} \middle| W < \infty\right) - p^* e^{-r\Delta}\left(1 - e^{-\lambda\Delta}\right)}{1 - e^{-r\Delta}\left(1 - \mathrm{P}(W < \infty)\left(1 - e^{-\lambda\Delta}\right)\right)} \\ &= \frac{p^*}{\mathrm{P}(W < \infty)} \cdot \frac{1 - e^{-r\Delta}}{1 - e^{-r\Delta}\left(1 - \mathrm{P}(W < \infty)\left(1 - e^{-\lambda\Delta}\right)\right)} > 0. \end{aligned}$$

□

A subpoint we want to add motivated by underpricing in Proposition 5 (just proved) is that information-processing irrationalities that make everyone overpessimistic according to some criterion might justify short-sales bans. In Proposition 5, they do not matter for the semiratonal scenario, where everyone is willing to pay the equilibrium price, but short-sales constraints matter and bind in the irrational cases with underpricing ($\mathrm{P}(W = \infty) > 0$). This prompts the question of the effect of short-sales constraints on this kind of mispricing. In such special cases, short-sales constraints guarantee that the equilibrium price is not as low as the most overpessimistic estimates of the present value of the dividend (conditional expectations in Definition 1 with infinite holding durations). In fact, since waiting for the (one-shot) dividend forever is nobody's equilibrium strategy (part (b) of Proposition 4), the equilibrium price is higher than absolutely everyone's estimate of the present value of the dividend. Nonetheless, it is underpricing relative to the semirational beliefs (part (c) of Proposition 5), where everyone has the same estimate of the present value of the dividend as the equilibrium price (part (b) of Proposition 5). In other words, in the irrational cases short-sales constraints guarantee that the equilibrium price, which is lower than the semirational estimate of the present value of the dividend, is not as low as any one of the irrational estimates.

Our intention though is not to dwell on underpricing relative to that benchmark price further but to place the joint possibility of mispricing and trade between essentially identical individuals into a wide perspective. An alternative price benchmark with long history is simply the most optimistic investor's estimate of the present value of dividends, keeping the specification of the model fixed, due (like our solution concept) to *Harrison and Kreps* (1978). As we have already indirectly noted along the way, in the irrational cases in Proposition 5 the equilibrium price exceeds this latter benchmark, since waiting for the dividend forever is nobody's equilibrium strategy (part (b) of Proposition 4). This makes our point about the joint possibility of mispricing and trade between essentially identical individuals valid relative to more price benchmarks than one.

In fact, in another (irrational) special case of our trade results (Section 4) we can make the equilibrium price "beat" any benchmark except for the maximum admissible price—\$1—simply by making the trade-frequency parameter Δ small. Here it is first helpful to think intuitively of what happens in the limit as the market starts opening continuously. If the price were \$1, everyone would rather sell the asset and get \$1 immediately than commit to waiting for the one-shot \$1-dividend for a while before selling conditionally on the dividend remaining unpaid. Still, we can make the equilibrium price approach \$1 in the limit as the market starts opening continuously. The idea is to choose W in Proposition 4 so that for any lower price someone is optimistic enough to be willing to buy the asset and commit to waiting for the dividend for at least a little while before, if still possible, reselling. This is so if we let W come in some sense close to assigning (prior) probability one to the dividend waiting time turning out to be zero, although per se this (degenerate) distribution is inadmissible in our model. A notion of being close that suffices here and does make the equilibrium price approach \$1 is the distribution function of W becoming infinitely steep at zero:

Proposition 6. *Suppose that W satisfies the hypothesis* (7) *of Proposition 3, is finite-valued, as well as has a distribution function F which is continuously differentiable on* $(0, \infty)$ *and such that*

$$\lim_{w \to 0^+} F'(w) = \infty. \tag{11}$$

Consider the equilibrium price p^* given by (9) *as a function of Δ on $(0, T)$. We have*

$$\lim_{\Delta \to 0^+} p^*(\Delta) = 1.$$

Proof. L'Hôpital Rule. □

A concrete example of trade between essentially identical individuals that even feautures overpricing relative to any benchmark except for the maximum admissible price is under any gamma distribution satisfying (11). Under such a gamma distribution with parameter values corresponding to (11), the remaining hypotheses of Proposition 6, except for perhaps (7), are trivially true. For (7), note that (11) corresponds to the gamma distribution having a strictly decreasing hazard rate, which we can interpret as perceived instantaneous dividend likelihood being strictly decreasing in experience (for hazard-rate shapes, see, for example, *Klugman et al.*, 2019). Now pessimism at higher experience levels in the sense of stochastic dominance—condition (7)—follows from this hazard-rate shape, after differentiating the left-hand side of (3) with respect to x. Along the way, we actually see the equivalence of conditions (7) and (11) under a gamma distribution. But, in general, condition (11) can fail despite condition (7) being true, as, for example, when W has a distribution function F defined by

$$F(w) = \frac{w}{w+1} \text{ for } w \geq 0, \; F(w) = 0 \text{ for } w < 0$$

(a special case of the loglogistic distribution).

In conclusion of this section on the joint possibility of mispricing and trade between essentially identical individuals apart from age differences, we mention how it speaks to the empirical and experimental literature on mispricing and experience (see, e.g., *Dufwenberg et al.*, 2005; *Greenwood and Nagel*, 2009; *Akiyama et al.*, 2014; *Xie and Zhang*, 2016). While this literature has documented that experience matters, it mostly sidelined the question of why exactly and how to coherently reconcile relevant economic theories with empirical evidence on various experience effects. An exception is *Greenwood and Nagel* (2009), who discuss potential channels through which experience may play a role in mispricing and whether those channels are consistent with some of existing economic theories. Their conclusion is that their results "fit well with theories of adaptive learning" and "do not rule out that herding could help explain differences between young and old managers' investment choices", but they "doubt that human capital theories help". Essential parts of what they mean by adaptive learning are various ways to extrapolate from the past and possibly excessive extrapolation by the young. But what they did not consider is whether and how important it actually is in various learning theories that the way of information processing itself changes with experience for mispricing to occur. At first sight it may seem that a mispricing argument would require more experienced investors to process information more rationally, to be able to take advantage of less experienced investors' irrationality, e.g., excessive extrapolation or limited attention. However, as a byproduct of our analysis we see that mispricing under age (experience) heterogeneity can in principle occur even when the way of information processing stays the same throughout investors' lifetimes.

6 Conclusion

We studied whether a common (identical across investors) irrationality in information processing can be enough to create nontrivial trade between them. Our answer is yes. In our model, an irrationality with an interpretation as limited attention to what happened before one was born is enough to create trade even though every generation sees exactly the same thing when they look at what happened before they were born. This identically limited attention to identical histories takes all information into account but overweights learning from personal experience after being born. As a result, experience heterogeneity, as measured by age, transforms this identically irrational information processing into disagreement and disagreement into trade. In addition to this point about the possibility of trade between essentially identical individuals with the very same irrationality, we made an analogous point about the possibility of mispricing.

References

Adam, K., and A. Marcet (2011), Internal rationality, imperfect market knowledge and asset prices, *Journal of Economic Theory*, *146*(3), 1224–1252.

Akiyama, E., N. Hanaki, and R. Ishikawa (2014), How do experienced traders respond to inflows of inexperienced traders? An experimental analysis, *Journal of Economic Dynamics and Control*, *45*, 1–18.

Allen, F., S. Morris, and A. Postlewaite (1993), Finite bubbles with short sale constraints and asymmetric information, *Journal of Economic Theory*, *61*(2), 206–229.

Billingsley, P. (1995), *Probability and Measure*, A Wiley-Interscience Publication, Wiley.

Bolton, P., J. A. Scheinkman, and W. Xiong (2006), Executive compensation and short-termist behaviour in speculative markets, *Review of Economic Studies*, *73*(3), 577–610.

Chung, K. L. (2001), *A Course in Probability Theory*, Academic Press.

Dufwenberg, M., T. Lindqvist, and E. Moore (2005), Bubbles and experience: An experiment, *American Economic Review*, *95*(5), 1731–1737.

Falmagne, J., and C. Doble (2015), *On Meaningful Scientific Laws*, Springer.

Gilboa, I. (2015), Rationality and the Bayesian paradigm, *Journal of Economic Methodology*, *22*(3), 312–334.

Gilboa, I., A. Postlewaite, and D. Schmeidler (2012), Rationality of belief or: Why Savage's axioms are neither necessary nor sufficient for rationality, *Synthese*, *187*(1), 11–31, doi: 10.1007/s11229-011-0034-2.

Greenwood, R., and S. Nagel (2009), Inexperienced investors and bubbles, *Journal of Financial Economics*, *93*(2), 239–258.

Harris, M., and A. Raviv (1993), Differences of opinion make a horse race, *Review of Financial Studies*, *6*(3), 473–506.

Harrison, J. M., and D. M. Kreps (1978), Speculative investor behavior in a stock market with heterogeneous expectations, *Quarterly Journal of Economics*, *92*(2), 323–336.

Hong, H., and J. C. Stein (2007), Disagreement and the stock market, *Journal of Economic Perspectives*, *21*(2), 109–128.

Khan, U., and M. B. Stinchcombe (2015), The virtues of hesitation: Optimal timing in a non-stationary world, *American Economic Review*, *105*(3), 1147–1176.

Klugman, S., H. Panjer, and G. Willmot (2019), *Loss Models: From Data to Decisions*, Wiley Series in Probability and Statistics, Wiley.

Liu, F., and J. R. Conlon (2018), The simplest rational greater-fool bubble model, *Journal of Economic Theory*, *175*, 38–57.

Morris, S. (1996), Speculative investor behavior and learning, *Quarterly Journal of Economics*, *111*(4), 1111–1133.

Scheinkman, J. A., and W. Xiong (2003), Overconfidence and speculative bubbles, *Journal of Political Economy*, *111*(6), 1183–1220.

Steiner, J., and C. Stewart (2015), Price distortions under coarse reasoning with frequent trade, *Journal of Economic Theory*, *159*, 574–595.

Xie, H., and J. Zhang (2016), Bubbles and experience: An experiment with a steady inflow of new traders, *Southern Economic Journal*, *82*(4), 1349–1373.