Robust Contracting in General Contract Spaces

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ROBUST CONTRACTING IN GENERAL CONTRACT SPACES

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Abstract. We consider a general framework of optimal mechanism design under adverse selection and ambiguity about the type distribution of agents. We prove the existence of optimal mechanisms under minimal assumptions on the contract space and prove that centralized contracting implemented via mechanisms is equivalent to delegated contracting implemented via a contract menu under these assumptions. Our abstract existence results are applied to a series of applications that include models of optimal risk sharing and of optimal portfolio delegation.

Keywords: robust contracts, nonmetrizable contract spaces, ambiguity, financial markets
JEL subject classification: C02, D82

1. Introduction

Questions of contracting are often based on parties that have a narrow idea about the opponent. This applies to many real-world situations, such as investing in another part of the world (lack of information) or delegating the management of pension schemes (lack of experience). This suggests the need to incorporate ambiguous beliefs about the agent's characteristics in the principal-agent problem, and to assume that the principal evaluates contracts by the worst-case performance with respect to (w.r.t.) different beliefs. Recent studies that analyze such problems in which the principal is uncertain about agents' characteristics include Bodoh-Creed (2012); Carroll (2015); Auster (2018); De Castro and Yannelis (2018).

We consider a principal-agent model that has incomplete information on one side: the agent's preferences are type-dependent and private information to the agent. The principal holds ambiguous beliefs about the distribution of types. The principal's preferences for ambiguity can be captured by the maxmin expected utility model of Gilboa and Schmeidler (1989) or, more generally, via variational preferences à la Maccheroni, Marinacci, and Rustichini (2006). Our framework allows for a very general set of possible beliefs; in particular, we do *not* assume that the principal has a reference belief, nor we capture model discrepancy by means of relative entropy as in Hansen and Sargent (2001).

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Our main contribution is to establish the existence of an optimal contract under minimal assumptions on the contract space. In particular, we do not require the contract space to be metrizable. Nonmetrizable contract spaces arise naturally in principal-agent models when the utility levels of the agent are the relevant contract variables. The contracting problem over the original contracting space, and the one over the space of the resulting utility levels, are equivalent under very mild conditions on the agent's utility function. Quantifying the decision variable of the principal accounted in utility units of the agent is a well-known approach; see Spear and Srivastava (1987), Schättler and Sung (1993) and more recently Sannikov (2008). In our case, the transformation has a different purpose. After the transformation, the agent's utility, as a function of type and contract, reduces to a bilinear form on the contract/type space pair. This significantly simplifies the incentive compatibility constraint. Moreover, conditions on the continuity of the agent's utility boil down to the joint continuity of this bilinear form. As a result, the following trade-off appears: a solvable model of optimal contracts with large type (contract) space typically only allows for a small contract (type) space. This dichotomy is well known for the commodity-price duality in general equilibrium analysis. We show in a series of applications that this seemingly mathematical generalization is of relevance. Our applications include models of optimal reinsurance and market optimized risk sharing, and models of optimal portfolio delegation. The latter models combine adverse selection and moral hazard effects.

Our abstract existence result extends Page (1992)[1] by dropping the assumption of a metrizable contract space. Without a metric on the contract space, we substitute the usual Hausdorff metric topology on the set of contract menus by the *Fell topology* (the topology of closed convergence). The appeal of this topology is that the hyperspace of all closed subsets is compact, as long as the original space is compact.[2] Working with the space of closed subsets under the Fell topology, we can show continuity of the utility functions defined on sets and hence the existence of an optimal contract menu. Subsequently, we prove that delegated contracting implemented via contract menus is equivalent to centralized contracting implemented via contract mechanisms. This equivalence is not an obvious result, because the usual measurable selection theorems on which the proof of the revelation principle is based, do not apply in nonmetrizable spaces.

The rest of the paper is organized as follows. Section 2 provides several examples that account for robustness and a contract space that cannot be captured by existing results. Section 3 provides the results on the existence of an optimal contract. Section 4 provides a series of applications. Section 5 concludes by comparing the results of the present work with those in the literature. Appendix A recalls a series of abstract topological results; Appendix B presents the proofs.

2. Motivating Examples

In this section, we present three motivating examples that illustrate the way in which nonmetrizable *compact* contract spaces over *utility units* arise naturally in

[1] In this study, a characterization of incentive compatibility is established in a Polish type space combined with a compact metric contract space.

[2] Appendix A provides a detailed account of all involved topological concepts.

models of reinsurance and optimal risk sharing with *non-compact* metric contract spaces. We analyze each example in greater detail in Section 4.

In what follows we denote by $L^0(Q), L^1(Q)$, and $L^\infty(Q)$ the class of all almost surely finite, integrable, respectively essentially bounded random variables defined on some probability space $(\Omega, \mathcal{F}, Q)$. On $L^0(Q)$ we consider the topology of convergence in Q-probability; the set $L^p(Q)$, $0 < p \leq \infty$ is equipped with the usual L^p-norm $\|\cdot\|_p$. A set of random variables X is called uniformly norm bounded if $\sup_{x\in X} \|x\|_\infty < \infty$. It is called bounded in Q-probability if $\lim_{R\nearrow\infty} \sup_{x\in X} Q(|x| \geq R) = 0$.

2.1. **A reinsurance model with utilities on the positive half line.** Let us consider a reinsurance model between an uninformed principal and an informed agent. Risk exchange can occur over a set $\mathcal{X}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, Q)$. The principal's endowment is given by a random variable $e_p \in L^1(Q)$. She may hedge her risk by exchanging payoffs $x \in \mathcal{X}$ with the agent. Her utility from a risk transfer x is given by

$$E_{P'}[v(e_p - x)]$$

for some bounded, concave utility function $v : \mathbb{R}_+ \to \mathbb{R}$ on the positive half line, and some belief P' about the distribution of the states of the world. We assume that the belief is equivalent to Q with bounded density.

The agent is endowed with a claim $e_a \in L^1(Q)$. His utility from a risk transfer $x \in \mathcal{X}$ is given by

$$E_P\left[u(e_a + x)\right]$$

for some strictly increasing, concave, continuous utility function $u : \mathbb{R}_+ \to \mathbb{R}$ on the positive half line and some belief P that is also equivalent to Q with bounded density. We refer to P as the agent's type. Types are private knowledge to the agent; the principal only knows the set of possible types $\mathcal{Q}$. We identify $\mathcal{Q}$ with the set of densities $\frac{dP}{dQ}$. Incorporating the non-negativity constraint on both parties' payoff, the set of admissible transfers (the contract space in payoff units) is given by

$$X := \left\{x \in \mathcal{X} : -e_a \leq x \leq e_p\right\}.$$

2.1.1. *The contracting problem over contingent claims.* The principal faces the problem of designing an optimal risk-sharing rule $x : \mathcal{Q} \to X$ that assigns a transfer to any agent type $P \in \mathcal{Q}$ and that maximizes her utility, subject to the usual individual rationality and incentive compatibility constraint. We assume that the principal's utility from a risk-sharing mechanism $x : \mathcal{Q} \to X$ is given by a variational utility function (see Maccheroni, Marinacci, and Rustichini (2006)) of the form

$$\inf_{\kappa\in\mathbb{K}} \left\{ \int_{\mathcal{Q}} E_{P'}\left[v\left(e_p - x_P\right)\right]\kappa(dP) + \alpha(\kappa) \right\} \tag{1}$$

where $\mathbb{K}\subset \Delta(\mathcal{Q})$ is a set of probability measures on $\mathcal{Q}$, and $\alpha : \mathbb{K} \to \mathbb{R}$ is a convex penalty function. Our choice of utility function allows the principal to be uncertain about the distribution of the agent type. The principal's optimization problem is thus given by: find a menu $x : \mathcal{Q} \to X$ that maximizes

$$\inf_{\kappa \in \mathbb{K}} \left\{ \int_{\mathcal{Q}} E_{P'} \left[v \left(e_p - x_P \right) \right] \kappa(P) + \alpha(\kappa) \right\}$$

$$\text{(2)} \qquad \begin{array}{lll} \text{subject to} & P \mapsto E_{P'} \left[v \left(e_p - x_P \right) \right] & \text{is measurable,} \\ & E_P[u(x_P + e_a) - u(e_a)] \geq 0, & P \in \mathcal{Q} \\ & E_P[u(x_P + e_a) - u(x_{\hat{P}} + e_a)] \geq 0, & P, \hat{P} \in \mathcal{Q}. \end{array}$$

Set X is $\sigma(L^1(Q), L^\infty(Q))$-compact, owing to the Dunford-Pettis theorem.[3] However, it is usually not norm compact.[4] At the same time, the agent's utility function is continuous w.r.t. the norm topology under mild technical conditions but usually fails to be continuous w.r.t. the $\sigma(L^1(Q), L^\infty(Q))$-topology (unless the agent is risk neutral).[5]

2.1.2. *The contracting problem over utility units.* Without continuity assumptions on the agent's utility function and compactness conditions on the contract space, it is difficult to establish the existence of a solution for the contracting problem over contingent claims. To overcome this problem, we follow an approach that goes back at least to Spear and Srivastava (1987) and Schättler and Sung (1993) and that has more recently been used by Sannikov (2008) and many others, and consider instead the following set

$$C := \{u(e_a + x) : x \in X\}$$

of the agent's utility levels as the new contract space. Since u is continuous, C is $\sigma(L^1(Q), L^\infty(Q))$-compact. This leads to the following equivalent problem over a compact contract space: find a menu $c : \mathcal{Q} \to C$ that maximizes

$$\inf_{\kappa \in \mathbb{K}} \left\{ \int_{\mathcal{Q}} E_{P'} \left[v \left(e_p + e_a - u^{-1}(c_P) \right) \right] \kappa(dP) + \alpha(\kappa) \right\}$$

$$\text{(3)} \qquad \begin{array}{lll} \text{subject to} & P \mapsto E_{P'} \left[v \left(e_p + e_a - u^{-1}(c_P) \right) \right] & \text{is measurable,} \\ & E_P[c_P - u(e_a)] \geq 0, & P \in \mathcal{Q} \\ & E_P[c_P - c_{\hat{P}}] \geq 0, & P, \hat{P} \in \mathcal{Q}. \end{array}$$

Although the change of variables helps to overcome the continuity problem (at the level of utility units, the agent's utility functional is linear), a new problem emerges: space $L^\infty(Q)$ is essentially never separable, and hence, set C cannot be expected to be metrizable when equipped with the weak topology. We show in Section 4 how our general existence result from Section 3 can be used to overcome this problem, and to establish the existence of an optimal risk-sharing rule under the preceding

[3] The Dunford-Pettis theorem states that a family of random variables $(X_i)_{i \in I} \subset L^1(Q)$ where I is an arbitrary index set is uniformly integrable if and only if it is relatively compact for the weak topology $\sigma(L^1(Q), L^\infty(Q))$; see (Föllmer and Schied, 2011, Theorem A.45).

[4] For the lack of norm compactness, observe as an illustration that when $-e_a < e_p$ are both constants, $\Omega = [0, 1]$, and Q is Lebesgue measure on $[0, 1]$, then X contains a sequence of $\{-e_a, e_p\}$-valued functions that oscillate increasingly, thus having no accumulation point w.r.t. Q-a.s. convergence and hence neither in $L^1(Q)$-norm. By the Riesz lemma, the unit ball $\{x : \|x\| \leq 1\}$ in a normed vector space $(X, \|\cdot\|)$ is compact if and only if X is finite dimensional.

[5] Although (2) is a static problem, an extension to continuous time, as in Mirrlees and Raimondo (2013), is possible.

assumptions on utility functions u and v if all the densities in $\mathcal{Q}$ are uniformly $L^\infty(Q)$ bounded, that is, uniformly norm bounded.

2.2. **A reinsurance model with utilities on the whole real line.** Let us now consider a modification of the preceding reinsurance model. We assume again that risk transfer occurs over a set of random variables $\mathcal{X}$. The agent is endowed with a *bounded* initial claim e_a, his utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is still defined on the positive half-line but the principal's utility function $v : \mathbb{R} \to \mathbb{R}$ is now defined on the whole real line, for example, it is the exponential utility function. Thus, in contrast to Subsection 2.1, the principal's payoff is not subject to a non-negativity constraint. In this case, we choose as our set of admissible transfers a set

$$X \subset \{x \in \mathcal{X} : -e_a \leq x\} \tag{4}$$

that is closed in $L^0(Q)$, convex and bounded in Q-probability. For instance, if $\mathcal{X}$ is convex we may consider all transfers that are bounded below by $-e_a$ and bounded above by some positive random variable $y \in L^0(Q)$. Unlike in the previous subsection, set X is not necessarily weakly compact. Nonetheless, we prove in Subsection 4.1.2 that set $C := \{u(e_a + x) : x \in X\}$ of the agent's utility levels is again $\sigma(L^1(Q), L^\infty(Q))$-compact under the above assumptions on X if u satisfies a certain asymptotic elasticity condition. Moreover, we prove that the contracting problem (3) has a solution if the agent's maximal attainable utility from reinsurance is finite and the type space $\mathcal{Q}$ is $L^\infty(Q)$-norm compact.

Remark 2.1. If the principal's utility function v is bounded as it was assumed in the preceding subsection, then the type space only needs to be norm bounded. In the present setup, we can no longer assume that v is bounded because it is now defined on the whole real line.

2.3. **A hedging problem with financial markets.** We close this section with a financial market framework for the general setting of Section 2.2. Specifically, we consider an incomplete continuous-time financial market for which the asset price dynamics is described by a d-dimensional semi-martingale (S_t) defined on a filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F}, Q)$. We consider a contracting problem between an investor (the principal) and a broker (the agent). The broker's assessment of the financial market dynamics is private knowledge. We assume that the principal cannot (or does not want to) trade in the financial market, either because she has no access to the market or because she finds it too costly. Instead, she may hedge her risk by exchanging payoffs with the agent. The agent is endowed with a bounded claim e_a and accepts all transfers that he could super-replicate at zero initial investment by trading in the financial market over the time interval $[0, T]$. Here, a contingent claim $e \in L^0(Q)$ is called super-replicable at zero initial investment if an admissible trading strategy π exists such that the resulting gains from trading satisfy $e \leq (\pi \cdot S)_T := \int_0^T \pi_t dS_t$. An admissible trading strategy is an $(\mathcal{F}_t)$ adapted d-dimensional stochastic process (π_s) such that $(\pi \cdot S)_T$ is uniformly bounded from below. Let $\mathcal{X}$ be the set of super-replicable claims. We prove in Subsection 4.1.3 that the resulting set X in (4) is indeed closed in $L^0(Q)$, convex and bounded in Q-probability.

3. An Abstract Adverse Selection Problem

Let (Θ, τ^Θ) and (C, τ^C) be topological spaces. We endow each of them with their Borel sigma-algebras, denoted $\mathcal{B}(\Theta)$ and $\mathcal{B}(C)$ respectively. Space C is the *space of contracts* that the principal may offer the agent. Space Θ is the *space of possible types* that the agent may have.

3.1. Menus.
If the agent is of type $\theta \in \Theta$ and contract $c \in C$ is chosen, the agent's utility is denoted by $U(\theta, c)$ and the principal's utility is denoted by $V(\theta, c)$. Let

$$\mathbb{K} \neq \emptyset$$

be a set of regular Borel probability measures on Θ. Set $\mathbb{K}$ comprises the principal's beliefs about the agent's type distribution.

Assumption 3.1. Throughout, we make the following standing assumptions:

(a) (C, τ^C) and (Θ, τ^Θ) are compact Hausdorff spaces.
(b) $U : \Theta \times C \to \mathbb{R}$ is jointly continuous on $\Theta \times C$.
(c) $V : \Theta \times C \to \mathbb{R}$ is jointly upper semicontinuous on $\Theta \times C$.

Remark 3.2. The standard situation we focus on is one in which C and Θ are subsets of Banach spaces E and E' that form a dual pair $\langle E, E' \rangle$, where τ^C is the weak topology on C and τ^Θ is a norm topology on Θ, and where $U(\theta, c)$ is a continuous function of the duality product $\langle c, \theta \rangle$.

The family of all τ^C-closed subsets of C (excluding $\emptyset$) is denoted by $\mathrm{CL}(C)$.

Definition 3.3. A *menu* is a non-empty closed subset of contracts. In other words, menus are the elements of $\mathrm{CL}(C)$.

A menu $D \in \mathrm{CL}(C)$ is automatically compact. Whenever C is not metrizable, we cannot introduce the familiar Hausdorff distance on $\mathrm{CL}(C)$. Instead, we equip $\mathrm{CL}(C)$ with the Fell topology τ_F. We recall the Fell topology and the related concept of Kuratowski-Painlevé convergence[6] in Appendix A. Then, in Appendix B, we also prove the following well-known result for the reader's convenience:

Lemma 3.4. *Space* $(\mathrm{CL}(C), \tau_F)$ *is compact Hausdorff.*

3.2. Agent.
Let us consider the agent's utility over menus. Given a menu $D \in \mathrm{CL}(C)$, an agent of type θ derives as utility the maximum $U^*(\theta, D)$ he can attain out of the contracts in the menu:

$$U^* : \Theta \times \mathrm{CL}(C) \to \mathbb{R}, \qquad U^*(\theta, D) := \sup_{d \in D} U(\theta, d).$$

Proposition 3.5. U^* *is continuous on* $(\Theta \times \mathrm{CL}(C), \tau^\Theta \times \tau_F)$.

For the agent let Φ be the optimal contracts, depending on types and menu:

$$\Phi : \Theta \times \mathrm{CL}(C) \twoheadrightarrow C, \qquad \Phi(\theta, D) := \arg\max_{d \in D} U(\theta, d). \tag{5}$$

Set $\Phi(\theta, D)$ contains all contracts within menu D, which are optimal for the agent of type θ.

[6] Limits in this sense are where the sets accumulate. Note that we have to work with nets instead of sequences unless the contract set is metrizable.

Proposition 3.6. Φ *is a jointly upper hemicontinuous correspondence with compact non-empty values.*

To formulate the individual rationality constraint, we assume that the agent agrees with a proposed menu only if he is left better off than if he had just retained his given (type-dependent) reservation utility $\underline{u}$, that is, if

$$U^*(\theta, D) \;\geq\; \underline{u}(\theta), \qquad \text{for all } \theta \in \Theta. \tag{6}$$

We assume that function $\underline{u} : \Theta \to \mathbb{R}$ is common knowledge and the following standing assumption holds:[7]

Assumption 3.7. There exists $c^* \in C$ such that

$$U(\theta, c^*) \geq \underline{u}(\theta), \text{ for all } \theta \in \Theta.$$

To describe the set of individually rational menus, let $T : \Theta \twoheadrightarrow \mathrm{CL}(C)$ be the correspondence defined by

$$\begin{aligned} T(\theta) \;&:=\; \{D \in \mathrm{CL}(C) \,:\, U^*(\theta, D) \geq \underline{u}(\theta)\} \\ &=\; U^*(\theta, \cdot)^{-1}[\underline{u}(\theta), \infty), \end{aligned}$$

which is closed-valued (by Proposition 3.5) and hence compact-valued. By Assumption 3.7, $c^* \in C$ exists such that $\{c^*\} \in T(\theta)$ for all $\theta \in \Theta$. As a result, we might define, with a slight abuse of notation,

$$T \;:=\; \bigcap_{\theta \in \Theta} T(\theta) \;\subset \mathrm{CL}(C), \tag{7}$$

which is compact and non-empty. We call T the set of individually rational menus.

3.3. **Principal.** Given an agent's type $\theta \in \Theta$ and a menu D, the principal will only consider those contracts in D that are optimal w.r.t. her utility for this given type. This defines a type-dependent utility for the principal, namely:

$$V^* : \Theta \times \mathrm{CL}(C) \to \mathbb{R}, \qquad V^*(\theta, D) \;:=\; \sup_{d \in \Phi(\theta, D)} V(\theta, d).$$

The counterpart to Proposition 3.5 is the following:

Proposition 3.8. V^* *is upper semicontinuous on* $(\Theta \times \mathrm{CL}(C), \tau^{\Theta} \times \tau_F)$.

To specify the principal's utility, we recall that the (non-empty) set $\mathbb{K}$ consists of regular Borel probability measures on the type space and thus it collects the principal's possible priors over the type of the agent. We fix a penalty function

$$\alpha : \mathbb{K} \to \mathbb{R}$$

and define the principal's utility of a menu D in variational form as follows:

$$\inf_{\kappa \in \mathbb{K}} \left\{ \int_{\theta \in \Theta} V^*(\theta, D) d\kappa(\theta) + \alpha(\kappa) \right\}. \tag{8}$$

[7]The assumption states that at least one individually rational contract exists. For instance, one formally expects that the action "c^* =do nothing" (often $c^* = 0$ is a concrete model) belongs to C and $\underline{u}(\theta) := U(\theta, c^*)$ holds for all $\theta \in \Theta$.

We stress that these integrals are well-defined if we assume that the type space is compact, as we have assumed so far, or if merely the principal's utility function is bounded. The principal's optimization problem over menus reads:

$$\sup_{D\in T}\inf_{\kappa\in\mathbb{K}}\left\{\int_{\theta\in\Theta} V^*(\theta,D)\,d\kappa(\theta)+\alpha(\kappa)\right\} \tag{9}$$

The following result states a general existence of optimal contracts.

Theorem 3.9. *Under the standing Assumptions 3.1 and 3.7, an optimal solution $D^*\in T$ to problem (9) exists.*

We now present two extensions of the preceding theorem. The first deals with non-compact type spaces but assumes that the principal's utility is bounded.

Proposition 3.10. *Let V be uniformly bounded. Further suppose that Assumption 3.7 holds, as well as Assumptions 3.1 where the compactness of the type space (Θ,τ^Θ) is dropped. Then, an optimal solution $D^*\in T$ to the problem (9) exists.*

The second extension applies to a situation in which the principal evaluates her performance according to a worst-case approach. In this case, the existence of an optimal menu can be shown under considerably weaker conditions. The proof is a straightforward modification of the proof of Theorem 3.9.

Proposition 3.11. *Assume that Assumption 3.7 and the following conditions hold:*

(a') (C,τ^C) is a compact Hausdorff space.
(b') $U:\Theta\times C\to\mathbb{R}$ is continuous on C for any fixed type θ.
(c') $V:\Theta\times C\to\mathbb{R}$ is upper semicontinuous on C for any fixed type θ.

Then, an optimal solution $D^*\in T$ to the problem of maximizing $D\mapsto\inf_{\theta\in\Theta}V^*(\theta,D)$ over T exists.

3.4. **On optimal contract mechanisms.** A contract mechanism is a mapping $\varphi:\Theta\to C$ from the type space into the contract space. A contract mechanism is individually rational if

$$U(\theta,\varphi(\theta)) \;\geq\; \underline{u}(\theta),\qquad \text{for all } \theta\in\Theta. \tag{10}$$

and incentive compatible if

$$U(\theta,\varphi(\theta)) \;\geq\; U(\theta,\varphi(\theta')),\qquad \text{for all } \theta,\theta'\in\Theta. \tag{11}$$

We denote by M the set of all contract mechanisms that are individually rational and incentive compatible. In defining a principal-agent problem over optimal mechanisms, rather than over contract menus, the first difficulty is that the integral

$$\int_\Theta V(\theta,\varphi(\theta))d\kappa(\theta),$$

is not defined unless $\theta\mapsto V(\theta,\varphi(\theta))$ is measurable. Thus, we introduce set

$$\hat{M}:=\{\varphi\in M:\; V(\cdot,\varphi(\cdot)) \text{ is measurable}\}$$

and define the principal-agent problem over contract mechanisms as follows:

$$\sup_{\varphi\in\hat{M}}\inf_{\kappa\in\mathbb{K}}\left\{\int_\Theta V(\theta,\varphi(\theta))d\kappa(\theta)+\alpha(\kappa)\right\}. \tag{12}$$

Theorem 3.12. *Under the assumptions of either Theorem 3.9 or Proposition 3.10, an optimal solution $\varphi^* \in \hat{M}$ for problem (12) exists.*

The proof of Theorem 3.12 relies on Theorem 3.9/Proposition 3.10. In fact, this proof reveals that delegated contracting implemented via contract menus is equivalent to centralized contracting implemented via contract mechanisms.

4. Applications

In this section, we consider several examples for which we can establish the existence of an optimal contract by using the results of Section 3. We are mostly interested in the case in which the type space Θ is a subset of $L^\infty(Q)$ and the contract space C in utility units is a $\sigma(L^1(Q), L^\infty(Q))$-compact subset of $L^1(Q)$. The agent's utility function then depends on the duality product

$$\langle \cdot, \cdot \rangle : L^1(Q) \times L^\infty(Q) \to \mathbb{R}.$$

The duality product is *sequentially* continuous when $L^\infty(Q)$ is equipped with the $\sigma(L^\infty(Q), L^1(Q))$-topology (see (Aliprantis and Border, 2006, Theorem 9.37)) but may fail to be continuous in general. Therefore, to guarantee continuity of the duality product we need to consider the norm topology on $L^\infty(Q)$ and assume that Θ is subset of $L^\infty(Q)$.

First, we revisit the motivating examples presented in Section 2. Subsequently, we consider two examples in which the agent can trade in a financial market after transacting with the principal.

4.1. **The motivating examples revisited.**

4.1.1. *The reinsurance model of Section 2.1.* Since the agent's utility function is strictly increasing, the contracting problems (2) and (3) are equivalent. We apply our abstract existence results to the contracting problem over utility levels. In the notation of Section 3, τ^C is the $\sigma(L^1(Q), L^\infty(Q))$-topology on the contract space

$$C = \{u(e_a + x) : x \in X\}$$

of utility levels, τ^Θ is the $L^\infty(Q)$ norm-topology on the type space $\Theta = \mathcal{Q}$, whereas

$$U(P, c) = E_P[c] \quad \text{and} \quad V(P, c) = E_{P'}[v(e_p + e_a - u^{-1}(c))].$$

The utility function U is jointly continuous for our choice of topologies. To establish the upper semicontinuity of the utility function V, we recall the following result; see Proposition 2.10 in Barbu and Precupanu (2012).

Lemma 4.1. *Let L be a normed vector space and let $f : L \to \mathbb{R}$ be concave. Then, norm upper-semicontinuity of f is equivalent to the weak upper-semicontinuity of f.*

Since u^{-1} is convex by definition, $-u^{-1}(\cdot)$ is concave, and hence V is concave because v is concave. Since V is also $L^1(Q)$-norm upper semicontinuous by Fatou's lemma, it follows that V is (jointly) weakly upper semicontinuous. Moreover, if $\mathcal{Q} \subset L^\infty(Q)$ is norm-bounded then V is uniformly bounded because v is bounded. Thus, if an individually rational contract exists, then the existence of a solution to the contracting problem follows from Theorem 3.12. By reverting the change of variables from X to C, we obtain an optimal mechanism for problem (2).

4.1.2. *The reinsurance model of Section 2.2.* This section analyzes the model of Section 2.2 in a broader context. Our analysis is inspired by (Kramkov and Schachermayer, 2003, Section 2). We assume that the agent's utility function $u : \mathbb{R}_+ \to \mathbb{R}$ satisfies the following conditions.

Assumption 4.2. The utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is lower bounded, strictly increasing, strictly concave, continuously differentiable, and it satisfies the Inada condition and the following asymptotic elasticity (AE) condition:

$$\limsup_{z\to\infty} \frac{zu'(z)}{u(z)} < 1.$$

The (AE) condition was first introduced by Kramkov and Schachermayer (1999) to show that optimal solutions exist for a wide class of utility maximization problems.

Example 4.3. The CRRA class of utility functions $u(z) = \frac{z^p}{p}$ for $p \in (0,1)$ satisfies Assumption 4.2.

To handle utilities v on the whole real line, let $\mathcal{X} \subset L^0(Q)$ be the contract space in payoff units that satisfies the following condition:

Assumption 4.4. Set $\mathcal{X}$ is convex and closed in $L^0(Q)$. Further, for any $\lambda \in \mathbb{R}$ the following set is bounded in Q-probability:

$$\{x \in \mathcal{X} : x \geq \lambda\}.$$

We assume that the agent is endowed with a bounded random variable $e_a \in L^\infty(Q)$ and introduce the set of feasible risk transfers

$$\begin{aligned} \mathcal{X}(e_a) \; &:= \; \{x \in L^0(Q) : 0 \leq x \leq e_a + \bar{x} \text{ for some } \bar{x} \in \mathcal{X}\} \\ &= \; (e_a + \mathcal{X} - L^0_+(Q)) \cap L^0_+(Q). \end{aligned} \tag{13}$$

This set represents in abstract terms the set of risk transfers that are feasible to the agent. It follows from Assumption 4.4 and the Komlos lemma that $\mathcal{X}(e_a)$ is convex, solid,[8] and closed in $L^0(Q)$. Next, we introduce the polar set of $\mathcal{X}(e_a)$, namely,

$$\mathcal{X}(e_a)^0 := \{x^0 \in L^0_+(Q) : E_Q[xx^0] \leq 1 \text{ for all } x \in \mathcal{X}(e_a)\}$$

along with the convex conjugate u^* of u:

$$u^*(y) := \sup_{z\in\mathbb{R}}\{u(z) - zy\}.$$

Lemma 4.5. *Under Assumptions 4.2 and 4.4, assume that some $y > 0$ and a random variable $x^0 \in \mathcal{X}(e_a)^0$ exist such that*

$$E_Q[u^*(x^0 y)] < \infty. \tag{14}$$

Then, the contract space in utility units $C := u(\mathcal{X}(e_a)) = \{u(x) : x \in \mathcal{X}(e_a)\}$ is contained in $L^1(Q)$ and is $\sigma(L^1(Q), L^\infty(Q))$-compact.

The following lemma gives a sufficient condition that guarantees condition (14). It essentially states that the agent's maximal attainable utility from risk transfer is finite. The proofs of Lemmas 4.5 and 4.6 are presented in Appendix B.

[8] This means that if $x \in \mathcal{X}(e_a)$ and $y \leq x$, then $y \in \mathcal{X}(e_a)$.

Lemma 4.6. *Let $\|e_a\|$ be considered a constant function. If*

$$\sup_{x\in\mathcal{X}(\|e_a\|)} E_Q[u(x)] < \infty, \tag{15}$$

where $\mathcal{X}(\|e_a\|)$ has been introduced in (13), then condition (14) holds.

With Lemma 4.5 and 4.6 at hand, we can now use the same arguments as in the preceding subsection along with Theorem 3.12 to establish the following result. It shows that the contracting problem introduced in Section 2.2 has a solution if the agent's utility from risk sharing is finite.

Theorem 4.7. *Under Assumptions 4.2 and 4.4, if (15) holds and if $\mathcal{Q} \subset L^\infty(Q)$ is norm compact,*[9] *then the contracting problem of finding a mechanism $x : \mathcal{Q} \to \mathcal{X}(e_a)$ that maximizes*

$$\begin{aligned} &\inf_{\kappa\in\mathbb{K}} \left\{ \int_{\mathcal{Q}} E_{P'}\left[v\left(e_p - e_a + x_P\right)\right]\kappa(dP) + \alpha(\kappa) \right\} \\ \text{subject to}\quad & P \mapsto E_{P'}\left[v\left(e_p - e_a + x_P\right)\right], \qquad \text{is measurable} \\ & E_P[u(x_P) - u(e_a)] \geq 0, \qquad P \in \mathcal{Q} \\ & E_P[u(x_P) - u(x_{\hat{P}})] \geq 0, \qquad P, \hat{P} \in \mathcal{Q}, \end{aligned} \tag{16}$$

has a solution as soon as an individually rational contract exists (this is the case if, for instance, $0 \in \mathcal{X}$).

4.1.3. *The hedging problem of Section 2.3.* The analysis of the hedging problem is related to the analysis in Cvitanić, Schachermayer, and Wang (2001). A trading strategy π is called *admissible*, if the gain from trade, modelled as a stochastic integral $(\pi.S)_T = \int_0^T \pi_t dS_t$, is well–defined and lower bounded. We put

$$\mathcal{X}_0 := \{x : x \leq (\pi.S)_T,\ \pi \text{ is admissible}\},$$

and

$$u_{max} := \sup_{x\in\mathcal{X}_0} E_Q[u(x + e_a)].$$

For the considered financial market, we require a form of no-arbitrage:

Assumption 4.8. A probability measure $Q' \approx Q$ exists such that the process $t \mapsto (\pi.S)_t$ is a Q'-local martingale for every admissible trading strategy π. Moreover, $u_{max} < \infty$.

The key observation, made in (Kramkov and Schachermayer, 2003, Lemma 1) for the case without random endowments, but easily obtainable in our setting as well, is the following:

Lemma 4.9. *Assumptions 4.2 and 4.8 imply $\sigma(L^1(Q), L^\infty(Q))$-compactness of $C = \{u(x) : x \in (\mathcal{X}_0 + e_a) \cap L^0_+(Q)\}$.*

Again, in the notation of Section 3, τ^C is the $\sigma(L^1(Q), L^\infty(Q))$-topology on C of utility levels, τ^Θ is the $L^\infty(Q)$ norm-topology on set $\Theta = \mathcal{Q}$, and

$$U(P, c) = E_P[c] \quad \text{and} \quad V(P, c) = E_{P'}[v(e + e_a - u^{-1}(c))].$$

[9]Special cases of norm-compact subsets of $L^\infty(Q)$ can be constructed using Arzela and Ascoli's theorem when Ω is a topological space.

The agent's utility U is jointly continuous and the principal's utility V is upper semicontinuous. As a result, the hedging problem introduced in Section 2.3 has a solution if the set of densities in $\mathcal{Q}$ is $L^\infty(Q)$-norm-compact. This result is formally covered by Theorem 4.7, but not quite, since hypothesis $u_{max} < \infty$ is weaker than (15).

4.2. **Market optimized risk sharing.** In the previous examples, the agent had to decide whether to accept (or reject) a contract based on his portfolio after the risk transfer. In this section we consider two examples in which he decides to accept (or reject) a contract based on the "indirect utility" that arises after investing in a financial market about which he has private information. Specifically, we consider a continuous time financial market model with one risk-free and one risky asset. The price of the risk-free asset is normalized to one. The discounted price of the risky asset follows a geometric Brownian motion with drift:

$$\frac{dS_t}{S_t} = dW_t + f'(W_t)dt, \quad S_0 = s, \quad 0 \le t \le T$$

where W is a one-dimensional standard Brownian motion defined on the Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, Q)$ and $f \in \mathbb{F}$ where $\mathbb{F}$ is a class of real-valued twice continuously differentiable, uniformly bounded, uniformly equicontinuous functions $f : \mathbb{R} \to \mathbb{R}$ with common compact support that satisfy the normalization constraint $f(0) = 0$[10]. The closure $\bar{\mathbb{F}}$ of $\mathbb{F}$ w.r.t. the topology of uniform convergence is a compact subset of the set of continuous functions on $\mathbb{R}$ w.r.t. the supremum norm by the theorem of Arzela and Ascoli. By Girsanov's theorem, for any $f \in \bar{\mathbb{F}}$ there exists a unique equivalent martingale measure P_f whose density is given by

$$\frac{dP_f}{dQ} = e^{f(W_T)}.$$

We assume that the pricing kernel is private knowledge to the agent and choose as our type set the function space

$$\Theta := \bar{\mathbb{F}}$$

equipped with the topology of uniform convergence. In particular, the financial market is complete from the agent's point of view. The agent is endowed with a bounded claim $e_a \in L^\infty(Q)$. His preferences over payoffs are defined by an expected utility functional of the form $E_Q[u(\cdot)]$ for some Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$ that we choose to be the exponential (Subsection 4.2.1), respectively the logarithmic (Subsection 4.2.2) one. The principal is endowed with a bounded claim $e_p \in L^\infty(Q)$; her preferences over payoffs are defined by an expected utility functional of the form $E_Q[v(\cdot)]$ for some concave Bernoulli utility function $v : \mathbb{R} \to \mathbb{R}$. An admissible trading strategy for the agent is an adapted real-valued stochastic process vector ξ that satisfies $E_Q[\int_0^T \xi_t^2 dt] < \infty$.

[10]We think of f' as being a constant outside a large compact set. In this case, the price dynamics essentially reduces to geometric Brownian motion.

4.2.1. *Risk- sharing with hedging.* Let us first consider a risk-sharing problem in which the agent can hedge his risk by trading in the financial market after transacting with the principal. We assume that

$$u(y) = 1 - e^{-\alpha y}$$

with CARA parameter $\alpha > 0$. Given his initial endowment $e_a \in L^\infty(Q)$, the budget set of agent $f \in \bar{\mathbb{F}}$ is given by

$$\mathcal{B}(f) := \left\{x \in L^1(Q) : E_f[x - e_a] \leq 0\right\} \tag{17}$$

where E_f denotes the expectation under the pricing measure P_f. We notice that $E_f[e_a]$ is finite because P_f has a bounded density w.r.t. Q. The following result can be inferred from (Föllmer and Schied, 2011, Chapter 3, Example 3.37). It yields the optimal claim and the optimal utility from trading for any agent type.

Lemma 4.10. *The optimal attainable payoff over all admissible trading strategies for an agent of type f in the budget set (17) is*

$$x^* = -\frac{1}{\alpha}\log\frac{dP_f}{dQ} + E_f[e_a] + \frac{1}{\alpha}H(P_f|Q),$$

and the optimal utility from trading in the market is

$$E_Q[u(x^*)] = 1 - e^{-\alpha E_f[e_a] - H(P_f|Q)}.$$

Here, $H(P_f|Q) := E_f\left[\log(dP_f/dQ)\right] = E_Q\left[e^{f(W_T)}f(W_T)\right] < \infty$ denotes the relative entropy of P_f w.r.t. the Wiener measure Q.

Let $\mathcal{X} \subset L^1(Q)$ be a closed set of uniformly integrable financial positions $x : \Omega \to \mathbb{R}$. By the Dunford-Pettis theorem, this is equivalent to $\mathcal{X}$ being $\sigma(L^1(Q), L^\infty(Q))$-compact. We take $\mathcal{X}$, equipped with the $\sigma(L^1(Q), L^\infty(Q))$ topology as our set of contractible payoffs:

$$C = \mathcal{X}.$$

When offered a payoff $x \in \mathcal{X}$ the agent's endowment before trading is $e_a + x$. It follows from the above lemma that his optimal optimal utility after trading in the financial market, that is, his indirect utility function is given by

$$U\left(f, x\right) = 1 - e^{-\alpha E_f[e_a + x] - H(P_f|Q)}.$$

Since all functions $f \in \bar{\mathbb{F}}$ are uniformly bounded it follows from the dominated convergence theorem that the mapping $f \mapsto H(P_f|Q)$ is continuous. As a result, the mapping

$$(f, x) \mapsto U\left(f, x\right)$$

on $\Theta \times \mathcal{X}$ is jointly continuous. A direct computation shows that a mechanism $f \mapsto x_f$ is incentive compatible if and only if

$$E_f[x_f - x_{f'}] \geq 0, \qquad \text{for all } f, f' \in \bar{\mathbb{F}}.$$

We assume that at least one individually rational incentive compatible mechanism exists. The principal's utility from transacting with agent f is

$$V(x_f) = E_Q[v(e_P - x_f)].$$

Since v is concave, V is upper semicontinuous, and thus, it follows from Theorem 3.12 that the principal's optimization problem to find a mechanism $x : \bar{\mathbb{F}} \to C$ that maximizes

$$\inf_{\kappa \in \mathbb{K}} \left\{ \int_{f \in \bar{\mathbb{F}}} V(x_f)\, d\kappa(f) + \alpha(\kappa) \right\}$$

subject to the usual incentive compatibility, individual rationality, and measurability condition, has a solution.

4.2.2. *A model of optimal portfolio delegation.* In the previous applications, the principal's utility function was independent of the agent type ($f \in \bar{\mathbb{F}}$). In this section, we consider a simple model of optimal portfolio delegation in which the principal's utility function depends on the agent type. We now assume that the agent's Bernoulli utility function is

$$u(y) = \ln(y).$$

We retain the assumption that the drift of the stock price process is private information to the agent. The budget set of an agent of type f is the same as in (17). The following result can again be inferred from (Föllmer and Schied, 2011, Chapter 3, Example 3.43):

Lemma 4.11. *The optimal attainable payoff over all admissible trading strategies for an agent of type f in the budget set (17) is*

$$x^* = E_f[e_a] \frac{dQ}{dP_f},$$

and the optimal utility from trading in the market is

$$E_Q[u(x^*)] = \ln E_f[e_a] + H(Q|P_f) = \ln E_f[e_a] - E_Q[f(W_T)].$$

The principal, considered an investor, outsources her portfolio selection to a manager (the agent) who has private information about the financial market dynamics and whose investment decisions in the above-specified financial market cannot be monitored. Following Ou-Yang (2003); Backhoff and Horst (2016), we restrict the class of admissible contracts to linear ones: the principal can offer the agent some contingent claim plus a fraction of the gains or losses from trading. In particular, the agent is liable for possible losses. That is, a contract consists of some $\mathcal{F}$-measurable random variable x, the contingent claim, that we again assume to belong to some $\sigma(L^1(Q), L^\infty(Q))$-compact set $\mathcal{X}$ and some number $\beta \in [0,1]$, the proportion of wealth the agent can keep. That is,

$$C = \mathcal{X} \times [0,1],$$

equipped with the product topology. When offered a contract (x, β), the agent's wealth from an admissible trading strategy ξ is

$$e_a + x + \beta \int_0^T \xi_s dS_s.$$

By Lemma 4.11, since asset prices are martingales under measure P_f the agent's optimal utility after trading in the market can be described by the continuous indirect utility function

$$U(f, x) := \ln E_f[e_a + x] + H(Q|P_f).$$

In particular, the agent's optimal utility is independent of the performance part of his contract[11] and a mechanism $f \mapsto (x_f, \beta_f)$ is incentive compatible if and only if

$$E_f[x_f - x_{f'}] \geq 0 \quad \text{for all } f, f' \in \bar{\mathbb{F}}.$$

It is important to note that although the agent's optimal utility is independent of the performance part of his contract, the optimal trading strategy ξ^* and the income w^* to the principal do both depend on the agent's type. Since the optimal claim for the agent can be decomposed as $x^* = x + e_a + \beta \int_0^T \xi_t^* dS_t$ the optimal gains from trading are given by

$$\int_0^T \xi_t^* dS_t = \frac{1}{\beta}\left(E_f[e_a + x]\frac{dQ}{dP_f} - x - e_a\right)$$

and hence, the principal's income from trading is given by

$$w^* = \frac{1-\beta}{\beta}\left(E_f[e_a + x]\frac{dQ}{dP_f} - (e_a + x)\right).$$

Hence, unlike in the previous section, the principal's utility function now depends on the type of the agent with whom she interacts. Her utility when interacting with a type f agent and offering a contract (x, β) is given by

$$V(f, (x, \beta)) = E_Q\left[v\left(e_p - x + \frac{1-\beta}{\beta}\left(E_f[e_a + x]\frac{dQ}{dP_f} - (e_a + x)\right)\right)\right].$$

This function is jointly upper semicontinuous. We can now use the same arguments as in the previous subsection to conclude that the principal's optimization problem of finding a mechanism $(x, \beta) : \bar{\mathbb{F}} \to C$ that maximizes

$$\inf_{\kappa \in \mathbb{K}} \left\{ \int_{\bar{\mathbb{F}}} V(f, (x_f, \beta_f))\, d\kappa(f) + \alpha(\kappa) \right\}$$

subject to the usual incentive compatibility, individual rationality, and measurability condition, has a solution (provided a feasible mechanism exists).

5. Concluding Remarks

We considered a very general setting for a mechanism design problem in the presence of adverse selection. Under mild hypotheses we proved that optimal contracts exist and that centralized contracting implemented via contract mechanisms is equivalent to delegated contracting implemented via contract menus. The guiding principle of our work was to use the utility levels of the agent as the relevant contract variables. When doing so, the agent's utility function is given by a bilinear form. This does not only significantly simplify the incentive compatibility condition but also yields a natural duality between the type and the contract type. Simultaneously, it naturally results in a framework in which the relevant topological spaces for the contracting problem may lack metrizability (and possibly separability too), as we illustrated with various examples.

Our study was greatly influenced by the works Page (1992, 1997), and in particular, by Page's idea of considering contract menus rather than mechanisms. However, the lack of metrizability precludes us from applying the results therein, because it

[11] This has previously been observed in Ou-Yang (2003) and Backhoff and Horst (2016).

rules out the use of measurable selection results. For instance, in Page (1992), the contract space is given by sequentially compact subsets of L^∞. Under an additional separability assumption that would not be needed in our setting, the contract space is metrizable. Compared with Page (1992), we reverse the role of the contract and type space: in all our examples, the contract spaces are subsets of L^1, which is very natural when the agent is an expected utility maximizer, whereas the type spaces are subsets of L^∞. The choice of norm-bounded, respectively compact, type spaces in L^∞ is an immediate consequence of the fact that the agent's utility function is given by a bilinear form after the transformation of variables.

To the best of our knowledge, the existence of mechanisms does not follow from existing Komlos-type results used or established in other studies, such as Page (1991); Balder (1996, 1990); Balder and Hess (1996). Our setting does not seem to fulfill the hypotheses needed for such arguments. Under various topological assumptions on the contract space, it has been shown that if the principal knows the distribution μ of the agent types and if the set of incentive compatible mechanisms is convex,[12] then a sequence $\{\varphi_k\}$ of strongly measurable incentive compatible mechanisms admits an almost surely Cesaro convergent subsequence, that is, a subsequence $\{n_k\}$ exists such that

$$\frac{1}{n_k}\sum_{i=1}^{n_k}\varphi_i(\theta) \to \varphi^*(\theta) \quad \mu\text{-a.s. as } k \to \infty.$$

Our setting does not seem to fulfill the hypotheses needed for such arguments. More strikingly, our framework allows to consider very general sets of beliefs about the type distribution. By contrast, Komlos-type arguments *always* need a reference distribution w.r.t. which the principal's beliefs are absolutely continuous. Such a reference distribution does not exist if the principal evaluates her performance according to a worst-case approach w.r.t. the agent's type. A further advantage of the approach used here is that we show that suitably-measurable contract mechanisms, and contract menus, are largely equivalent even though we cannot use measurable selection arguments. In other words, we do not require mechanisms to be measurable functions; only the principal's utility from implementing these mechanisms needs to be measurable.

Appendix A. Abstract Results

A.1. **The Fell Topology.** We denote by C a compact Hausdorff topological space and recall that $\mathrm{CL}(C)$ stands for the family of all non-empty closed subsets of C.

Definition A.1. The *Fell topology* on $\mathrm{CL}(C)$, which we denote τ_F, is the topology generated by the subbase consisting of all sets of the form

$$V^- := \{A \in \mathrm{CL}(C) : A \cap V \neq \emptyset\},$$

and

$$W^+ := \{A \in \mathrm{CL}(C) : A \subset W\},$$

where V and W are non-empty open subsets of C.

[12]This is the case if the contract space is convex and the agent's utility function is affine. The latter is guaranteed if mixed contracts are considered.

The definition of Fell topology when C is only Hausdorff can be found in (Beer, 1993, Definition 5.1.1.), and it reduces to Definition A.1 in the present case of C being compact.

Let us now recall the notions of lower and upper closed limits for nets of sets. For a net $\{A_\lambda\}_{\lambda\in\Lambda}$ in $\mathrm{CL}(C)$, let $\mathrm{Li}(A_\lambda)$ denote the set of all limit points of $\{A_\lambda\}_{\lambda\in\Lambda}$. These are the points $c \in C$ such that each neighborhood of c intersects A_λ for all λ in some residual subset of Λ. By contrast, $\mathrm{Ls}(A_\lambda)$ denotes the set of all cluster points of $\{A_\lambda\}_{\lambda\in\Lambda}$. These are the points $c \in C$ such that each neighborhood of c intersects A_λ for all λ in some cofinal subset of Λ. We always have $\mathrm{Li}(A_\lambda) \subset \mathrm{Ls}(A_\lambda)$.

Since C is a compact Hausdorff space, convergence w.r.t. the Fell topology and the notion of Kuratowski-Painlevé convergence of nets coincide (Beer, 1993, Theorem 5.2.6). Hence, we focus on the latter convergence, which is easier to apply in the proofs.

Definition A.2. Let$\{A_\lambda\}_{\lambda\in\Lambda}$ be a net in $\mathrm{CL}(C)$ and $A \in \mathrm{CL}(C)$. We say that $\{A_\lambda\}_{\lambda\in\Lambda}$ converges in the *Kuratowski-Painlevé sense* to A (denoted $A = {}^K\lim_\lambda A_\lambda$) if $\mathrm{Li}(A_\lambda) = A = \mathrm{Ls}(A_\lambda)$.

A.2. **Semicontinuity of an integral functional.** We made use of the following result, which is nontrivial in the present nonmetrizable setting:

Lemma A.3. *Let X, Y be compact Hausdorff spaces, λ a regular probability measure on $\mathcal{B}(X)$, and $g : X \times Y \to \mathbb{R}$ a jointly upper semicontinuous function. Then,*

$$Y \ni y \mapsto \int_X g(x,y)d\lambda(x),$$

is well-defined and upper semicontinuous.

Proof. Compact Hausdorff spaces are completely regular, and in a completely regular space every finite upper semicontinuous function equals the pointwise infimum of its continuous majorants:

$$g(x,y) = \inf_{c\in S} c(x,y), \quad S := \{c : c \geq g \text{ everywhere}, c \text{ continuous}\};$$

see for example (Bourbaki, 1958, Proposition 7, No 7, Chapter IX.10). Let us assume for the moment[13] that for $c : X \times Y \to \mathbb{R}$ continuous we have

$$y_\alpha \to y \text{ implies } \sup_x |c(x,y_\alpha) - c(x,y)| \to 0. \tag{18}$$

From the duality of continuous functions and finite measures, we find that

$$y \mapsto \int_X c(x,y)d\lambda(x),$$

is continuous. Next, note that set S is downwards directed, and hence, in particular, it can be seen as a decreasing net. In line with (Baranov and Woracek, 2009, Proposition 2.13), where the regularity of λ is needed, applied to the (upper-bounded)

[13] Actually, this follows immediately from the fact that the topologies on X and Y are generated by uniformities: we prefer to give self-contained arguments, perhaps paving the way for future extensions.

upper semicontinuous function g, we easily deduce

$$\int_X g(x,y)d\lambda(x) = \int_X \left(\inf_{c\in S} c(x,y)\right) d\lambda(x) = \inf_{c\in S} \int_X c(x,y)d\lambda(x).$$

Thus, the function on the l.h.s. is an infimum of continuous functions and therefore upper semicontinuous.

Let us return to (18), which would be trivial in the metrizable setting. We fix y and let $\epsilon > 0$. For each x we have by continuity the existence of neighborhoods U_x^X and V_x^Y of x and y respectively, such that

$$|c(\bar{x},\bar{y}) - c(x,y)| \leq \epsilon/2 \text{ for all } \bar{x} \in U_x^X, \forall \bar{y} \in V_x^Y.$$

We now cover $X \times \{y\}$, a compact, with $\bigcup_{x\in X} U_x^X \times V_x^Y$. Therefore, we obtain $x_1, \dots, x_n$ such that

$$X \times \{y\} \subset \bigcup_{i\leq n} U_{x_i}^X \times V_{x_i}^Y,$$

and we introduce $V := \cap_{i\leq n} V_{x_i}^Y$. This is an open set containing y, and thus,

$$X \times \{y\} \subset \bigcup_{i\leq n} U_{x_i}^X \times V.$$

Therefore, given $\bar{x}$ arbitrary and $\bar{y} \in V$, there is some $i \leq n$ such that $(\bar{x},\bar{y}) \in U_{x_i}^X \times V$ and thus

$$\begin{aligned} |c(\bar{x},y) - c(\bar{x},\bar{y})| &\leq |c(\bar{x},y) - c(x_i,y)| + |c(x_i,y) - c(\bar{x},\bar{y})| \\ &\leq \epsilon/2 + \epsilon/2. \end{aligned}$$

This shows that $\sup_{x\in X} |c(x,y) - c(x,\bar{y})| \leq \epsilon$ for all $\bar{y} \in V$, as desired. □

Corollary A.4. *Let X be a Hausdorff space, and let Y be a compact Hausdorff space. Let λ a regular probability measure on $\mathcal{B}(X)$, and $g : X \times Y \to \mathbb{R}$ a jointly upper semicontinuous bounded function. Then*

$$Y \ni y \mapsto \int_X g(x,y)d\lambda(x),$$

is well-defined and upper semicontinuous.

Proof. Let (X_n) be an increasing sequence of compact subsets of X such that $\lambda(X_n) \uparrow 1$, the existence of which is guaranteed by the (inner) regularity of λ. By Lemma A.3, the mappings

$$G_n(y) := \int_{X_n} g(x,y)d\lambda(x)$$

are well-defined and upper semicontinuous for each $n \in \mathbb{N}$. Subtracting from g its lower bound, we may assume that g is non-negative. For any y and λ a.e. x we have $1_{X_n}(x)g(x,y) \nearrow g(x,y)$, and hence, by sequential monotone convergence

$$\lim_{n\to\infty} G_n(y) = \sup_n G_n(y) = \int_X g(x,y)d\lambda(x) =: G(y).$$

Thus, G is well-defined, and its upper semicontinuity remains to be proved. Since g is bounded, we have $\|G_n - G\|_\infty \to 0$ as $n \to \infty$. Hence, if $y_\alpha \to y$ is a net, then for any $\epsilon > 0$ and $n = n(\epsilon)$ big enough

$$\limsup G(y_\alpha) \leq \epsilon + \limsup G_n(y_\alpha) \leq \epsilon + G_n(y) \leq \epsilon + G(y),$$

which concludes the proof. ☐

Appendix B. Proofs

We now present the proof of Lemma 3.4, which actually follows by (Beer, 1993, Proposition 5.1.2 and Exercise 5.1.4(a)). For the reader's convenience, we provide the complete argument:

Proof of Lemma 3.4. Since C is compact Haussdorf, for each $A_1, A_2 \in \mathrm{CL}(C)$ disjoint, we can find $U_1, U_2 \in \tau^C$ disjoint such that $A_i \subset U_i$. Then, U_1^+ and U_2^+ are disjoint open neighborhoods in τ_F of A_1 and A_2 respectively. Thus τ_F is Hausdorff.

To prove that τ_F is compact, we follow (Beer, 1993, Theorem 5.1.3). Let $\{V_\lambda : \lambda \in \Lambda\}$ and $\{W_\sigma : \sigma \in \Sigma\}$ be two families of non-empty open sets of C, such that

$$\mathrm{CL}(C) = \{V_\lambda^- : \lambda \in \Lambda\} \cup \{W_\sigma^+ : \sigma \in \Sigma\}.$$

By the Alexander subbase theorem, it suffices to check the existence of a finite subcovering for this kind of coverings, to obtain proof of compactness of τ_F. Note that if Λ is empty, then $C \in \cup_\sigma W_\sigma^+$, therefore, for some $\bar{\sigma} \in \Sigma$ we must have $W_{\bar{\sigma}} = C$ and consequently $W_{\bar{\sigma}}^+ = \mathrm{CL}(C)$ is a finite subcovering. Hence, we now assume that Λ is non-empty. If Σ is empty, then $\forall c \in C : \{c\} \in \cup_\lambda V_\lambda^-$, and thus, these V_λ's form an open covering of C. Then, we find $C = \cup_{i=1}^n V_{\lambda^n}$ and get

$$\mathrm{CL}(C) = \cup_{i=1}^n V_{\lambda^n}^-$$

is a finite subcovering.

Finally, if both Λ and Σ are non-empty, some $\sigma_0 \in \Sigma$ must exist such that $(W_{\sigma_0})^c \subset \cup_\lambda V_\lambda$. Indeed, if this was not the case, we may choose $c_\sigma \in (W_\sigma)^c \backslash \cup_\lambda V_\lambda$ for each σ, and then the closed set $\overline{\{c_\sigma : \sigma \in \Sigma\}}^\tau$ intersects no V_λ and is not contained in any W_σ, contradicting the covering assumption. Since $(W_{\sigma_0})^c$ is compact, it follows $(W_{\sigma_0})^c \subset \cup_{k=1}^m V_{\lambda_k}$ and then clearly $W_{\sigma_0}^+ \cup_k V_{\lambda_k}^- = \mathrm{CL}(C)$. ☐

Proof of Proposition 3.5. Let $I : \Theta \times \mathrm{CL}(C) \twoheadrightarrow \Theta \times C$ be the *identity* correspondence

$$(\theta, D) \mapsto I(\theta, D) := \{\theta\} \times D.$$

It is easy to see that I is a continuous correspondence (see (Aliprantis and Border, 2006, Lemmata 17.4-17.5)) where the domain is given the topology $(\tau^\Theta \times \tau_F)$ and the range is given $(\tau^\Theta \times \tau^C)$. Since by Assumption 3.1, function U is jointly continuous, it follows by (Aliprantis and Border, 2006, Berge Maximum Theorem 17.31) that U^* is continuous too. ☐

Proof of Proposition 3.6. Since $D \in \mathrm{CL}(C)$ is compact, we have $\Phi(\theta, D) \neq \emptyset$ by the continuity of $U(\theta, \cdot)$. Further, since set $D \cap U(\theta, \cdot)^{-1}(\{U^*(\theta, D)\}) \subset C$ is closed, it must be compact too, showing that $\Phi(\theta, D)$ is compact.

We now prove upper hemicontinuity. Since C is compact, by (Aliprantis and Border, 2006, 17.11 Closed Graph Theorem) we need only check that the graph of Φ is closed. Let $\{(\theta_\lambda, D_\lambda)\}_{\lambda \in \Lambda}$ be a net with $(\theta_\lambda, D_\lambda) \to (\theta, D)$, and let $f_\lambda \in \Phi(\theta_\lambda, D_\lambda)$ with $f_\lambda \to f$. We must show that $f \in \Phi(\theta, D)$. Since f is a limit point of $\{f_\lambda\}$, with $f_\lambda \in D_\lambda$ and $D = \mathrm{Li}(D_\lambda)$, we obtain $f \in D$. On the other hand, by definition

$$U(\theta_\lambda, f_\lambda) = U^*(\theta_\lambda, D_\lambda).$$

Since U and U^* are continuous (Proposition 3.5), we can go into the limit, finding

$$U(\theta, f) = U^*(\theta, D).$$

This finding and $f \in D$ conclude the proof. □

Proof of Proposition 3.8. By Proposition 3.6, the correspondence $(\theta, D) \mapsto \Phi(\theta, D)$ is upper hemicontinuous and has non-empty compact values. On the other hand, by Assumption 3.1, the function $(\theta, f) \mapsto V(\theta, f)$ is upper semicontinuous. It follows that the mapping $V^*(\cdot, \cdot)$ is upper semicontinuous; see (Berge, 1963, Theorem 2 (p. 116)) or (Aliprantis and Border, 2006, Lemma 17.30). □

Proof of Theorem 3.9. First, define $F^\kappa : \mathrm{CL}(C) \to \mathbb{R}$ by

$$F^\kappa(D) := \int_\Theta V^*(\theta, D) d\kappa(\theta),$$

for $\kappa \in \mathbb{K}$. We have $V^*(\theta, D) \le \sup_{\theta \in \Theta, c \in C} V(\theta, c)$, the r.h.s. of which is finite by compactness and u.s.c. of V. By Lemma A.3, taking $g := V^*$, we have that F^κ is well-defined and, in fact, upper semicontinuous in $\mathrm{CL}(C)$. It follows that $\inf_{\kappa \in \mathbb{K}} F^\kappa(\cdot)$ is also upper semicontinuous. The set T is non-empty and compact, see (7). The result follows. □

Proof of Proposition 3.10. Since V is assumed bounded, the proof follows from the same arguments as for Theorem 3.9, using Corollary A.4 instead of Lemma A.3. □

Proof of Theorem 3.12. Let D^* be the optimizer of (3.9). Since set $\Phi(\theta, D^*)$ is non-empty and compact for each θ, we deduce that set

$$\arg\max_{d \in \Phi(\theta, D^*)} V(\theta, d)$$

is non-empty. By axiom of choice, we select $\varphi^*(\theta) \in \arg\max_{d \in \Phi(\theta, D^*)} V(\theta, d)$, thus, by definition,

$$V(\theta, \varphi^*(\theta)) = V^*(\theta, D^*), \text{ for all } \theta.$$

Observe that $V(\cdot, \varphi^*(\cdot))$ is measurable, because $V^*(\cdot, D^*)$ is measurable. In addition, since $\varphi^*(\theta) \in D^*$ for each θ, we deduce in particular

$$\varphi^*(\theta) \in \Phi(\theta, D^*),$$

which yields that

$$U(\theta, \varphi^*(\theta)) \ge U(\theta, \varphi^*(\theta')), \text{ for all } \theta' \in \Theta.$$

Finally, since $D^* \in T$ and $\varphi^*(\theta) \in \Phi(\theta, D^*)$ we conclude that $U(\theta, \varphi^*(\theta)) \ge \underline{u}(\theta)$. Thus far, we have established that $\varphi^* \in \hat{M}$. Now, let $\varphi \in \hat{M}$ arbitrary, and let D be the closure of $\varphi(\Theta)$. It follows that $\varphi(\theta) \in \Phi(\theta, D)$, since φ is incentive compatible and by continuity of U. Thus $V(\theta, \varphi(\theta)) \le V^*(\theta, D)$ by definition. Since φ is individually rational, it follows $D \in T$. All in all,

$$\begin{aligned}\int_\Theta V(\theta, \varphi(\theta)) d\kappa(\theta) &\le \int_\Theta V^*(\theta, D) d\kappa(\theta) \\ &\le \int_\Theta V^*(\theta, D^*) d\kappa(\theta) = \int_\Theta V(\theta, \varphi^*(\theta)) d\kappa(\theta),\end{aligned}$$

from which φ is optimal for Problem (12).

□

Proof of Lemma 4.5. Without loss of generality, we may assume that $u \geq 0$. By (14) and the definitions of v and $\mathcal{X}(e_a)^0$ we obtain that C is bounded in $L^1(Q)$. By Eberlein-Smulian's theorem, it is enough to check sequential compactness. By contradiction suppose that C is not $\sigma(L^1(Q), L^\infty(Q))$-relatively compact, and hence not uniformly integrable. As in the proof of (Kramkov and Schachermayer, 2003, Lemma 1), we obtain the existence of a sequence $\{g_n\} \subset \mathcal{X}(e_a)$, a sequence of disjoint measurable events $\{A_n\}$ and an $\alpha > 0$ such that

$$E_Q[u(g_n)1_{A_n}] \geq \alpha.$$

Accordingly, if we define $H_k := \sum_{n<k} u(g_n)1_{A_n}$ we find $E_Q[H_k] \geq k\alpha$. Note that as in (Kramkov and Schachermayer, 1999, Lemma 6.3), $u^*(y/2) \leq au^*(y) + b$, and thus, in particular,

$$u^*(y2^{-m}) \leq a_m u^*(y) + b_m.$$

Now, clearly

$$H_k = u\left(\sum_{n\leq k} g_n 1_{A_n}\right) \leq u^*(Y) + Y\sum_{n\leq k} g_n 1_{A_n},$$

for every non-negative random variable Y. Let us temporarily assume that we could choose Y^* independent of k and such that for some $L < \alpha$:

$$E_Q\left[Y^* \sum_{n\leq k} g_n 1_{A_n}\right] \leq Lk. \tag{19}$$

Then from the above computations we would have

$$k\alpha \leq E_Q[H_k] \leq E_Q[u^*(Y^*)] + Lk,$$

and hence, if further

$$E_Q[u^*(Y^*)] < \infty, \tag{20}$$

this would yield a contradiction with $L < \alpha$. We now show the existence of Y^* fulfilling (19) and (20). By Assumption (14), there exists $y > 0, Y \in \mathcal{X}(e_a)^0$ such that $E_Q[u^*(yY)] < \infty$. However, then

$$E_Q[u^*(2^{-m}yY)] < \infty,$$

for all $m > 0$. We now take m large so that $L := y2^{-m} < \alpha$ and define $Y^* := LY$. Then

$$E_Q\left[Y^* \sum_{n\leq k} g_n 1_{A_n}\right] \leq E_Q\left[Y^* \sum_{n\leq k} g_n\right] \leq Lk.$$

Hence Y^* fulfills (19)-(20) as desired.

It remains to show that C is weakly closed. First we show that C is convex. Since u^{-1} is increasing and u concave, for $X, \bar{X} \in \mathcal{X}(e_a)$ and $\beta \in [0,1]$ we have

$$\begin{aligned} u^{-1}\left(\beta u(X) + (1-\beta)u(\bar{X})\right) &\leq u^{-1}\left(u(\beta X + (1-\beta)\bar{X})\right) \\ &= \beta X + (1-\beta)\bar{X} \in \mathcal{X}(e_a). \end{aligned}$$

Because $\mathcal{X}(e_a)$ is solid, then $u^{-1}\left(\beta u(X)+(1-\beta)u(\bar{X})\right) \in \mathcal{X}(e_a)$, and therefore $\beta u(X)+(1-\beta)u(\bar{X}) \in C$. Since C is convex, it is weakly closed if and only if it is strongly closed. Let us recall the bipolar theorem of (Brannath and Schachermayer, 1999, Theorem 1.3), which applies here since $\mathcal{X}(e_a)$ is convex, solid, and closed in $L^0(Q)$, that states

$$X \in \mathcal{X}(e_a) \iff E_Q[XY] \leq 1 \text{for all } Y \in \mathcal{X}(e_a)^0.$$

Now let $\{U(X_n)\}_n \subset C$ converge in L^1-norm to Z, and let $Y \in \mathcal{X}(e_a)^0$ be arbitrary. Then,

$$E_Q[Yu^{-1} \circ u(X_n)] = E_Q[YX_n] \leq 1,$$

and therefore, by Fatou's lemma $E_Q[Yu^{-1}(Z)] \leq 1$ too, and by the bipolar theorem $u^{-1}(Z) \in \mathcal{X}(e_a)$. Consequently, $Z \in C$, finishing the proof. □

Proof of Lemma 4.6. Observe that $\mathcal{X}(e_a) \subset \mathcal{X}(\|e_a\|)$, and, as a consequence,

$$\forall X \in \mathcal{X}(\|e_a\|) : E_Q[XY] \leq 1 \Rightarrow \forall X \in \mathcal{X}(e_a) : E_Q[XY] \leq 1.$$

Assumption (15) allows us to apply (Kramkov and Schachermayer, 1999, Theorem 3.1) to the pairing between $\mathcal{X}(\|e_a\|)$ and its polar. As a particular consequence, there exists $y \geq 0$ and Y in the polar of $\mathcal{X}(\|e_a\|)$ such that

$$E_Q[u^*(yY)] < \infty.$$

Since the polar of $\mathcal{X}(e_a)$ is larger, we conclude the proof. □

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